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Generalized Fractional Integral Operators Pertaining to Galue Type Struve Function

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Abstract: The Marichev-Saigo-Maeda operators are applied to investigates certain fractional integral image formulas associated with the product of the unified Galué type Struve function and the general class of polynomials. The main formulas are expressed in terms of the generalized Fox-Wright function. Utility or importance are also discussed by giving particular cases to the main results.

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1. Introduction and Preliminaries

In recent year, the interest in the fractional calculus operators containing different types of special functions has found very useful in various fields of engineering and science. Because of the significant importance of fractional calculus operators, many research papers have studied and investigated the verity of extensions and applications for these operators. It is fairly well-known that there are a number of different definitions of fractional calculus operators and their applications. Each definition has its own advantages and suitable for applications to different type of scientific or engineering problems. For more details about verity of operators of fractional calculus, one can see the monographs of Miller and Ross [6], and Samko et al. [17]. The unified fractional integrals involving the Saigo operators ([14, 15]) given in [5] (see also ([17], p. 194, (10.47) and whole section 10.3)) and its extension in term of any complex order with Appell function $F_3(.)$ as the kernel found in ([16], p.393, eq. (4.12) and (4.13)). The fractional integral operators of Marichev-Saigo-Maeda (MSM) type are given by:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\delta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt, \tag{1}$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\delta)} \int_x^\infty (t-x)^{\delta-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\delta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)dt, \tag{2}$$

where $\alpha,\alpha',\beta,\beta',\delta \in \mathbb{C}$, $(\Re(\delta) > 0)$ and $x > 0$. For the definition of the Appell function $F_3(.)$ the interested reader may refer to the monograph by Srivastava and Karlson [22], (see Erdélyi et al. [2] and Prudnikov et al. [13]). Now, we recall the

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general class of polynomial (cf. [21], p.1. eq.(1)):

$$S_w^u[x] = \sum_{s=0}^{[w/u]} \frac{(-w)_{u.s}}{s!} A_{w.s} x^s, \ (w = 0, 1, 2, ...), \tag{3}$$

for any positive integer u and the arbitrary coefficients $A_{w.s}(w, s) \geq 0$ (constants, real or complex). Recently, Nisar et al. [7] introduced a new generalization of Struve function and defined as the following power series

$${}_a w_{p,b,c,\xi}^{\lambda,\mu}(z) = \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(\lambda r + \mu)\Gamma\left(ar + \frac{p}{\xi} + \frac{b+2}{2}\right)} \left(\frac{z}{2}\right)^{2r+p+1}, \quad a \in \mathbb{N}, \quad p, b, c \in \mathbb{C}, \tag{4}$$

where $\lambda > 0, \xi > 0$ and μ is an arbitrary parameter. In the same paper, they also investigated certain fractional integral transforms and the solutions of certain general classes of fractional kinetic equations associated with the generalized Galué type Struve function. The numerical comparison between solutions of these kinetic equations involving generalized Bessel function and generalized Galué type generalization of Struve function are also presented in [7]. Further, the Pathway fractional integral image formulas and solution of fractional kinetic equation involving generalized Struve function (defined in [11]), were investigated in [8] and [9], respectively. For more details and applications about the Struve function, one can refer ([1, 3, 8–10, 18–20, 23]).

Here, our aim to establish certain new image formulas associated with the MSM integral operators for the product of the unified Galué type Sturve function and the general class of polynomials. The main formulas obtained here are represented in terms of the generalized Wright hypergeometric function and is given by the series ([24]) (see, for detail, [22]):

$${}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k) k!}. \tag{5}$$

where $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \Re$, $(\alpha_i, \beta_j \neq 0; i = 1, 2, ..., p; j = 1, 2, ..., q)$. The generalized hypergeometric function $a_i, b_j \in \mathbb{C}$ and $b_j \neq 0, -1, \cdots (i = 1, \cdots, p;\ j = 1, \cdots, q)$ is given by the power series ([2], Section 4.1(1)):

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r z^r}{(b_1)_r \cdots (b_q)_r r!}, \tag{6}$$

where for convergence, we have $|z| < 1$ if $p = q + 1$ and for any z if $p \leq q$. If we take $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, then

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} \, {}_p\psi_q \left[\begin{matrix} (a_i, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{matrix} \middle| z \right]. \tag{7}$$

2. Generalized Fractional Integral Formulas

The following Lemma due to Saigo and Maeda [16]:

Lemma 2.1.

