Int. J. Math. And Appl., 8(1)(2020), 231–237
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Universal Minimal Resolving Functions in Graphs

Varughese Mathew[1,*] and S. Arumugam[2,3,4]

1 Department of Mathematics, Mar Thoma College Tiruvalla, Kerala, India.

2 National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH), Kalasalingam University, Tamil Nadu, India.

3 Adjunct Professor, Amrita Vishwa Vidyapeetham, Coimbatore, Tamil Nadu, India.

4 Adjunct Professor, Ball State University, Muncie, USA.

Abstract: A vertex x in a connected graph $G = (V, E)$ is said to resolve a pair $\{u, v\}$ of vertices of G if the distance from u to x is not equal to the distance from v to x. For the pair $\{u, v\}$ of vertices of G the collection of all resolving vertices is denoted by $R\{u, v\}$ and is called the resolving neighborhood for the pair $\{u, v\}$. A real valued function $g : V \to [0, 1]$ is a resolving function (RF) of G if $g(R\{u, v\}) \geq 1$ for all distinct pair $u, v \in V$. A resolving function g is minimal (MRF) if any function $f : V \to [0, 1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$ is nota resolving function of G. A minimal resolving function (MRF) is called a universal minimal resolving function $(UMRF)$ if its convex combination with every other MRF is again an MRF. Minimal resolving functions are related to the fractional metric dimension of graphs. In this paper, we initiate a study of universal minimal resolving functions of a connected graph G.

MSC: 05C12.

Keywords: Metric dimension, Fractional metric dimension, Resolving set, Resolving function, Universal minimal resolving function.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected and connected graph with neither loops nor parallel edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology, we refer to Chartrand and Lesniak [5]. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest u-v path in G. By an ordered set of vertices we mean a set $W = \{w_1, w_2, \cdots, w_k\}$ on which the ordering $(w_1, w_2, \cdots, w_k)$ has been imposed. For an ordered subset $W = \{w_1, w_2, \cdots, w_k\}$ of V, we refer to the k-vector (ordered k-tuple) $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ as the (metric) representation of v with respect to W. The set W is called a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$ for all $u, v \in V(G)$. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v|W) : v \in V\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality for a graph G is called a basis for G and the metric dimension of G is defined to be the cardinality of a basis of G and is denoted by $dim(G)$. A resolving set W of G is a minimal resolving set if no proper subset of W is a resolving set.

A vertex $x \in V$ is said to resolve a pair of vertices $\{u, v\}$ in G if $d(u, x) \neq d(v, x)$. Let V_p denote the collection of all $\binom{n}{2}$ pairs of vertices of G. Fehr et al. [9] have defined the Resolving graph $R(G)$ of a connected graph $G = (V, E)$ as a bipartite graph with bipartition (V, V_p) where a vertex $x \in V$ is joined to a pair $\{u, v\} \in V_p$ if and only if x resolves $\{u, v\}$

* E-mail: varughesemathewmtc@gmail.com

in G. Then the minimum cardinality of a subset S of V such that $N(S) = V_p$ in $R(G)$ is the metric dimension of G, where $N(u) = \{v \in V | uv \in E(G)$.

The idea of resolving sets has appeared in the literature previously. In [16] and later in [17], Slater introduced the concept of a resolving set for a connected graph G under the term locating set. He referred to a minimum resolving set as a reference set for G. He called the cardinality of a minimum resolving set (reference set) the location number of G. Independently, Harary and Melter [10], discovered these concepts as well but used the term metric dimension.

Applications of resolving sets arise in various areas including coin weighing problem [15], drug discovery [4], robot navigation [11], network discovery and verification [3], connected joins in graphs [14] and strategies for the mastermind game [7]. For a survey of results in metric dimension, we refer to Chartrand and Ping [6]. Chartrand et al. [4] formulated the problem of finding the metric dimension of a graph as an integer programming problem. Fehr et al. [9], used this idea to formulate the fractional version of metric dimension as follows.

