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Generalized Sixth Order Mock Theta Functions and Some Identities

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Abstract: We have given a new generalization of sixth order mock theta functions by introducing four independent variables. We found some identities for these generalized mock theta functions. We also have given q-integral representation and multibasic expansion for these functions.

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1. Introduction

In his last letter to Hardy [12] Ramanujan gave 17 functions 4 of order 3, 10 of order 5 and 3 of order 7 and called them mock theta functions. Later Andrews discovered eight more mock theta functions in his note book and called it "Lost" Notebook. Andrews and Hickerson [2] called these mock theta functions of order six. In this paper we have given a generalization of sixth order mock theta functions by introducing four independent variables. The advantage of having four independent variables generalization that by specializing the parameters we get some known functions and some new identities. The sixth order mock theta functions of Ramanujan are;

$$\phi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \qquad \psi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}},$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}}, \qquad \sigma(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$

$$\lambda(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^n(q;q^2)_n}{(-q;q)_n}, \qquad \mu(q)=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n}{(-q;q)_n},$$

$$\phi_-(q)=\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n}, \qquad \psi_-(q)=\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n},$$

where, $(a;q^k)_n=\prod_{j=1}^{n}(1-aq^{k(j-1)})$, $(a;q^k)_\infty=\prod_{j=1}^{\infty}(1-aq^{k(j-1)})$ and $(a;q^k)_0=1$.

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2. Generalized Sixth Order Mock Theta Functions

We generalize the sixth order mock theta functions by introducing four independent variables, these are;

$$\phi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{n^2-n+n\beta}(zq;q^2)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}}, \tag{1}$$

$$\psi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{(n+1)^2-n+n\beta}(zq;q^2)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n+1}}, \tag{2}$$

$$\rho(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n+1)/2-n+n\beta}(-zq;q)_n z^{2n}\alpha^n}{(\alpha q;q^2)_{n+1}}, \tag{3}$$

$$\sigma(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{(n+1)(n+2)/2-n+n\beta}(-zq;q)_n z^{2n}\alpha^n}{(\alpha q;q^2)_{n+1}}, \tag{4}$$

$$\lambda(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{n-n+n\beta}(q/\alpha;q^2)_n}{(-q/z;q)_n z^{3n+3}}, \tag{5}$$

$$\mu(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n (q/\alpha;q^2)_n q^{n\beta-n}}{(-q/z;q)_n z^{3n+3}}, \tag{6}$$

$$\phi_-(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=1}^{\infty}\frac{(t)_n q^{n-n+n\beta}(-q/\alpha;q)_{2n-1}\alpha^n}{(q/z;q^2)_n z^{3n}}, \tag{7}$$

$$\psi_-(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=1}^{\infty}\frac{(t)_n q^{n-n+n\beta}(-q/\alpha;q)_{2n-2}\alpha^n}{(q/z;q^2)_n z^{3n}}. \tag{8}$$

For $t=0, \alpha=1, \beta=1$ and $z=1$ these functions reduce to classical mock theta functions of sixth order.

3. Some Identities for Generalized Mock Theta Function

We prove some identities for the generalized sixth order mock theta functions. We state them in the following theorem.

Theorem 3.1.

$$\textit{(i).}\ \sigma(0,\alpha,1,z;q) + \frac{z}{\alpha(1+z)}\mu(0,\alpha,1,z;q) = \frac{1}{(1-\alpha q)}\frac{(1/\alpha z^6 q;q^2)_\infty (q^2z^2\alpha;q)_\infty}{(-1/z^3,-1/\alpha q z^3,q^3z^3\alpha;q)_\infty}$$

$$\times\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^3 q^{2r+2})(-zq;q)_r(\alpha z^6 q^3;q^2)_r(-1)^r z^{2r}\alpha^{2r}q^{(3r^2+5r+2)/2}}{(1-\alpha z^3q^2)(\alpha^2q^3;q^2)_r(-z^2\alpha q^2;q)_r}, \tag{9}$$

$$\textit{(ii).}\ \phi(0,\alpha,1,z;q) + \frac{1+\alpha}{(\alpha)}\phi_-(0,\alpha,1,z;q) = \frac{(-1/z^3,-q^2/z^3,qz^2\alpha;q^2)_\infty}{(\alpha q/z^3,q^2/\alpha z^3,q^2z^3\alpha;q^2)_\infty}$$