(a). If $\Re(\delta) > 0$ *and* $\Re(\sigma) > max\{0, \Re(\alpha + \alpha' + \beta - \delta), \Re(\alpha' - \beta')\}$, *then*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta} x^{\sigma-1}\right)(x) = \Gamma \left[\begin{matrix} \sigma, \sigma + \gamma - \alpha - \alpha' - \beta, \sigma + \beta' - \alpha' \\ \sigma + \beta', \sigma + \gamma - \alpha - \alpha', \sigma + \gamma - \alpha' - \beta \end{matrix} \right] T, \tag{8}$$

(b). If $\Re(\delta) > 0$ *and* $\Re(\sigma) < 1 + min\{\Re(-\beta), \Re(\alpha + \alpha' - \delta), \Re(\alpha + \beta' - \delta)\}$, *then*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta} x^{\sigma-1}\right)(x) = \Gamma\left[\begin{matrix} 1-\sigma-\beta, 1-\sigma-\delta+\alpha+\alpha', 1-\sigma+\alpha+\beta'-\delta \\ 1-\sigma, 1-\sigma+\alpha+\alpha'+\beta'-\delta, 1-\sigma+\alpha-\beta \end{matrix}\right] T, \tag{9}$$

where $T = x^{\sigma-\alpha-\alpha'+\delta-1}$.

The symbol occurring in (8) and (9) is given by:

$$\Gamma\left[\begin{matrix} \alpha_1,\ \beta_1,\ \delta_1 \\ \alpha_2,\ \beta_2,\ \delta_2 \end{matrix}\right] = \frac{\Gamma(\alpha_1)\,\Gamma(\beta_1)\Gamma(\delta_1)}{\Gamma(\alpha_2)\,\Gamma(\beta_2)\Gamma(\delta_2)}.$$

The MSM operator (1) of the product of (3) and (4) are as follows:

Theorem 2.2. *Let* $a \in \mathbb{N}$, $\alpha, \alpha', \beta, \beta', \delta, \sigma, \tau, b, c \in \mathbb{C}$, *and the conditions on* λ, μ *same as* (4) *be such that* $\Re(\sigma+\tau+1) > \max[0, \Re(\alpha+\alpha'+\beta-\delta), \Re(\alpha'-\beta')]$, $\Re(\delta) > 0$, $\frac{\tau}{\xi} + \frac{b}{2} \neq -1, -2, -3, ...$, *then there hold the formula:*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1} S_w^u(t)\, {}_a w_{\tau,b,c,\xi}^{\lambda,\mu}(t)\right]\right)(x) = \frac{x^{l+\sigma-\alpha-\alpha'+\delta+\tau}}{2^{\tau+1}} \sum_{s=0}^{[w/u]} \frac{(-w)_{u.s}}{s!} A_{w.s}(x)^s$$

$$\times\, {}_4\psi_5\left[\begin{matrix} (1+\sigma+\tau+s,2),(1+\sigma+\delta-\alpha-\alpha'-\beta+\tau+s,2),(1+\sigma+\beta'-\alpha'+\tau+s,2),(1,1) \\ (\frac{\tau}{\xi}+\frac{b+2}{2},a),(1+\sigma+\beta'+\tau+s,2),(1+\sigma+\delta-\alpha-\alpha'+\tau+s,2),(1+\sigma+\delta-\alpha'-\beta+\tau+s,2),(\mu,\lambda) \end{matrix}\middle| \frac{-cx^2}{4}\right]. \tag{10}$$