Suppose $V = \{v_1, v_2, \cdots, v_n\}$ and $V_p = \{s_1, s_2, \cdots, s_{\binom{n}{2}}\}$. Let $A = (a_{ij})$ be the $\binom{n}{2} \times n$ matrix with $a_{ij} = 1$ if $s_i v_j \in E(R(G))$ and 0 otherwise, where $1 \leq i \leq \binom{n}{2}$ and $1 \leq j \leq n$. The integer programming formulation of the metric dimension is given by Minimize $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$

Subject to $A\overline{x} \geq \overline{1}$ where $\overline{x} = (x_1, x_2, \cdots, x_n)^T$, $x_i \in \{0,1\}$ and $\overline{1}$ is the $\binom{n}{2} \times 1$ column vector all of whose entries are 1. The optimal solution of the linear programming relaxation of the above I.P, where we replace $x_i \in \{0,1\}$ by $0 \leq x_i \leq 1$, gives the fractional metric dimension of G, which we denote by $dim_f(G)$. The credity of obtaining basic results on fractional metric dimension of graphs goes to Arumugam and Mathew ([1, 2]).

For a detailed study of fractional graph theory and fractionalization of various graph parameters, we refer to Scheinerman and Ullman [13].

In this paper we develop a theory for universal minimal resolving functions analogous to minimal dominating functions [8] of a graph.

2. Basic Results

Definition 2.1 ([1]). *Let $G = (V, E)$ be a connected graph of order n. A function $f : V \to [0,1]$ is called a resolving function (RF) of G if $f(R\{u,v\}) \geq 1$ for any two distinct vertices $u, v \in V$, where $f(R\{u,v\}) = \sum_{x \in R\{u,v\}} f(x)$. A resolving function g of a graph G is minimal (MRF) if any function $f : V \to [0,1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$ is not a resolving function of G.*

$MRFs$ generalise the concept of minimal resolving sets of vertices, since the integer valued (i.e. 0 or 1) $MRFs$ are precisely the characteristic functions of the minimal resolving sets of a graph. Mathew and Arumugam [12], initiated a study of minimal resolving functions of a connected graph G and defined the Resolving convexity graph $C_R(G)$. We need the follwing definitions and theorems.

Theorem 2.2 ([12]). *Let f be a resolving function of a connected graph $G = (V, E)$. Then f is a minimal resolving function of G if and only if whenever $f(x) > 0$ there exists $\{u, v\} \in V_P$ such that $x \in R\{u,v\}$ and $f(R\{u,v\}) = 1$.*

Definition 2.3 ([12]). *Let f be a RF of a graph G. The boundary set $\mathcal{B}_f$ and the positive set $\mathcal{P}_f$ of f are defined by $\mathcal{B}_f = \{\{u,v\} \in V_p : f(R\{u,v\}) = 1\}$ and $\mathcal{P}_f = \{u \in V(G) : f(u) > 0\}$.*

Definition 2.4 ([12]). *Let $x \in V$ and $D \subseteq V_P$. We say that x resolves D, if there exists a pair $\{u,v\} \in D$ of such that $d(u,x) \neq d(v,x)$ and write $x \xrightarrow[r]{} D$. Let $S \subseteq V(G)$ and $D \subseteq V_p$. We say S resolves D if $x \xrightarrow[r]{} D$ for all $x \in S$ and write $S \xrightarrow[r]{} D$.*

Theorem 2.5 ([12]). *A resolving function f of a graph G is a minimal resolving function if and only if $\mathcal{P}_f \underset{r}{\rightarrow} \mathcal{B}_f$.*

Example 2.6. *For the cycle $C_5 = (u_1u_2u_3u_4u_5u_1)$, the different $R\{u, v\}$ sets are $R\{u_1, u_2\} = \{u_1, u_2, u_3, u_5\}$, $R\{u_1, u_3\} = \{u_1, u_3, u_4, u_5\}$, $R\{u_1, u_4\} = \{u_1, u_2, u_3, u_4\}$, $R\{u_1, u_5\} = \{u_1, u_2, u_4, u_5\}$, $R\{u_2, u_5\} = \{u_2, u_3, u_4, u_5\}$. Define $f : V(C_5) \to [0, 1]$ defined by $f(u_1) = 0.8$, $f(u_2) = 0.25$, $f(u_3) = 0.75$ and $f(u_4) = 0 = f(u_5)$. Then f is a resolving function of C_5. Here, $\mathcal{P}_f = \{u_1, u_2, u_3\}$ and $\mathcal{B}_f = \{\{u_2, u_5\}\}$. Also $\mathcal{P}_f$ does not resolve $\mathcal{B}_f$, since $d(u_1, u_2) = d(u_1, u_5) = 1$. Hence f is not an MRF of G.*