$$\times\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^3q^{4r})(zq;q^2)_r(-z^3q,-z^3;q^2)_r z^{2r}\alpha^{2r}(-1)^r q^{3r^2-r}}{(1-\alpha z^3)(-\alpha q;q)_{2r}(z^2\alpha q;q^2)_r}, \tag{10}$$

$$\textit{(iii).}\ \psi(0,\alpha,1,z;q) + \frac{1+\alpha}{\alpha^2}\psi_-(0,\alpha,1,z;q) = \frac{1}{(1+\alpha q)}\frac{(-1/z^3,-q/z^3,q^3z^2\alpha;q^2)_\infty}{(q\alpha/z^3,1/\alpha z^3,q^4z^3\alpha;q^2)_\infty}$$

$$\times\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^3q^{4r+2})(zq,-z^3q^2,-z^3q;q^2)_r z^{2r}\alpha^{3r}(-1)^r q^{3r^2+3r+1}}{(1-\alpha z^3q^2)(-\alpha q^2,-\alpha q^3,zq^3\alpha;q^2)_r}, \tag{11}$$

$$\textit{(iv).}\ \rho(0,\alpha,1,z;q) + \frac{z}{\alpha(1+z)}\lambda(0,\alpha,1,z;q) = \frac{1}{(1-\alpha q)}\frac{(q/\alpha z^6;q^2)_\infty(-qz^2\alpha;q)_\infty}{(-q/z^3,1/\alpha z^3,q^2z^3\alpha;q)_\infty}$$

$$\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^3q^{2r+1})(-zq;q)_r(z^6\alpha q;q^2)_r z^{2r}\alpha^{2r}q^{(3r^2+3r)/2}}{(1-\alpha z^3q)(\alpha q^3;q^2)_r(-z^2q\alpha;q)_r}. \tag{12}$$

Proof of (i). We shall required the following transformation formula [10] for proving the theorem. If $|q|, |bd/azq| < 1$, then

$$\sum_{r=-\infty}^{\infty} \frac{(a;q^2)_r z^r q^{r^2}}{(b,d;q^2)_r} = \frac{(-bq/az,-dq/az,-qz;q^2)_\infty}{(-bd/aqz,-q^3/az,-aqz;q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1+azq^{4r-1})(a,-azq/b,-azq/d;q^2)_r (bdz)^r q^{3r^2-4r}}{(1+az/q)(b,d,-zq;q^2)_r}. \tag{13}$$

Putting $z = z^2\alpha q^3,\ a = -zq^2,\ b = \sqrt{\alpha}q^3$, and $d = -\sqrt{\alpha}q^3$ in the transformation formula (13), the left side is

$$= \sum_{r=-\infty}^{\infty} \frac{(-zq^2;q^2)_r (z^2\alpha q^3)^r q^{r^2}}{(\sqrt{\alpha}q^3, -\sqrt{\alpha}q^3; q^2)_r}$$

$$= \sum_{r=-\infty}^{\infty} \frac{(-zq^2;q^2)_r z^{2r}\alpha^r q^{r^2+3r}}{(\alpha q^6;q^4)_r}$$

$$= \frac{(1-\alpha q^2)}{q^2} \sum_{r=-\infty}^{\infty} \frac{(-zq^2;q^2)_r z^{2r}\alpha^r q^{r^2+3r+2}}{(\alpha q^2;q^4)_{r+1}}$$

$$= \frac{(1-\alpha q^2)}{q^2} \left[\sum_{r=0}^{\infty} \frac{(-zq^2;q^2)_r z^{2r}\alpha^r q^{r^2+3r+2}}{(\alpha q^2;q^4)_{r+1}} + \sum_{r=-1}^{-\infty} \frac{(-zq^2;q^2)_r z^{2r}\alpha^r q^{r^2+3r+2}}{(\alpha q^2;q^4)_{r+1}} \right]$$

$$= \frac{(1-\alpha q^2)}{q^2} \left[\sum_{r=0}^{\infty} \frac{(-zq^2;q^2)_r z^{2r}\alpha^r q^{r^2+3r+2}}{(\alpha q^2;q^4)_{r+1}} + \sum_{r=1}^{\infty} \frac{(-zq^2;q^2)_{-r} z^{-2r}\alpha^{-r} q^{r^2-3r+2}}{(\alpha q^2;q^4)_{-r+1}} \right] \tag{14}$$