Proof. Denoting L by left-hand side (L.H.S) of (10). Then, using (3) and (4), and then interchanging the order of integration and summation, under the valid given conditions in Theorem 2.1, we have

$$L = \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \frac{(-w)_{u.s}}{s!} A_{w.s} \frac{(-c)^k}{\Gamma(\lambda k+\mu)\,\Gamma\left(ak+\frac{\tau}{\xi}+\frac{b+2}{2}\right) 2^{2k+\tau+1}} \times \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta}\left(t^{\sigma+2k+\tau+s}\right)\right)(x). \tag{11}$$

Here, on making use of the result (8), we obtain

$$\begin{aligned} &= \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \frac{(-w)_{u.s}}{s!} A_{w.s} \frac{(-c)^k}{\Gamma(\lambda k+\mu)\,\Gamma\left(ak+\frac{l}{\xi}+\frac{b+2}{2}\right)(2)^{2k+\tau+1}} x^{\sigma-\alpha-\alpha'+\delta+l+\tau+s+2k} \\ &\quad \times \frac{\Gamma(1+\sigma+\tau+s+2k)\Gamma(1+\sigma+\delta-\alpha-\alpha'-\beta+\tau+s+2k)}{\Gamma(1+\sigma+\beta'+\tau+s+2k)\Gamma(1+\sigma+\delta-\alpha-\alpha'+\tau+s+2k)} \\ &\quad \times \frac{\Gamma(1+\sigma+\beta'-\alpha'+\tau+s+2k)}{\Gamma(1+\sigma+\delta-\alpha'-\beta+\tau+s+2k)}, \end{aligned} \tag{12}$$

$$\begin{aligned} &= \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \frac{(-w)_{u.s}}{s!} A_{w.s} \frac{1}{\Gamma(\lambda k+\mu)\,\Gamma\left(ak+\frac{\tau}{\xi}+\frac{b+2}{2}\right)} x^{\sigma-\alpha-\alpha'+\delta+l+\tau+s} \\ &\quad \times \frac{\Gamma(1+\sigma+\tau+s+2k)\,\Gamma(1+\sigma+\delta-\alpha-\alpha'-\beta+\tau+s+2k)}{\Gamma(1+\sigma+\beta'+\tau+s+2k)\Gamma(1+\sigma+\delta-\alpha-\alpha'+\tau+s+2k)} \\ &\quad \times \frac{\Gamma(1+\sigma+\beta'-\alpha'+\tau+s+2k)}{\Gamma(1+\sigma+\delta-\alpha'-\beta+\tau+s+2k)} \left(\frac{-cx^2}{4}\right)^k. \end{aligned} \tag{13}$$

In accordance with the definition of (5), we obtain (10). □

If we set $w = 0$, $A_{0,0} = 1$, then $s_w^u = 1$ in Theorem 2.1, then we get,

Corollary 2.3. *Let the conditions given in Theorem 2.1 are satisfied, then there hold the formula:*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1}{}_a w_{\tau,b,c,\xi}^{\lambda,\mu}(t)\right]\right)(x)=\frac{x^{l+\sigma-\alpha-\alpha'+\delta+\tau}}{2^{l+\tau}} \times_4\psi_5\left[\begin{matrix}(1+\sigma+\tau,2),(1+\sigma+\delta-\alpha-\alpha'-\beta+\tau,2),(1+\sigma+\beta'-\alpha'+\tau,2),(1,1)\\ (\frac{\tau}{\xi}+\frac{b+2}{2},a),(1+\rho+\beta'+\tau,2),(1+\sigma+\delta-\alpha-\alpha'+\tau,2),(1+\sigma+\delta-\alpha'-\beta+\tau,2),(\mu,\lambda)\end{matrix}\middle|\frac{-cx^2}{4}\right]. \tag{14}$$

Some more special case of (1) *are given below.*

When $\alpha=\alpha+\beta$, $\alpha'=\beta'=0$, $\beta=-\eta$, $\delta=\alpha$, *then we get the following relation.*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta}\right)(x)=\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x), \tag{15}$$

which is given by

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-t/x)f(t)dt,\ \Re(\alpha)>0. \tag{16}$$

Which is of the Saigo fractional integral operator [14].