Theorem 2.7. *Let S be a minimal resolving set of a connected graph $G = (V, E)$. Then $f = \chi_S$ is a minimal resolving function of G.*

Proof. Clearly, $f = \chi_S$ is a resolving function of G. Suppose f is not minimal. Then P_f does not resolve B_f. That is, S does not resolve B_f since $P_f = S$. This implies, there exist $y \in S$ such that $d(u, y) = d(v, y)$ for all $\{u, v\} \in B_f$......(1). But S is minimal and $y \in S$ implies, there exists $\{x, w\} \in V_p$ such that $R\{x, w\} \cap S = \{y\}$. Hence $f(R\{x, w\}) = 1$ and thus $\{x, w\} \in B_f$ and $d(x, y) \neq d(w, y)$, which is a contradiction to (1). Hence $f = \chi_S$ is a minimal resolving function of G. □

Definition 2.8 ([8]). *Let f and g be RFs of G and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1 - \lambda)g$ is called a convex combination of f and g.*

Theorem 2.9 ([12]). *A convex combination of two resolving functions of a graph G is again a resolving function of G.*

Remark 2.10 ([12]). *A convex combination of two $MRFs$ of a graph G need not be an MRF of G.*

For example, consider the cycle $G = C_5 = (u_1u_2u_3u_4u_5u_1)$, the different $R\{u, v\}$ sets are $R\{u_1, u_2\} = \{u_1, u_2, u_3, u_5\}$, $R\{u_1, u_3\} = \{u_1, u_3, u_4, u_5\}$, $R\{u_1, u_4\} = \{u_1, u_2, u_3, u_4\}$, $R\{u_1, u_5\} = \{u_1, u_2, u_4, u_5\}$, $R\{u_2, u_5\} = \{u_2, u_3, u_4, u_5\}$. The function $f : V(G) \to [0, 1]$ defined by $f(u_1) = 1 = f(u_2)$ and $f(u_3) = 0 = f(u_4) = f(u_5)$ is a MRF of G. Also, the function $g : V(G) \to [0, 1]$ defined by $g(u_1) = 0 = g(u_2) = g(u_3)$ and $g(u_4) = 1 = g(u_5)$ is a MRF of G. Let $h = \frac{1}{2}f + \frac{1}{2}g$. Then $h(u_1) = \frac{1}{2}$, $h(u_2) = \frac{1}{2}$, $h(u_3) = 0$, $h(u_4) = \frac{1}{2}$,and $h(u_5) = \frac{1}{2}$. So, h is a resolving function. But $\mathcal{P}_h = \{u_1, u_2, u_4, u_5\}$ and $\mathcal{B}_h = \phi$. Clearly, $\mathcal{P}_h$ does not resolve $\mathcal{B}_h$. Hence, h is not minimal. Note that in this example $\mathcal{P}_h = \mathcal{P}_f \cup \mathcal{P}_g$ and $\mathcal{B}_h = \mathcal{B}_f \cap \mathcal{B}_g$.

The following theorem gives a necessary and sufficient condition for the convex combination of two minimal resolving functions to be minimal.

Theorem 2.11 ([12]). *Let $G = (V, E)$ be a connected graph. Let f and g be two $MRFs$ of G. Then any convex combination of f and g is again a MRF of G if and only if $\mathcal{P}_f \cup \mathcal{P}_g \underset{r}{\rightarrow} \mathcal{B}_f \cap \mathcal{B}_g$*

3. Main Section

Cockayne et al. [8] introduced the concept of universal dominating functions of a graph and investigated the existence of such functions. We now introduce the analogous concept of universal minimal resolving function.

Let $\mathcal{F}_R$ denote the set of all minimal resolving functions of a graph G. We define a binary relation $\mathcal{R}$ on the set $\mathcal{F}_R$ as follows: For $f, g \in \mathcal{F}_R$, $f\mathcal{R}g$ if and only if $h_\lambda = \lambda f + (1 - \lambda)g$ is an MRF of G for all $\lambda \in (0, 1)$.

