Int. J. Math. And Appl., 7(4)(2019), 31–35

ISSN: 2347-1557

Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Research Schur-Zassenhaus Theorem by Using Anti-homomorphism

Zhaoquan Wang[1,*], Shouxu Du[1] and Tao Wang[1]

1 Basic College of Arts and Sciences, Qingdao Binhai University, Qingdao 266000, China.

Abstract: Let $f : G \to K$ be a function between finite groups. When the function f is a anti-homomorphism it may preserve group structure. In this paper, we consider measures of how nearly the group structure is preserved by an arbitrary function. We first define anti-distributor which is a new way to build anti-homomorphism from arbitrary function. we demonstrate the applicability of this theory by constructing anti-homomorphism to prove Schur-Zassenhaus theorem.

Keywords: anti-distributor; anti-homomorphism; Schur-Zassenhaus theorem.

1. Introduction

Commutativity is one of the important properties in the study of finite group theory. As a measure of commutativity, the commutator and commutator group have been defined in group theory. Let G be a group, $a, b \in G$, write $[a, b] := a^{-1}b^{-1}ab$. It is called the commutator of a and b. Also write $G' := \langle [a, b] | a, b \in G \rangle$ is the commutator subgroup of G. In order to study the p-commutativity, C.Hobby [1] have defined the concept of p-commutator, let p be a prime number, $a, b \in G$, write $[a, b; p] := a^{-p}b^{-p}(ab)^p$. It is called the p-commutator of a and b. In [2], I.Hawthorn and Y.Guo have generalized the concept of p-commutator to the general situation. Let G and K be groups, $f : G \to K$ be function, $x, y \in G$, write $[x, y; f] := f(y)^{-1}f(x)^{-1}f(xy)$. It is called the f-distributor of x and y. They have studied the finite groups function and f-distributor on influencing the group structure, and obtained some important results. In this paper, We first define anti-distributor which is a new way to build anti-homomorphism from arbitrary function. As an application, we present a new proof of the Schur-Zassenhaus theorem by constructing anti-homomorphism and group inverse action.

2. Preliminaries

Definition 2.1. *Let G and K be groups, $f : G \longrightarrow K$ be a function, $x, a \in G$. Write $f^a(x) = f(a)^{-1}f(xa)$, then $f^a : G \longrightarrow K$ is also a function, f^a is called the anti-conjugate of the function f under a.*

Definition 2.2. *Let G and K be groups, $f : G \to K$ be function, for any $x, y \in G$, $f(xy) = f(y)f(x)$, then f is an anti-homomorphism.*

* *E-mail: zhaoquanwang2019@163.com*

Definition 2.3. *Let φ be an anti-homomorphism from G to K. Then $\alpha : G/Ker(\varphi) \to K$ with $(Ker(\varphi)x \mapsto \varphi(x))$ is an anti-homomorphism and injective. In particular $G/Ker(\varphi) \cong Im(\varphi)$.*

Lemma 2.4. *Let G and K be groups, $f : G \longrightarrow K$ be a function. Then f is an anti-homomorphism if and only if $f^a = f$, for any $a \in G$.*

Proof. If f is an anti-homomorphism, then for any $x, a \in G$, we have $f(xa) = f(a)f(x)$. And by definition 2.1, we obtain $f^a(x) = f(a)^{-1}f(xa)$, hence $f^a(x) = f(x)$;
On the other hand, by Definition 2.1, we obtain $f^a(x) = f(a)^{-1}f(xa)$, and since $f^a(x) = f(x)$, then $f(a)^{-1}f(xa) = f(x)$, hence $f(xa) = f(a)f(x)$, this f is an anti-homomorphism. □

Definition 2.5. *Let G and K be groups, $f : G \longrightarrow K$ be a function, $x, y \in G$. Write $[x, y; f] := f(x)^{-1}f(y)^{-1}f(xy)$. It is called the f anti-distributor of x and y. It follows that $f^y(x) = f(x)[x, y; f]$.*

Lemma 2.6. *Let G and K be groups, $f : G \longrightarrow K$ be a function. Let $x, y, z, \in G$, then $[y, z; f]^{f(x)} = [x, y; f][xy, z; f][x, yz; f]^{-1}$.*

Proof. we expand $f(xyz)$ in two different ways to obtain.

$$f(xyz) = f(yz)f(x)[x, yz; f] = f(z)f(y)[y.z; f]f(x)[x, yz; f],$$

and

$$f(xyz) = f(z)f(xy)[xy, z; f] = f(z)f(y)f(x)[x, y; f][xy, z; f].$$

So

$$[y, z; f]^{f(x)} = [x, y; f][xy, z; f][x, yz; f]^{-1}.$$

□

Lemma 2.7. *Let G and K be groups, $f_i : G \longrightarrow K$ be function $(1 \le i \le n)$. Let $x, y \in G$, write $(f_i \cdot f_j)(x) = f_i(x)f_j(x)$, then $[x, y; f_i \cdot f_j] = f_j(x)^{-1}f_i(x)^{-1}f_j(y)^{-1}f_i(x)[x, y; f_i]f_j(y)f_j(x)[x, y; f_j]$.*

Proof. For any $x, y \in G$, by Definition 2.5, we have

$$[x, y; f_i \cdot f_j] = f_i \cdot f_j(x)^{-1} f_i \cdot f_j(y)^{-1} f_i \cdot f_j(xy) = f_j(x)^{-1}f_i(x)^{-1}f_j(y)^{-1}f_i(y)^{-1}f_i(xy)f_j(xy)$$