Now, on using (15) the Theorem 2.1 reduces to the following Corollary:

Corollary 2.4. *The following formula holds:*

$$\left(I_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}S_w^u(t)\,{}_a w_{\tau,b,c,\xi}^{\lambda,\mu}(t)\right]\right)(x)=\frac{x^{l+\sigma-\beta+\tau}}{2^{l+1}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^s \times{}_3\psi_4\left[\begin{matrix}(1+\sigma+\tau+s,2),(1+\sigma+\tau+\eta-\beta+s,2),(1,1)\\ (\frac{\tau}{\xi}+\frac{b+2}{2},a),(1+\sigma-\beta+\tau+s,2),(1+\sigma+\alpha+\eta+\tau+s,2),(\mu,\alpha)\end{matrix}\middle|\frac{-cx^2}{4}\right]. \tag{17}$$

Marichev-Saigo-Maeda right-sided fractional integration (2) of the product of the general class of polynomials (3) and the unified Galué type Sturve function (4) is given as follows:

Theorem 2.5. *Suppose* $a\in\mathbb{N}$, $\alpha,\alpha',\beta,\beta',\delta,\sigma,\tau,b,c\in\mathbb{C}$, λ *and* μ *is same as in* (4) *be such that* $\Re(\delta)>0,\Re(\sigma-l)<2+\min\left[\Re(-\beta),\Re(\alpha+\alpha'-\delta),\Re(\alpha+\beta'-\delta)\right]$, $\frac{\tau}{\xi}+\frac{b}{2}\neq-1,-2,-3,...$, *then there hold the formula:*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1}S_w^u(t)\,{}_a w_{\tau,b,c,\xi}^{\lambda,\mu}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{\sigma-\tau-\alpha-\alpha'+\delta+s}}{2^{l+\tau}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^s \times{}_4\psi_5\left[\begin{matrix}(l-\sigma-\beta-s+\tau,2),(\alpha+\alpha'-\delta-\sigma+l-s+\tau,2),(\alpha+\beta'-\delta-\sigma+l-s+\tau,2),(1,1)\\ (\frac{\tau}{\xi}+\frac{b+2}{2},a),(l-\sigma-s+\tau,2),(\alpha+\alpha'+\beta'-\delta-\sigma+l-s+\tau,2),(\alpha-\beta-\sigma+l-s+\tau,2),(\mu,\lambda)\end{matrix}\middle|\frac{-cx^2}{4}\right]. \tag{18}$$

Proof. Let L denotes the L.H.S of (18). Then, by means of (3) and (4), and then interchanging the order of integration and summation, valid with the given conditions in Theorem 2.2, we have

$$L=\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\frac{(-w)_{u.s}}{s!}A_{w.s}\frac{(-c)^k}{\Gamma(\lambda k+\mu)\Gamma\left(ak+\frac{\tau}{\xi}+\frac{b+2}{2}\right)(2)^{2k+\tau+1}}\times\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta}\left(t^{\sigma-2k-l-\tau+s}\right)\right)(x), \tag{19}$$

Again on making use of the result (9), we obtain

$$L=\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\frac{(-w)_{u.s}}{s!}A_{w.s}\frac{(-c)^k}{\Gamma(\lambda k+\mu)\Gamma\left(ak+\frac{\tau}{\xi}+\frac{b+2}{2}\right)(2)^{2k+\tau+1}}x^{\sigma-\alpha-\alpha'+\delta-l-2k-\tau+s} \times\frac{\Gamma(l-\sigma-\beta-s+\tau+2k)\Gamma(\alpha+\alpha'-\delta-\sigma+l-s+\tau+2k)}{\Gamma(l-\sigma-s+\tau+2k)\Gamma(\alpha+\alpha'+\beta'-\delta-\sigma+l-s+\tau+2k)}$$

