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Using Gaussian Function to Construct the Appropriate Wavelet Function for Accurate Computation of Derivative

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Abstract: Maurice Hasson has constructed wavelet function using maxican hat function and richardson extrapolation technique has been used in his work. In this paper the author have constructing the appropriate wavelet function by using gaussian function. This wavelet is usfull for accurate computation of derivative. we also use Richardson Extrapolation technique in our construction. We present many vanishing moment condition in our construction when being convolved in a precise manner.

Keywords: Richardson extrapolation technique, Error Estimate, Taylor Series.

1. Introduction and Preliminaries

Wavelet transform for derivative calculation has been reported based on the property of specific wavelet function: Gaussian wavelets. The underlying principle of the wavelet transform for derivative calculation is investigated, and a general approach is proposed. By theoretical analysis, it can be found that wavelet transform with commonly used wavelet functions can be regarded as a smoothing and a differentiation process and the order of differentiation is determined by the property of the wavelet function. Derivatives of different simulated signals by using all the commonly used wavelet functions are investigated and compared with the conventional numerical differentiation method. It is shown that differentiation is a common property of all these wavelet functions, and $n^{th}-$order derivative can be simply obtained through just one transform procedure, instead of repeated transform, by using an appropriate wavelet function.

In the same way, Maurice Hasson has constructed wavelet function using Maxican hat function and Richardson extrapolation technique has been used in his work. In this present work we construct the appropriate wavelet function by using Gaussian function. This wavelet is useful for accurate computation of derivative. We also use Richardson extra-polarisation technique in our construction. We present many vanishing moment condition in our construction when being convolved in a precise manner.

In this paper we use the Gaussian function for construct the appropriate wavelet function.

Definition 1.1. *In mathematics, a Gaussian function has the following form*

$$f(x) = ae^{\frac{-(x-b)^2}{2c^2}} \tag{1}$$

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where a, b and c are arbitrary real constants. The graph of Gaussian is a characteristic symmetric "bell curve" shape in which a represent the hight of the curve's peak, b is the position of the center of the peak, and c (standard deviation) is the parameter for controlling the width of the "bell", and we use the following notations, Fourier transform of function $f(x)$ is

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{2}$$

and the inverse formula takes the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \tag{3}$$

2. Construction of Wavelet Function

In this section we construct the wavelet $\psi(x)$ satisfying the following properties 1 and 2.

$$\int_{-\infty}^{\infty} t^k \psi_2(t) dt = \begin{cases} 0, & \text{if} \quad k = 1, 2, 3, 5, 7, 9, \\ 1, & \text{if} \ \ k = 0 \end{cases} . \tag{4}$$

The required wavelet is built by using the classical Richardson extrapolation method. We begin with the Gaussian function

$$\psi(x) = e^{\frac{-x^2}{2}}.$$

The Fourier transform of $\psi(x)$ is

$$\hat{\psi}(\omega) = e^{\frac{-\omega^2}{2}}.$$

Now let

$$\hat{\psi}_1(\omega) = \frac{\hat{\psi}(\omega)}{2} = \frac{e^{\frac{-\omega^2}{2}}}{\sqrt{2\pi}}$$

and the Inverse Fourier transform of $\psi_1(\omega)$ is

$$\psi_1(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}.$$

Let

$$\hat{\psi}_1(\omega) = \frac{e^{\frac{\omega^2}{2}}}{\sqrt{2\pi}} \quad \textit{and} \quad \hat{\psi}_1(0) = \frac{1}{\sqrt{2\pi}}$$

then

$$\hat{\psi}_1'(\omega) = \frac{-\omega e^{\frac{-\omega^2}{2}}}{\sqrt{2\pi}} \quad \textit{and} \quad \hat{\psi}_1'(0) = 0$$

and

$$\hat{\psi}_1''(0) = -\frac{1}{\sqrt{2\pi}} \quad \textit{and} \quad \hat{\psi}_1'''(0) = 0.$$

Hence we have

$$\int_{-\infty}^{\infty} \frac{1}{h} \psi_1(\frac{x}{h}) dx = 1.$$

We have

$$\hat{\psi}_1(\omega) = -\frac{e^{\frac{-\omega^2}{2}}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \left(1 - \frac{\omega^2}{2} + \frac{\omega^4}{8} - \frac{\omega^6}{48} + \dots\right)$$

$$\hat{\psi}_1\left(\frac{\omega}{2}\right) = \frac{e^{\frac{\omega^2}{8}}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \left(1 - \frac{\omega^2}{8} + \frac{\omega^4}{128} - \frac{\omega^6}{3072} \dots\right).$$

Next we use the Richardson extrapolation technique we get

$$\hat{\psi}_2(\omega) = \frac{\hat{\psi}_1(\omega) - 4\hat{\psi}_1(\frac{\omega}{2})}{-3}.$$

We have

$$\hat{\psi}_2(0) = \frac{1}{\sqrt{2\pi}}, \quad \hat{\psi}_2'(0) = 0 \quad and \quad \hat{\psi}_2''(0) = 0.$$