By Theorem 2.9, for all $\lambda \in (0, 1)$, the convex combination $\lambda f + (1 - \lambda)g$ of resolving functions f, g of G, is also a resolving function. However a convex combination of two minimal resolving functions need not be minimal (Remark 2.10). This fact led to the concept of a universal minimal resolving function. The study of universal $MRFs$ has an answer to the following interpolation problem.

The weight of a function f is $|f| = f(V) = \sum_{u \in V} f(u)$. Let f and g be minimal resolving functions of G with weights $|f| = \alpha$ and $|g| = \beta$. Suppose $t \in (\alpha, \beta)$. Does G have a minimal resolving function h with $|h| = t$?. The answer is affirmative if G has a universal MRF.

Definition 3.1. *A universal minimal resolving function (UMRF) is an MRF f whose convex combinations with any other MRF is also minimal, or equivalently $f\mathcal{R}g$ for all $g \in \mathcal{F}_R$. That is, an MRF g is universal if $f\mathcal{R}g \in \mathcal{F}_R$ for all $f \in \mathcal{F}_R$.*

Proposition 3.2. *If an MRF g of a graph $G = (V, E)$ satisfies $\mathcal{B}_g = V_p$ and for all MRFs f of G, $V \underset{r}{\rightarrow} \mathcal{B}_f$ then g is an universal MRF of G.*

Proof. For any MRF f, we have $\mathcal{B}_g \cap \mathcal{B}_f = V_p \cap \mathcal{B}_f = \mathcal{B}_f$. Also, we have $\mathcal{P}_f \cup \mathcal{P}_g \subseteq V$. Thus $V \underset{r}{\rightarrow} \mathcal{B}_f$ implies $\mathcal{P}_f \cup \mathcal{P}_g \underset{r}{\rightarrow} \mathcal{B}_f \cap \mathcal{B}_g$ and hence by Theorem 2.11, g is a universal MRF. □

Using the above proposition, we prove the existence of universal MRFs for certain classes of graphs.

Theorem 3.3. *The path $P_n, n \geq 3$ has a universal MRF.*

Proof. Let $P_n = (u_1, u_2, \ldots, u_n)$. Clearly $u_1, u_n \in R\{u, v\}$ for all $u, v \in V(P_n)$. Also for any $u, v \in V(P_n)$ we have

$$R\{u, v\} = \begin{cases} V(P_n) & \text{if } d(u, v) \text{ is odd} \\ V(P_n) - \{u_r\} & \text{if } d(u, v) \text{ is even,} \end{cases}$$

where u_r is the central vertex of the u-v path if $d(u, v)$ is even. Hence the function $g : V(P_n) \to [0, 1]$ defined by $g(u_1) = \frac{1}{2} = g(u_n)$ and $g(v) = 0$ for all $v \in V - \{u_1, u_n\}$, is an MRF of P_n with $\mathcal{B}_g = V_p$. We now claim that $V(P_n) \underset{r}{\rightarrow} \mathcal{B}_f$, for all MRF f. Suppose not. Then there exists an MRF f and a vertex $x \in V(P_n)$ such that x does not resolve $\mathcal{B}_f$. By Theorem 2.5, we have $\mathcal{P}_f \underset{r}{\rightarrow} \mathcal{B}_f$ and hence $f(x) = 0$. Let $\{u, v\} \in \mathcal{B}_f$. Then $f(R\{u, v\}) = 1$ and $x \notin R\{u, v\}$. Let y be a vertex adjacent to x. Then $R\{x, y\} = V(P_n)$ and hence $f(R\{x, y\}) = f(R\{u, v\}) + f(x) = 1$. Thus $\{x, y\} \in \mathcal{B}_f$ and $x \underset{r}{\rightarrow} \mathcal{B}_f$, which is a contradiction. Thus for all MRF f, $V(P_n) \underset{r}{\rightarrow} \mathcal{B}_f$ and hence by Proposition 3.2, P_n has a universal MRF. □

Theorem 3.4. *Any odd cycle C_n has a universal MRF.*

Proof. Let $C_n = (u_1 u_2 u_3 \ldots u_n u_1)$ where n is odd. Clearly for any $u, v \in V(C_n)$ we have $R\{u, v\} = V(C_n) - \{u_r\}$ where u_r is the central vertex of the u-v section of C_n having even length and hence $|R\{u, v\}| = n - 1$. Hence the function $g : V(C_n) \to [0, 1]$ defined by $g(v) = \frac{1}{n-1}$ for all $v \in V(C_n)$ is an MRF of C_n with $\mathcal{B}_g = V_p$.