$$\times \frac{\Gamma(\alpha+\beta'-\delta-\sigma+l-s+\tau+2k)}{\Gamma(\alpha-\beta-\sigma+l-s+\tau+2k)}, \tag{20}$$

$$L=\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\frac{(-w)_{u.s}}{s!}A_{w.s}\frac{1}{\Gamma(\lambda k+\mu)\Gamma\left(ak+\frac{\tau}{\xi}+\frac{b+2}{2}\right)}x^{\sigma-\alpha-\alpha'+\delta-l-\tau+s}$$
$$\times \frac{\Gamma(l-\sigma-\beta-s+\tau+2k)\Gamma(\alpha+\alpha'-\delta-\sigma+l-s+\tau+2k)}{\Gamma(l-\sigma-s+\tau+2k)\Gamma(\alpha+\alpha'+\beta'-\delta-\sigma+l-s+\tau+2k)}$$
$$\times \frac{\Gamma(\alpha+\beta'-\delta-\sigma+l-s+\tau+2k)}{\Gamma(\alpha-\beta-\sigma+l-s+\tau+2k)}\left(\frac{-cx^2}{4}\right)^k. \tag{21}$$

Further, in accordance with the definition of (5), we reached the required result (17). □

Interestingly, if we set $w=0$, $A_{0,0}=1$ that is $s_w^u=1$ in the Theorem 2.2, then it yields the integral formula:

Corollary 2.6. *Let the conditions of Theorem 2.2 is satisfied, then*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1}{}_a w_{\tau,b,c,\xi}^{\lambda,\mu}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{\sigma-l-\alpha-\alpha'+\delta-\tau}}{2^{l+1}}$$
$$\times {}_4\psi_5\left[\begin{matrix}(l-\sigma-\beta+\tau,2),(\alpha+\alpha'-\delta-\sigma+l+\tau,2),(\alpha+\beta'-\delta-\sigma+l+\tau,2),(1,1)\\(\frac{\tau}{\xi}+\frac{b+2}{2},a),(l-\sigma+\tau,2),(\alpha+\alpha'+\beta'-\delta-\sigma+l+\tau,2),(\alpha-\beta-\sigma+l+\tau,2),(\mu,\lambda)\end{matrix}\middle|\frac{-cx^2}{4}\right]. \tag{22}$$

Now, we give a particular case of the operator (2) as under:

For $\alpha=\alpha+\beta,\alpha'=\beta'=0,\beta=-\eta,\delta=\alpha$, we have the relation from Prudnikov [13].

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta}\right)(x)=\left(I_{-}^{\alpha,\beta,\eta}f\right)(x), \tag{23}$$

where,

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-x/t)f(t)dt,\ \Re(\alpha)>0. \tag{24}$$

Now, on using the relation (23), from the Theorem 2.2, we have,

Corollary 2.7. *From Theorem 2.2, we have*

$$\left(I_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}S_w^u(t)\,{}_a w_{\tau,b,c,\xi}^{\lambda,\mu}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{\sigma-\beta-l-\tau}}{2^{\tau+1}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^s$$
$$\times {}_3\psi_4\left[\begin{matrix}(l-\sigma+\beta-s+\tau,2),(l+\eta-\sigma-s+\tau,2),(1,1)\\(\frac{\tau}{\xi}+\frac{b+2}{2},a),(l-\sigma-s+\tau,2),(l-\sigma+\alpha+\beta+\eta-s+\tau,2),(\mu,\alpha)\end{matrix}\middle|\frac{-cx^2}{4}\right]. \tag{25}$$

3. Special Cases

If we set $\alpha=a=1,\mu=3/2$ and $\xi=1$ in definition (4), we obtain the generalized Struve function (cf. [11, 12]) as under:

$$H_{p,b,c}(z)=\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma(k+3/2)\Gamma\left(k+p+\frac{b+2}{2}\right)}\left(\frac{z}{2}\right)^{2k+p+1},\qquad p,b,c\in\mathbb{C}. \tag{26}$$