The inverse Fourier transform of $\hat{\psi}_2(\omega)$ is

$$\begin{aligned}
\psi_2(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_2(\omega) e^{i\omega x} d\omega \\
\psi_2(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{\psi}_1(\omega) - 4\hat{\psi}_1(\frac{\omega}{2})}{-3} e^{i\omega t} d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{\psi}_1(\omega) e^{i\omega x}}{-3} d\omega - \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{\psi}_1(\frac{\omega}{2}) e^{i\omega x}}{-3} d\omega \\
&= \frac{1}{-3} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_1(\omega) e^{i\omega x} d\omega - \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_1(\frac{\omega}{2}) e^{i\omega x} d\omega \right] \\
&= \frac{1}{-3} \left[\psi_1(x) - \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_1(\frac{\omega}{2}) e^{i\omega x} d\omega \right]
\end{aligned}$$

on taking $\frac{\omega}{2} = u$ then $d\omega = 2du$ and by using equation (3), we get

$$\begin{aligned}
\psi_2(x) &= \frac{1}{-3} \left[\psi_1(x) - \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_(2u) e^{i2ux} \right] 2du \\
&= \frac{1}{-3} \left[\psi_1(x) - \frac{8}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_1(2u) e^{i2ux} \right] du \\
&= \frac{1}{-3} \left[\psi_1(x) - \frac{8}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}_1(2u) e^{i2ux} du \right] \\
&= \frac{1}{-3} \left[\psi_1(x) - \psi_1(2x) \right] \\
\psi_2(x) &= \frac{\psi_1(x) - 8\psi_1(2x)}{-3} \qquad (5) \\
\psi_2(x) &= \frac{e^{-\frac{x^2}{2}} - 8e^{-2x^2}}{-3\sqrt{2\pi}} \qquad (6)
\end{aligned}$$

Theorem 2.1. *For the wavelet $\psi_2(x)$ defined by*

$$\psi_2(x) = \frac{e^{-\frac{x^2}{2}} - 8e^{-2x^2}}{-3\sqrt{2\pi}}$$

equation (4) hold for

$$\int_{-\infty}^{\infty} x^k \psi_2(x) dx = \begin{cases} 0, & if \quad k = 1, 2, 3, 5, 7, 9 \\ 1, & if \quad k = 0 \end{cases}. \qquad (7)$$

Theorem 2.2. *Let $f(x)$ be a smooth function. Then in infinite precision arithmetic*

$$\frac{1}{h} \int_{-\infty}^{\infty} f(x-t) \psi_2(\frac{t}{h}) dt - f(x) = C_1 f^4(x) h^4 + C_2 f^6(x) h^6 + O(h^8)$$

where $\psi_2(x)$ is given in the equation (6).

$$C_1 = \frac{1}{4!} \int_{-\infty}^{\infty} t^4 \psi_2(t) dt = 0.09375$$

and

$$C_2 = \frac{1}{6!} \int_{-\infty}^{\infty} t^6 \psi_2(t) dt = 0.01953125$$

Proof. Consider

$$\frac{1}{h}\int_{-\infty}^{\infty} f(x-t)\psi_2(\frac{t}{h})dt$$

and taking $\frac{t}{h}=u$ then $dt=hdu$, we get

$$\int_{-\infty}^{\infty} f(x-ht)\psi_2(t)dt$$

$$\int_{-\infty}^{\infty}\left(f(x)-thf'(x)\right)\psi_2(t)dt \quad +O(h^2)$$

Now

$$\lim_{h\to 0}\frac{1}{h}\int_{-\infty}^{\infty} f(x-t)\psi_2\left(\frac{t}{h}\right)dt=f(x)$$

$$\int_{-\infty}^{\infty}\frac{1}{h}f(x-t)\psi_2\left(\frac{t}{h}\right)dt-f(x)$$

$$\int_{-\infty}^{\infty}\left(\frac{1}{h}f(x-t)\psi_2\left(\frac{t}{h}\right)-\frac{1}{h}\psi_2\left(\frac{t}{h}\right)f(x)\right)dt$$

Now we use equation (4), Hence

$$\frac{1}{h}f(x-t)\psi_2(\frac{t}{h})dt-f(x)=\int_{-\infty}^{\infty}(f(x-th)-f(x))\psi_2(t)dt \tag{8}$$

Now use

$$f(x-ht)=\sum_{k=0}^{7}(-1)^k\frac{f^k(x)}{k!}t^kh^k+O(t^8h^8)$$

Now together with equation (4), (8) to conclude that

$$\int_{-\infty}^{\infty}\frac{1}{h}f(x-t)\psi_2(\frac{t}{h})dt-f(x)=\int_{-\infty}^{\infty}(\frac{t^4h^4}{4!}f^4(x))\psi_2(t)dt+\int_{-\infty}^{\infty}(\frac{t^6h^6}{6!}f^6(x))\psi_2(t)dt+O(h^8) \tag{9}$$