Using

$$(a;q)_{-n} = \frac{(-q/a)^n q^{n(n-1)/2}}{(q/a;q)_n}$$

we get

$$= \frac{(1-\alpha q^2)}{q^2} \left[\sum_{r=0}^{\infty} \frac{(-zq^2;q^2)_r (-1)^r z^{2r}\alpha^r q^{r^2+3r+2}}{(\alpha q^2;q^4)_{r+1}} + \frac{z}{\alpha(1+z)} \sum_{r=0}^{\infty} \frac{(-1)^r (q^2/\alpha;q^4)_r}{(-q^2/z;q^2)_r z^{3r+3}} \right]$$

$$= \frac{(1-\alpha q^2)}{q^2} \left[\sigma(0,\alpha,1,z;q^2) + \frac{z}{\alpha(1+z)} \mu(0,\alpha,1,z;q^2) \right]. \tag{15}$$

Now the right side is

$$= \frac{(1/\alpha z^6 q^2;q^4)_\infty (-q^4 z^2\alpha;q^2)_\infty}{(-1/z^3, -1/\alpha q^2 z^3, q^6 z^3\alpha; q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-\alpha z^3 q^{4r+4})(-zq^2;q^2)_r (\alpha z^6 q^6;q^4)_r (-1)^r z^{2r}\alpha^{2r} q^{(3r^2+5r+2)}}{(1-\alpha z^3 q^4)(\alpha q^6;q^4)_r (-z^2\alpha q^4;q^2)_r}. \tag{16}$$

From (15) and (16), we have

$$\frac{(1-\alpha q^2)}{q^2} \left[\sigma(0,\alpha,1,z;q^2) + \frac{1}{(1+z)} \mu(0,\alpha,1,z;q^2) \right] = \frac{(1/\alpha z^6 q^2;q^4)_\infty (-q^4 z^2\alpha;q^2)_\infty}{(-1/z^3, -1/\alpha q^2 z^3, q^6 z^3\alpha; q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-\alpha z^3 q^{4r+4})(-zq^2;q^2)_r (\alpha z^6 q^6;q^4)_r (-1)^r z^{2r}\alpha^{2r} q^{(3r^2+5r+2)}}{(1-\alpha z^3 q^4)(\alpha q^6;q^4)_r (-z^2\alpha q^4;q^2)_r} \tag{17}$$

Writting q for q^2 in (17), we get (i).

Proof of (ii). Put $z = -z^2\alpha,\ a = zq,\ b = -\alpha q$, and $d = -\alpha q^2$ in the transformation formula (13) to get (ii).

Proof of (iii). Put $z = -z^2 q^2\alpha,\ a = zq,\ b = -\alpha q^2$, and $d = -\alpha q^3$ in the transformation formula(13) to get (iii).

Proof of (iv). Put $z = z^2 q\alpha,\ a = -zq^2,\ b = \sqrt{\alpha}q^3$, and $d = -\sqrt{\alpha}q^3$ in the transformation formula (13) to get (iv). □

4. q-Integral Representation for the Generalized Sixth Order Mock Theta Functions.

We give the integral representation for the generalized sixth order mock theta function. Thomae and Jackson [7, p.19] defined q-integral.

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n$$

limiting case of q-beta integral we have

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty d_qt \tag{18}$$

We give the integral representation for the generalized sixth order mock theta functions in the following theorem.

Theorem 4.1.

(i). $\phi(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\phi(0,\alpha,pu,z;q)d_qu.$

(ii). $\psi(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\psi(0,\alpha,pu,z;q)d_qu.$

(iii). $\rho(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\rho(0,\alpha,pu,z;q)d_qu.$

(iv). $\sigma(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\sigma(0,\alpha,pu,z;q)d_qu.$

(v). $\lambda(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\lambda(0,\alpha,pu,z;q)d_qu.$

(vi). $\mu(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\mu(0,\alpha,pu,z;q)d_qu.$

(vii). $\phi_-(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\phi_-(0,\alpha,pu,z;q)d_qu.$

(viii). $\psi_-(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty\psi_-(0,\alpha,pu,z;q)d_qu.$

Proof. We will give the detailed proof of (i), the rest are on similar lines.