In the following corollaries (Corollary 3.1-Corollary 3.4), we suppose that $\alpha,\alpha',\ \beta,\beta',\delta,\sigma,l,b,c\in\mathbb{C}$ with $(\tau+b/2)\neq -1,-2,-3,...$

Corollary 3.1. *Let $\Re(\delta)>0,\Re(\sigma+\tau+1)>0$ then there hold the formula:*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1}S_w^u(t)\,H_{\tau,b,c}(t)\right]\right)(x)=\frac{x^{l+\sigma-\alpha-\alpha'+\delta+\tau}}{2^{\tau+1}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^s$$

$$\times {}_4\psi_5\left[\begin{matrix}(1+\sigma+\tau+s,2),(1+\sigma+\delta-\alpha-\alpha'-\beta+\tau+s,2),(1+\sigma+\beta'-\alpha'+\tau+s,2),(1,1)\\(l+\frac{b+2}{2},1),(1+\sigma+\beta'+\tau+s,2),(1+\sigma+\delta-\alpha-\alpha'+\tau+s,2),(1+\sigma+\delta-\alpha'-\beta+\tau+s,2),(3/2,1)\end{matrix}\middle|\frac{-cx^2}{4}\right]. \tag{27}$$

Corollary 3.2. *Let* $\Re(\delta)>0$, $\Re(\sigma-l)<2+\min\left[\Re(-\beta),\Re(\alpha+\alpha'-\delta),\Re(\alpha+\beta'-\sigma)\right]$, *then there hold the formula:*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1}S_{w}^{u}\left(t\right)H_{\tau,b,c}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{\sigma-l-\alpha-\alpha'+\delta-\tau}}{2^{\tau+1}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^{s}$$

$$\times\,{}_{4}\psi_{5}\left[\begin{matrix}(l-\sigma-\beta-s+\tau,2),(\alpha+\alpha'-\delta-\sigma+l-s+\tau,2),(\alpha+\beta'-\delta-\sigma+l-s+\tau,2),(1,1)\\(l+\frac{b+2}{2},1),(l-\sigma-s+\tau,2),(\alpha+\alpha'+\beta'-\delta-\sigma+l-s+\tau,2),(\alpha-\beta-\sigma+l-s+\tau,2),(3/2,1)\end{matrix}\middle|\frac{-cx^{2}}{4}\right]. \tag{28}$$

Further, if we set $w=0$, $A_{0,0}=1$ (or $s_{w}^{u}=1$) in the Corollaries 3.1 and 3.2, then we get the following result as:

Corollary 3.3. *Suppose that* $\Re(\delta)>0,\Re(\sigma+\tau+1)>0$ *then there hold the formula:*

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\sigma-1}H_{\tau,b,c}(t)\right]\right)(x)=\frac{x^{l+\sigma-\alpha-\alpha'+\tau+\delta}}{2^{\tau+1}}$$

$$\times\,{}_{4}\psi_{5}\left[\begin{matrix}(1+\sigma+l,2),(1+\sigma+\delta-\alpha-\alpha'-\beta+\tau,2),(1+\sigma+\beta'-\alpha'+\tau,2),(1,1)\\(l+\frac{b+2}{2},1),(1+\sigma+\beta'+\tau,2),(1+\sigma+\delta-\alpha-\alpha'+\tau,2),(1+\sigma+\delta-\alpha'-\beta+\tau,2),(3/2,1)\end{matrix}\middle|\frac{-cx^{2}}{4}\right]. \tag{29}$$