Now calculate the values of the quantities C_1 and C_2 We have

$$\int_{-\infty}^{\infty}t^4\psi_2(t)dt=\sqrt{2\pi}\hat{\psi}_2^4(0) \tag{10}$$

By using equation (10). We find that

$$\begin{aligned}
\hat{\psi}_2(\omega) &= \frac{1}{\sqrt{2\pi}}\left(-3+\frac{3\omega^4}{32}-\frac{15\omega^6}{768}+\dots\right) \\
\hat{\psi}_2(\omega) &= \frac{1}{\sqrt{2\pi}}\left(\frac{3.4.3.2.1}{32}-\frac{15.6.5.4.3.\omega^2}{768}\right) \\
\hat{\psi}_2^4(0) &= \frac{1}{\sqrt{2\pi}}\left(\frac{72}{32}-0\right)=\frac{2.25}{\sqrt{2\pi}}
\end{aligned} \tag{11}$$

Put the value of $\hat{\psi}_2^4(0)$ in equation (9). We get

$$\int_{-\infty}^{\infty}t^4\psi_2(t)dt=\frac{\sqrt{2\pi}2.25}{\sqrt{2\pi}}$$

$$\int_{-\infty}^{\infty}t^4\psi_2(t)dt=2.25$$

In similar manner we find

$$\int_{-\infty}^{\infty}t^6\psi_2(t)dt=\frac{\sqrt{2\pi}14.0625}{\sqrt{2\pi}}$$

$$\int_{-\infty}^{-\infty} t^6\psi_2(t)dt = 14.0625$$

The proof of the Theorem 2.2 is completed. Let us proof the stability of the system $\psi_2(x)$.

"An arbitary system is said to be stable iff every bounded input produces a bounded output."

If $f(x)$ (input function) is bounded, there exist a constant $'M'$ such that

$$|f(x)| \leq M \qquad for\ all\ \ x.$$

Now the convolution formula

$$g(x) = \int_{-\infty}^{\infty} f(x-t)\psi_2(t)dt$$

where $g(x)$ is the output signal, therefore

$$\begin{aligned}
|g(x)| &= |\int_{-\infty}^{\infty} f(x-t)\psi_2(t)dt| \\
&\leq \int_{-\infty}^{\infty} |f(x-t)||\psi_2(t)|dt \\
|g(x)| &\leq M\int_{-\infty}^{\infty} |\psi_2(t)|dt \qquad (12)
\end{aligned}$$

Now

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi_2(t)|dt &= \int_{-\infty}^{\infty} |\frac{e^{-\frac{t^2}{2}} - 8e^{-2t^2}}{-3\sqrt{2\pi}}dt| \\
\int_{-\infty}^{\infty} |\psi_2(t)|dt &\leq \frac{1}{3\sqrt{2\pi}}(\int_{-\infty}^{\infty} |e^{-\frac{t^2}{2}}dt| + 8\int_{-\infty}^{\infty} |e^{-2t^2}dt| \\
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq 2\int_{0}^{\infty} |e^{-\frac{t^2}{2}}dt| + 16\int_{0}^{\infty} |e^{-2t^2dt}|
\end{aligned}$$

Let us consider $\frac{t^2}{2} = u$ then $tdt = du$ and $2t^2 = v$ also $4tdt = dv$

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq \frac{1}{3\sqrt{2\pi}}\left(2\int_{o}^{\infty} |\frac{e^{-u}}{\sqrt{2\pi}}du| + 16\int_{0}^{\infty} |\frac{e^{-v}}{4\sqrt{\frac{v}{2}}}dv|\right) \\
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq \frac{1}{3\sqrt{2\pi}}\left(\frac{2}{\sqrt{2}}\int_{0}^{\infty} u^{\frac{1}{2}-1}e^{-u}du + \frac{16\sqrt{2}}{4}\int_{0}^{\infty} v^{\frac{1}{2}-1}e^{-v}dv\right)
\end{aligned}$$

By the definition and properties of Gamma function we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq \frac{1}{3\sqrt{2\pi}}\left(\frac{2}{\sqrt{2}}\Gamma\frac{1}{2} + 8\sqrt{2}\Gamma\frac{1}{2}\right) \\
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq \frac{1}{3\sqrt{2\pi}}\left(\frac{2\sqrt{\pi}}{\sqrt{2}} + 8\sqrt{2\pi}\right) \\
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq \frac{1}{3}(1+8) \\
\int_{-\infty}^{\infty} |\psi_2(t)dt| &\leq 3.
\end{aligned}$$

This together with our output $g(x)$ is also bounded. Hence system $\psi_2(x)$ is stable. □

3. Conclusion

In this paper, we generalized Richardson extrapolation technique by using the Gaussian function to construct the appropriate wavelet function. This wavelet is useful for accurate computation of derivative. We also use Richardson extrapolation technique in our construction and shows existence of many vanishing moment when being convolved in a precise manner with the function.

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