We now claim that $V(C_n) \underset{r}{\rightarrow} \mathcal{B}_f$, for all MRF f. Suppose not. Then there exists an MRF f and a vertex $x \in V(C_n)$ such that x does not resolve $\mathcal{B}_f$. By Theorem 2.5, we have $\mathcal{P}_f \underset{r}{\rightarrow} \mathcal{B}_f$ and hence $f(x) = 0$. Let $\{u, v\} \in \mathcal{B}_f$. Then $f(R\{u, v\}) = 1$ and $x \notin R\{u, v\}$. Let $u_r \in R\{u, v\}$ be such that $f(u_r) = \epsilon > 0$. Then $x \in R\{u_{r-1}, u_{r+1}\}$ and $u_r \notin R\{u_{r-1}, u_{r+1}\}$ and hence $R\{u_{r-1}, u_{r+1}\} = R\{u, v\} \cup \{x\} - \{u_r\}$. Now $f(R\{u_{r-1}, u_{r+1}\}) = f(R\{u, v\}) + f(x) - f(u_r) = 1 - \epsilon < 1$, which is a contradiction. Thus for all MRF f, $V(C_n) \underset{r}{\rightarrow} \mathcal{B}_f$ and hence by Proposition 3.2, C_n has a universal MRF. □

Theorem 3.5. *For $n \geq 3$, the complete graph $G = K_n$ has a universal MRF.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Clearly $R\{u, v\} = \{u, v\}$ for all $u, v \in V(G)$ and hence the function $g(u) = \frac{1}{2}$ for all $u \in V(G)$ is an MRF of G with $\mathcal{B}_g = V_p$.

We now claim that $V(G) \underset{r}{\rightarrow} \mathcal{B}_f$, for all MRF f. Suppose not. Then there exists an MRF f and a vertex $x \in V(G)$ such that x does not resolve $\mathcal{B}_f$. By Theorem 2.5, we have $\mathcal{P}_f \underset{r}{\rightarrow} \mathcal{B}_f$ and hence $f(x) = 0$. Let $\{u, v\} \in \mathcal{B}_f$. Then $f(R\{u, v\}) = 1$ and $x \notin R\{u, v\}$. Thus $f(u) + f(v) = 1$. Let $f(u) = \lambda$ and $f(v) = 1 - \lambda$ where $0 < \lambda \leq 1$. If $0 < \lambda < 1$, then

$f(R\{u,x\}) = f(\{u,x\}) = f(u) + f(x) = \lambda + 0 < 1$, which is a contradiction. Similarly if $\lambda = 1$, then $f(R\{v,x\}) = f(\{v,x\}) = f(v) + f(x) = 0 + 0 < 1$, which is again a contradiction. Thus for all MRF f, $V(G)\underset{r}{\rightarrow}\mathcal{B}_f$ and hence by Proposition 3.2, G has a universal MRF. □

Proposition 3.6. *Let g be an MRF of a connected graph $G = (V,E)$. If there exists some $v \in V(G)$ such that v does not resolve B_g, then g is not a universal MRF of G.*

Proof. Let S be any minimal resolving set of G containing v and consider $f = \chi_S$, the characteristic function of S. Since $v \in S$, we have $v \in P_f$ and so $v \in P_f \cup P_g$. Also, we have $\mathcal{B}_f \cap \mathcal{B}_g \subseteq \mathcal{B}_g$. Since $v \in P_f \cup P_g$ and v does not resolve B_g, we get $P_f \cup P_g$ does not resolve $\mathcal{B}_f \cap \mathcal{B}_g$ and hence g is not a universal MRF of G. □

Corollary 3.7. *If g is a universal MRF of a connected graph $G = (V,E)$, then $V\underset{r}{\rightarrow}B_g$.*

Proof. Suppose V does not resolve $\mathcal{B}_g$. Then there exists some $v \in V(G)$ such that v does not resolve $\mathcal{B}_g$ and so g is not a universal MRF, which is a contradiction. Hence $V\underset{r}{\rightarrow}B_g$. □