Corollary 3.4. *Let* $\Re(\sigma-\tau)<2+\min\left[\Re(-\beta),\Re(\alpha+\alpha'-\delta),\Re(\alpha+\beta'-\delta)\right]$, $\Re(\delta)>0$, *then there hold the formula:*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\delta}\left[t^{\rho-1}H_{\tau,b,c}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{\sigma-l-\alpha-\alpha'+\delta-\tau}}{2^{\tau+1}}$$

$$\times\,{}_{4}\psi_{5}\left[\begin{matrix}(l-\sigma-\beta+2,2),(\alpha+\alpha'-\delta-\sigma+l+\tau,2),(\alpha+\beta'-\delta-\sigma+l+\tau,2),(1,1)\\(l+\frac{b+2}{2},1),(l-\sigma+\tau,2),(\alpha+\alpha'+\beta'-\delta-\sigma+l+\tau,2),(\alpha-\beta-\sigma+l+\tau,2),(3/2,1)\end{matrix}\middle|\frac{-cx^{2}}{4}\right]. \tag{30}$$

Now, if we use the functional relations (15) and (23), we obtain the following image formulas associated with the Saigo operators, as under:

Corollary 3.5. *Assume that* $a\in\mathbb{N}$, $\alpha,\beta,\eta,\sigma,\tau,b,c\in\mathbb{C}$, $\alpha>0$, μ *is any parameter and* $\frac{\tau}{\xi}+\frac{b}{2}\neq-1,-2,-3,...,\Re(\alpha)>0,\Re(\sigma+\tau+1)>max\left[0,\Re(\beta-\eta)\right]$, *then the image formula hold:*

$$\left(I_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}S_{w}^{u}\left(t\right)H_{\tau,b,c}\left(t\right)\right]\right)(x)=\frac{x^{l+\sigma-\beta}}{2^{\tau+1}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^{s}$$

$$\times\,{}_{3}\psi_{4}\left[\begin{matrix}(1+\sigma+l+s,2),(1+\sigma+\tau+\eta-\beta+s,2),(1,1)\\(l+\frac{b+2}{2},1),(1+\sigma-\beta+\tau+s,2),(1+\sigma+\alpha+\eta+\tau+s,2),(3/2,1)\end{matrix}\middle|\frac{-cx^{2}}{4}\right]. \tag{31}$$

Corollary 3.6. *Suppose that* $a\in\mathbb{N}$, $\alpha,\beta,\eta,\sigma,\tau,b,c\in\mathbb{C}$, $\alpha>0$ *and* μ *be any parameter such that* $\frac{\tau}{\xi}+\frac{b}{2}\neq-1,-2,-3,...,\Re(\sigma-l)<2+min\left[\Re(\sigma),\Re(\eta)\right]$, $\Re(\alpha)>0$, *then the following formula holds true:*

$$\left(I_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}S_{w}^{u}\left(t\right)\,H_{\tau,b,c}\left(\frac{1}{t}\right)\right]\right)(x)=\frac{x^{\sigma-\beta-l-\tau}}{2^{\tau+1}}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}(x)^{s}$$

$$\times\,{}_{3}\psi_{4}\left[\begin{matrix}(l-\sigma+\beta-s+\tau,2),(l+\eta-\sigma-s+\tau,2),(1,1)\\(l+\frac{b+2}{2},1),(l-\sigma-s+2,2),(l-\sigma+\alpha+\beta+\eta-s+\tau,2),(3/2,1)\end{matrix}\middle|\frac{-cx^{2}}{4}\right]. \tag{32}$$

4. Concluding Remark

The MSM fractional integral operators have advantage that it generalizes the Saigo's, Erdélyi–Kober, Riemann-Liouville and Weyl fractional integral operators, therefore, several authors called this a general operator. So, we conclude this paper by emphasizing that many other interesting image formulas can be derived as the specific cases of our leading results Theorems 2.1 and 2.2, involving familiar fractional integral operators like Riemann-Liouville, Weyl, and Erdélyi–Kober fractional

integral operators. Further, the various types of Struve function are particular cases of (4). On the other hand, by putting the appropriate values to the arbitrary constant, the family of polynomials (defined by (1.3)) provide several well known classical orthogonal polynomial as its special cases, which includes Hermite, Laguerre, Jacobi, the Konhauser polynomials and so on. Hence, we notice that our key results can prompt yield number of other interesting fractional integrals including different types of Struve functions and orthogonal polynomials, by selecting the proper parameters in the theorems.

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