Proof of (i). By definition

$$\phi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{n^2-n+n\beta}(zq;q)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}}.$$

Replacing t by q^t we have

$$\phi(q^t,\alpha,\beta,z;q) = \frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_n(-1)^n q^{n^2-n+n\beta}(zq;q)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n+n\beta}(zq;q^2)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}(q^{n+t};q)_\infty}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n+n\beta}(zq;q^2)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{n+t-1}(uq;q)_\infty d_qu. \tag{19}$$

But

$$\phi(0,\alpha,\beta,z;q) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n+n\beta}(zq;q)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}},$$

putting $q^{\beta} = p$, we have

$$\phi(0,\alpha,p,z;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-n} p^n (zq;q)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}} \tag{20}$$

By (19) and (20), we finally have

$$\phi(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty \phi(0,\alpha,pu,z;q)d_q u.$$

which proves (i) .

The proof of all the other functions is similar, so omitted. □

5. Multibasic Expansion of Generalized Sixth Order Mock Theta Functions

The following bibasic expansion will be used to give multibasic expansion for the generalized functions.

Theorem 5.1.

$$\sum_{k=0}^{\infty} \frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_k q^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k} \sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(ap,bp;p)_m(cq,aq/bc;q)_m q^m}{(q,aq/b;q)_m(ap/c,bcp;p)_m}\alpha_m. \tag{21}$$

Using the summation formula [7, (3.6.7), p.71] and [11, Lemma 10, p.57], we have Theorem 5.1.

We will consider the following cases of Theorem 5.1.

Case I. Letting $q \to q^2$ and $c \to \infty$ in Theorem 5.1, we have

$$\sum_{k=0}^{\infty} \frac{(1-ap^kq^{2k})(1-bp^kq^{-2k})(a,b;p)_k q^{k^2+k}}{(1-a)(1-b)(q^2,aq^2/b;q^2)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(ap,bp;p)_m q^{m^2+m}}{(q^2,aq^2/b;q^2)_m b^m p^{\frac{m^2+m}{2}}}\alpha_m. \tag{22}$$

Case II. Letting $q \to q^3$ and $c \to \infty$ in Theorem 5.1, we have

$$\sum_{k=0}^{\infty} \frac{(1-ap^kq^{3k})(1-bp^kq^{-3k})(a,b;p)_k q^{\frac{3k^2+3k}{2}}}{(1-a)(1-b)(q^3,aq^3/b;q^3)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(ap,bp;p)_m q^{\frac{3m^2+3m}{2}}}{(q^3,aq^3/b;q^3)_m b^m p^{\frac{m^2+m}{2}}}\alpha_m. \tag{23}$$

Case III. Putting $p = q$ and $a = 0$ in Theorem 5.1, we have

$$\sum_{k=0}^{\infty} \frac{(b,c;q)_k q^k}{(q,bcq;q)_k} \sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty} \frac{(bq,cq;q)_m}{(q,bcq;q)_m}\alpha_m. \tag{24}$$

The multibasic expansion of generalized sixth order mock theta functions are.

Theorem 5.2.

$$(i).\ \phi(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(zq;q^2)_k(t;q)_{k-1}z^{2k}q^{k^2}\alpha^k}{(1-q^{k+2})(-\alpha q;q)_{2k}}$$
$$\times \phi\left[\begin{matrix} q;zq^{2k+1};tq^{3k},q^{3k+3} \\ q^{k+3};-\alpha q^{2k+1},-\alpha q^{2k+2};0 \end{matrix}; q,q^2,q^3;-\alpha qz^2\right],$$

$$(ii).\ \psi(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(zq;q^2)_k(t;q)_{k-1}q^{(k+1)^2}z^{2k}\alpha^k}{(1-q^{k+2})(-\alpha q;q)_{2k+1}}$$
$$\times \phi\left[\begin{matrix} q;zq^{2k+1};tq^{3k},q^{3k+3} \\ q^{k+3};-\alpha q^{2k+2},-\alpha q^{2k+3};0 \end{matrix}; q,q^2,q^3;-\alpha q^3z^2\right],$$