Proposition 3.8. *Let $G = (V,E)$ be a graph. If there exists $\{u,v\} \in V_p$ such that for each $x \in R\{u,v\}$ there exists an MRF f_x such that x does not resolve $\mathcal{B}_{f_x}$, then G does not have a universal MRF.*

Proof. Let $\{u,v\} \in V_p$ satisfies the hypotheses of the proposition. f_x is an MRF implies $\mathcal{P}_{f_x}\underset{r}{\rightarrow}\mathcal{B}_{f_x}$. Suppose g is a universal MRF of G. Then $\mathcal{P}_g \cup \mathcal{P}_{f_x}\underset{r}{\rightarrow} \mathcal{B}_g \cap \mathcal{B}_{f_x}$. We have $\mathcal{B}_g \cap \mathcal{B}_{f_x} \subseteq \mathcal{B}_{f_x}$ and hence x does not resolve $\mathcal{B}_g \cap \mathcal{B}_{f_x}$. Then $g(x) = 0$. For, suppose $g(x) > 0$. Then $x \in \mathcal{P}_g \cup \mathcal{P}_{f_x}$ and thus $x\underset{r}{\rightarrow}\mathcal{B}_g \cap \mathcal{B}_{f_x} \subseteq \mathcal{B}_{f_x}$, which is a contradiction. Since x is arbitrary we get $g(x) = 0$ for all $x \in R\{u,v\}$. Hence $g(R\{u,v\}) = 0 < 1$, which is a contradiction. Therefore, G does not have a universal MRF. □

Proposition 3.9 ([12])**.** *Let f be an MRF of a connected graph $G = (V,E)$. Let $\{u,v\}, \{x,y\} \in V_p$ with $\{u,v\} \in \mathcal{B}_f$ and $R\{x,y\} \subset R\{u,v\}$. Then*

(i). $\{x,y\} \in \mathcal{B}_f$ and

(ii). $f(w) = 0$ for all $w \in R\{u,v\} - R\{x,y\}$.

Proof.

(i). Since $\{u,v\} \in \mathcal{B}_f$, we have $f(R\{u,v\}) = 1$. Now f is an MRF of G and thus $1 \leq f(R\{x,y\}) \leq f(R\{u,v\}) = 1$ so that $f(R\{x,y\}) = 1$ and hence $\{x,y\} \in \mathcal{B}_f$.

(ii). Since $\{u,v\}, \{x,y\} \in \mathcal{B}_f$, we have $\sum\limits_{z \in R\{u,v\}} f(z) = 1 = \sum\limits_{z \in R\{x,y\}} f(z)$ so that $f(w) = 0$ for all $w \in R\{u,v\} - R\{x,y\}$. □

Definition 3.10. *Let $G = (V,E)$ be any connected graph. A pair $\{u,v\} \in V_p$ is said to absorb another pair $\{y,w\} \in V_p$ and $\{y,w\}$ is said to absorbed by $\{u,v\}$ if $R\{y,w\} \subset R\{u,v\}$, where $\subset$ denotes strict inclusion. In this case, $\{u,v\}$ is called an absorbing pair of vertices and $\{y,w\}$ an absorbed pair. Let $\mathcal{A}_G = \{\{u,v\} \in V_p : \{u,v\}$ is an absorbing pair$\}$ and $\Omega_G = \{\{y,w\} \in V_p : \{y,w\}$ is an absorbed pair$\}$. If there is no confusion regarding the graph G, we omit the subscript G and simply write $\mathcal{A}$ and Ω.*

Example 3.11.

(1). For the bistar $G = B(2,2)$ we have $\mathcal{A} = \{\{u_1,u\}, \{u_2,u\}, \{u_1,v_1\}, \{u_1,v_2\}, \{u_2,v_1\}, \{u_2,v_2\}, \{u,v\}, \{v,v_1\}, \{v,v_2\}, \{u_1,v\}, \{u_2,v\}, \{u,v_1\}, \{u,v_2\}\}$ and $\Omega = \{\{u_1,u_2\}, \{v_1,v_2\}, \{u_1,v\}, \{u_2,v\}, \{u,v_1\}, \{u,v_2\}\}$ where u and v are the non-pendant vertices of G, u_1,u_2 and v_1,v_2 are the pendant vertices adjacent to u and v respectively. In this case $\mathcal{A} \cap \Omega \neq \emptyset$.

(2). For the path $P_4 = (u_1, u_2, u_3, u_4)$ we have $\mathcal{A} = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_1, u_4\}\}$ and $\Omega = \{\{u_1, u_3\}, \{u_2, u_4\}\}$ and in this case $\mathcal{A} \cap \Omega = \emptyset$.

(3). For the complete graph $G = K_n$, $n \geq 3$, we have $\mathcal{A} = \emptyset$ and $\Omega = \emptyset$.

Note that $\mathcal{A}$ and Ω need not be disjoint. In the next proposition, we show that if $\mathcal{A}$ is non-empty then Ω is not contained in $\mathcal{A}$.

Proposition 3.12. *For any connected graph $G = (V, E)$ with $\mathcal{A} \neq \emptyset$ and for any pair $\{u, v\} \in \mathcal{A}$, there exists a pair $\{x, y\} \in \Omega - \mathcal{A}$ such that $R\{x, y\} \subset R\{u, v\}$.*

Proof. Let $\{u, v\} \in \mathcal{A}$. Then there exists $\{x_1, y_1\} \in V_p$ such that $R\{x_1, y_1\} \subset R\{u, v\}$. Clearly $\{x_1, y_1\} \in \Omega$. If $\{x_1, y_1\} \notin \mathcal{A}$, then the proof is complete. If $\{x_1, y_1\} \in \mathcal{A}$, choose $\{x_2, y_2\} \in V_p$ such that $R\{x_2, y_2\} \subset R\{x_1, y_1\} \subset R\{u, v\}$. By repeating this procedure we obtain a sequence $\{u, v\}, \{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_t, y_t\}$ in V_p with $R\{x_t, y_t\} \subset \cdots \subset R\{x_1, y_1\} \subset R\{u, v\}$, and since G is finite, the process terminates with a pair $\{x, y\}$ such that $\{x, y\} \notin \mathcal{A}$. □

Definition 3.13. *Let f be an MRF of a connected graph $G = (V, E)$. A vertex $w \in V$ is defined to be f-sharp, if $\mathcal{B}_f \cap R\{w\} \subseteq \mathcal{A}$. Also, w is said to be sharp if w is f-sharp for some MRF f of G.*

Lemma 3.14. *Let $G = (V, E)$ be any connected graph with $\mathcal{A} \neq \emptyset$. Let f be an MRF of G and let w be any f-sharp vertex of G. Then*

(i). there exists a pair $\{x, y\} \in \Omega - \mathcal{A}$ such that $w \notin R\{x, y\}$ and

(ii). $f(w) = 0$.

Proof.

(i). Since w is f-sharp, $\mathcal{B}_f \cap R\{w\} \subseteq \mathcal{A}$. Let $\{u, v\} \in \mathcal{B}_f \cap R\{w\}$ with $R\{x, y\} \subset R\{u, v\}$. Since $\{u, v\} \in \mathcal{B}_f$, by (i) of Proposition 3.9, we have $\{x, y\} \in \mathcal{B}_f$. Suppose $w \in R\{x, y\}$. Then $\{x, y\} \in R\{w\}$ and so $\{x, y\} \in \mathcal{B}_f \cap R\{w\} \subseteq \mathcal{A}$, which is a contradiction, since $\{x, y\} \in \Omega - \mathcal{A}$. Hence $w \notin R\{x, y\}$.

(ii). We have $w \notin R\{x, y\}, w \in R\{u, v\}, R\{x, y\} \subseteq R\{u, v\}$ and $\{u, v\} \in \mathcal{B}_f$. Hence it follows from (ii) of Proposition 3.9 that $f(w) = 0$. □

Theorem 3.15. *Let g be an MRF of a connected graph $G = (V, E)$ with $\mathcal{A} \neq \emptyset$. If*

(i). $V_p - \mathcal{A} \subseteq \mathcal{B}_g$ and

(ii). $g(w) = 0$ for each sharp vertex w of G,

then g is a universal MRF of G.