$$(iii).\ \rho(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty} \frac{(1-tq^{3k-1})(1-q^{-k+1})(-zq;q)_k(t;q)_{k-1}z^{2k}q^{\frac{k^2+k}{2}}\alpha^k}{(1-q^{k+1})(\alpha q;q^2)_{k+1}}$$
$$\times \phi\left[\begin{matrix} q,-zq^{k+1};q^{2k+2},tq^{2k};0 \\ q^{k+2};-\alpha q^{k+1} \end{matrix}; q,q^2;\alpha qz^2\right],$$

$$(iv).\ \sigma(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+1})(-zq;q)_k(t;q)_{k-1}z^{2k}q^{\frac{k^2+3k+2}{2}}\alpha^k}{(1-q^{k+1})(\alpha q;q^2)_{k+1}}$$
$$\times\,\phi\left[{q,-zq^{k+1};tq^{2k},q^{2k+2};0 \atop q^{k+3};\alpha q^{2k+3};0};q,q^2,q^3;\alpha q^2z^2\right],$$

$$(v).\ \lambda(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1-q)(t/q;q)_k(q/\alpha;q^2)_kq^{2k}}{(1-q^{k+1})(-q/z;q)_kz^{3k+3}}\times\phi\left[{q,q^{k+1},tq^{k+1};q^{2k+1}/\alpha \atop q^{k+3},-q^{k+1}/z;0};q,q^2;-qz^{-3}\right],$$

$$(vi).\ \mu(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1-q)(t/q;q)_k(q/\alpha;q^2)_kq^{2k}}{(1-q^{k+1})(-q/z;q)_kz^{3k+3}}\times\phi\left[{q,q^{k+1},tq^{k+1};q^{2k+1}/\alpha \atop q^{k+1},-q^{k+1}/z;0};q,q^2;-z^{-3}\right].$$

Proof. We shall give the detailed proof of (i) only, others are on similar lines.

Proof of (i). Taking $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m = \frac{(-1)^m(q^3;q^3)_m(t;q^3)_m(zq;q^2)_mq^mz^{2m}\alpha^m}{(q^3;q)_m(-\alpha q;q)_{2m}} \quad \text{in (23).}$$

We have

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q;q)_k(q^2;q)_kq^{\frac{3k^2+3k}{2}}}{(1-t/q)(1-q^2)(q^3,t;q^3)_kq^{2k}q^{\frac{k^2+k}{2}}}\times\sum_{m=0}^{\infty}\frac{(-1)^{m+k}(q^3;q^3)_{m+k}(t;q^3)_{m+k}(zq;q^2)_{m+k}q^{m+k}z^{2m+2k}\alpha^{m+k}}{(q^3;q)_{m+k}(-\alpha q;q)_{2m+2k}}$$
$$=\sum_{m=0}^{\infty}\frac{(t,q^3;q)_mq^{\frac{3m^2+3m}{2}}}{(q^3,t;q^3)_mq^{2m}q^{\frac{m^2+m}{2}}}\frac{(-1)^m(q^3;q^3)_mq^m(t;q^3)_m(zq;q^2)_mq^mz^{2m}\alpha^m}{(q^3;q)_m(-\alpha q;q)_{2m}}. \tag{25}$$

The right hand side of (25) is

$$=\sum_{m=0}^{\infty}\frac{(-1)^mq^{m^2}(t;q)_m(zq;q^2)_mz^{2m}\alpha^m}{(-\alpha q;q)_{2m}}$$
$$=(t)_\infty\phi(t,\alpha,1,z;q).$$

The left hand side of (25) is equal to

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q;q)_k(q^2;q)_kq^{k^2+k}}{(1-t/q)(1-q^2)(q^3,t;q^3)_kq^{2k}}$$
$$\sum_{m=0}^{\infty}\frac{(-1)^{m+k}(q^3;q^3)_k(q^{3k+3};q^3)_m(t;q^3)_k(tq^{3k};q^3)_m(zq;q^2)_k(zq^{2k+1};q^2)_mq^{m+k}z^{2m+2k}\alpha^{m+k}}{(q^3;q)_k(q^{k+3};q)_m(-\alpha q;q)_{2k}(-\alpha q^{2k+1};q)_{2m}}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(zq;q^2)_k(t;q)_{k-1}q^{k^2}z^{2k}\alpha^k}{(1-q^{k+2})(-\alpha q;q)_{2k}}$$
$$\times\sum_{m=0}^{\infty}\frac{(-1)^m(q^{3k+3};q^3)_m(tq^{3k};q^3)_m(zq^{2k+1};q^2)_mq^mz^{2m}\alpha^m}{(q^{k+3};q)_m(-\alpha q^{2k+1};q)_{2m}}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(zq;q^2)_k(t;q)_{k-1}q^{k^2}z^{2k}\alpha^k}{(1-q^{k+2})(-\alpha q;q)_{2k}}$$
$$\times\,\phi\left[{q;zq^{2k+1};tq^{3k},q^{3k+3} \atop q^{k+3};-\alpha q^{2k+1},-\alpha q^{2k+2};0};q,q^2,q^3;-\alpha qz^2\right]$$

which proves (i).