Proof. Let f be any MRF of G. Since g is also an MRF, we have $\mathcal{P}_g \xrightarrow[r]{} \mathcal{B}_g$ and $\mathcal{P}_f \xrightarrow[r]{} \mathcal{B}_f$. To show that g is universal, it is enough to show that $\mathcal{P}_f \cup \mathcal{P}_g \xrightarrow[r]{} \mathcal{B}_f \cap \mathcal{B}_g$. Let $w \in \mathcal{P}_f \cup \mathcal{P}_g$. If w is f-sharp then $\mathcal{B}_f \cap R\{w\} \subseteq \mathcal{A}$ and hence by Lemma 3.14, we have $f(w) = 0$. Also, by (ii), we have $g(w) = 0$. This is a contradiction since $w \in \mathcal{P}_f \cup \mathcal{P}_g$. Hence w is not f-sharp. Thus there exists a pair $\{u, v\} \in \mathcal{B}_f \cap R\{w\}$ such that $\{u, v\} \notin \mathcal{A}$. By, (i), we have $\{u, v\} \in \mathcal{B}_g$ and so $\{u, v\} \in \mathcal{B}_f \cap \mathcal{B}_g \cap R\{w\}$. This implies that $w \xrightarrow[r]{} \mathcal{B}_f \cap \mathcal{B}_g$ (Since $w \in R\{u, v\}$). Thus $\mathcal{P}_f \cup \mathcal{P}_g \xrightarrow[r]{} \mathcal{B}_f \cap \mathcal{B}_g$, which implies $f\mathcal{R}g$ and so g is universal MRF of G. □

Remark 3.16. *Mathew and Arumugam [12], defined the Resolving convexity graph $C_R(G)$ of a connected graph G and obtained the same for some families of graphs. It was observed that the resolving convexity graph $C_R(G)$ of G is complete if and only if every MRF of G is a universal MRF. Also G has no universal MRF if and only if $C_R(G)$ has no full degree vertex.*

The follwing are some problems for further investigation.

Problem 3.17. *Characterize connected graphs G with $\mathcal{A}_G \neq \emptyset$, which admits universal $MRFs$.*

Problem 3.18. *Characterize connected graphs G with $\mathcal{A}_G = \emptyset$, which admits universal $MRFs$.*

Problem 3.19. *Which trees admit universal $MRFs$?*

References

[1] S. Arumugam and Varughese Mathew, *The fractional metric dimension of graphs*, Discrete Math., 312(9)(2012), 1584-1590.

[2] S. Arumugam, Varughese Mathew and Jian Shen, *On fractional metric dimension of graphs*, Disc. Math. Algorithms and Appl., 5(2013), 1-8.

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffman, M. Mihalak and L. Ram, *Network discovery and verification*, IEEE J. on Selected Areas in Communications, 24(2006), 2168-2181.

[4] G. Chartrand, L. Eroh, M. Johnson and O. R. Oellermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math., 105(2000), 99-113.

[5] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Fourth Edition, Chapman & Hall/CRC, (2005).

[6] G. Chartrand and Ping Zhang, *The theory and applications of resolvability in graphs: A Survey*, Congressus Numerantium, 160(2003), 47-68.

[7] V. Chvátal, *Mastermind*, Combinatorica, 3(1983), 325-329.

[8] E. J. Cockayne, G. Fricke, S. T. Hedetniemi and C. M. Mynhardt, *Properties of minimal dominating functions of graphs*, Ars Combinatoria, 41(1995), 107-115.

[9] M. Fehr, Shonda Gosselin and Ortrud R. Oellermann, *The metric dimension of cayley digraphs*, Discrete Mathematics, 306(2006), 31-41.

[10] F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin., 2(1976), 191-195.

[11] S. Khuller, B. Raghavachari and A. Rosenfield, *Landmarks in graphs*, Disc. Appl. Math., 70(1996), 217-229.

[12] V. Mathew and S. Arumugam, *Convexity of Minimal Resolving Functions in Graphs*, Rational Discourse, 24(1)(2018), 8 pages.

[13] Scheinerman, E. R. and D. H. Ullman, *Fractional Graph Theory: A Rational Approch to the Theory of Graphs*, John wiley & sons, New York, (1997).

[14] A. Sebö and E. Tannier, *On metric generators of graphs*, Math. Oper. Res., 29(2004), 383-393.

[15] H. Shapiro and S. Soderberg, *A combinatory detection problem*, Amer. Math. Monthly, 70(1963), 1066-1070.

[16] P. J. Slater, *Leaves of trees*, Congressus Numerantium, 14(1975), 549-559.

[17] P. J. Slater, *Domination and location in acyclic graphs*, Networks, 17(1987), 55-64.