Proof of (ii). Take

$$a=t/q,\ b=q^2,\ p=q \text{ and } \alpha_m = \frac{(-1)^m(q^3;q^3)_m(t;q^3)_m(zq;q^2)_mq^{3m+1}z^{2m}\alpha^m}{(q^3;q)_m(-\alpha q;q)_{2m+1}} \quad \text{in (23).}$$

Proof of (iii). Take

$$a=t/q,\ b=q,\ p=q \text{ and } \alpha_m = \frac{(-1)^m(q^2;q^2)_m(t;q^2)_m(-zq;q)_mq^mz^{2m}\alpha^m}{(q^2;q)_m(\alpha q;q^2)_{m+1}} \quad \text{in (22).}$$

Proof of (iv). Take

$$a = t/q,\ b = q,\ p = q \text{ and } \alpha_m = \frac{(q^2;q^2)_m(t;q^2)_m(-zq;q)_m q^{2m+1} z^{2m}\alpha^m}{(q^3;q)_m(\alpha q;q^2)_{m+1}} \text{ in (22).}$$

Proof of (v). Take

$$b = t/q,\ c = q^2 \text{ and } \alpha_m = \frac{(-1)^m(q;q)_m(tq^2;q)_m(q/\alpha;q^2)_m q^m}{(q^3;q)_m(-q/z;q)_m z^{3m+3}} \text{ in (24).}$$

Proof of (vi). Take

$$b = t/q,\ c = q^2 \text{ and } \alpha_m = \frac{(-1)^m(q;q)_m(tq^2;q)_m(q/\alpha;q^2)_m}{(q^3;q)_m(-q/z;q)_m z^{3m+3}} \text{ in (24).}$$

□

6. Special Cases.

We have following known results for special cases of generalized sixth order mock theta functions

(i). $\phi(0,-1,1,-1;q) = \frac{f(q^2,q^4)}{\psi(-q)}$,

(ii). $\phi(0,-1,1,-1;q) = \frac{f(-q^6,-q^6)}{(q)_\infty}$,

(iii). $\psi(0,-1,1,-1;q) = q\frac{f(-q^2,-q^{10})}{(q)_\infty}$.

Proof of (i). By definition

$$\phi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{n^2-n+n\beta}(zq;q^2)_n z^{2n}\alpha^n}{(-\alpha q;q)_{2n}}$$

For $t = 0$, $\alpha = -1$, $\beta = 1$, and $z = -1$, we have

$$\phi(0,-1,1,-1;q) = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}} \tag{26}$$

From [13, eqn.(A.64), p.176]

$$\frac{f(q^2,q^4)}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}}, \tag{27}$$

By equation (26) and (27), we have case (i).

Proof of (ii). From [13, eqn.(A.104), p.180]

$$\frac{f(-q^6,-q^6)}{(q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}}, \tag{28}$$

By equation (26) and (28), we have

$$\phi(0,-1,1,-1;q) = \frac{f(-q^6,-q^6)}{(q)_\infty} \tag{29}$$

Proof of (iii). For $t = 0$, $\alpha = -1$, $\beta = 1$, and $z = -1$, we have

$$\psi(0,-1,1,-1;q) = \sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q;q)_{2n+1}} \tag{30}$$

From [13, eqn.(A.100), p.180]

$$\frac{f(-q^2,-q^{10})}{(q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}}. \tag{31}$$

By (30) and (31), we have

$$\psi(0,-1,1,-1;q) = q\frac{f(-q^2,-q^{10})}{(q)_\infty} \tag{32}$$

7. Conclusion

I have made comprehensive study of generalized mock theta functions of third order, fifth order, eighth order, tenth order and the papers have been communicated for publication.

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