By Definition 2.5, we observed that $f_i(xy) = f_i(y)f_i(x)[x, y; f_i]$ and $f_j(xy) = f_j(y)f_j(x)[x, y; f_j]$, then

$$[x, y; f_i \cdot f_j] = f_j(x)^{-1}f_i(x)^{-1}f_j(y)^{-1}f_i(y)^{-1}f_i(y)f_i(x)[x, y; f_i]f_j(y)f_j(x)[x, y; f_j],$$

therefore we obtain

$$[x, y; f_i \cdot f_j] = f_j(x)^{-1}f_i(x)^{-1}f_j(y)^{-1}f_i(x)[x, y; f_i]f_j(y)f_j(x)[x, y; f_j].$$

□

Lemma 2.8. *Let G and K be groups, $f_i : G \longrightarrow K$ be function $(1 \le i \le n)$. and K is an abelian group. Let $x, y \in G$, then function $F(x) = \prod_i^n f_i(x)$ is a product function from G to K, and $[x, y; F] = \prod_i^n [x, y; f_i]$.*

Proof. Firstly, if $n=1$, it is obvious that $[x,y;F]=[x,y;f_1]$. For case $n=2$, we get $F(x)=f_1(x)f_2(x)$, by Lemma 2.7, then

$$[x,y;F]=[x,y;f_1\cdot f_2]=f_2(x)^{-1}f_1(x)^{-1}f_2(y)^{-1}f_1(x)[x,y;f_1]f_2(y)f_2(x)[x,y;f_2].$$

Note that in the case that K is abelian group, then we obtain $[x,y;F]=[x,y;f_1\cdot f_2]=[x,y;f_1][x,y;f_2]$. For case $n\geq 2$, we can prove it by induction. Hence $[x,y;F]=\prod_i^n[x,y;f_i]$. □

Lemma 2.9. *Let G and K be groups, $f_i:G\longrightarrow K$ be function $(1\leq i\leq n)$. and K is an abelian group,write $F(x)=\prod_i^n f_i(x)$, $x,a\in G$, then F is a function from G to K, and $F^a(x)=\prod_i^n f_i^a(x)$;*

Proof. By the Definition 2.5 and Lemma 2.8, then $F^a(x)=F(x)[x,a;F]=\prod_i^n f_i(x)\prod_i^n[x,a;f_i]$, since K is an abelian group, then $F^a(x)=\prod_i^n f_i(x)[x,a;f_i]=\prod_i^n f_i^a(x)$. □

Lemma 2.10. *Let G and K be groups, $f_i:G\longrightarrow K$ be function $(1\leq i\leq n)$. Let $x,a\in G$, write $(f_i\cdot f_j)(x)=f_i(x)f_j(x)$, then $(f_i\cdot f_j)^a(x)=(f_i^a(x))^{f_j(a)}f_j^a(x)$.*

Proof. By Definition 2.1, we get

$$\begin{aligned}(f_i\cdot f_j)^a(x)&=(f_i\cdot f_j)(a)^{-1}(f_i\cdot f_j)(xa)\\&=f_j(a)^{-1}f_i(a)^{-1}f_i(xa)f_j(xa)\\&=f_j(a)^{-1}f_i^a(x)f_j(xa)\\&=f_j(a)^{-1}f_i^a(x)f_j(a)f_j(a)^{-1}f_j(xa)\\&=(f_i^a(x))^{f_j(a)}f_j^a(x).\end{aligned}$$

□

3. Proof of Theorem

Theorem 3.1 (Theorem of Schur-Zassenhaus)**.** *Let H be an Abelian normal subgroup of G such that $(|H|,|G/H|)=1$.*

(1). Then H has a complement in G.

(2). Suppose that K_0 and K_1 are two complements of H in G . Then K_0 and K_1 are conjugate in G.

Proof. For convenience, we use the following notations: $\bar{G}=G/H$, $\Omega=\{f|f:\bar{G}\longrightarrow G$ be function, $\pi\circ f=id_{\bar{G}}, f(1)=1\}$, where π is a natural anti-homomorphism from G to $\bar{G}$ with $\pi(g)=gH$, for any $g\in G$. We shall confirm the assertion by proving the following eight claims.

Claim 1. The group $\bar{G}$ inverse action on the set Ω of functions by conjugation.

Let $f\in\Omega$, and $\bar{a}\in\bar{G}$. By the Definition 2.1, $f^{\bar{a}}$ is a function from $\bar{G}$ to G, and $f^{\bar{a}}(1)=1$. For any $\bar{a}\in\bar{G}$, we have

$$\pi\circ f^{\bar{a}}(\bar{x})=\pi(f^{\bar{a}}(\bar{x}))=\pi(f(\bar{a})^{-1}f(\bar{x}\bar{a}))=\pi(f(\bar{x}\bar{a}))\pi(f(\bar{a})^{-1})=\pi\circ f(\bar{x}\bar{a})\pi\circ f(\bar{a})^{-1}=\bar{x}\bar{a}\bar{a}^{-1}=\bar{x},$$

it follows that $f^{\bar{a}}\in\Omega$, therefore the group $\bar{G}$ inverse action on the set Ω of functions by conjugation.

Claim 2. Let $|H|=n, |\bar{G}|=m$ and $a\in H$, then there exists positive integers k with $a^{km}=a$.

Since $(|H|,|G/H|)=1$, it follows that $(m,n)=1$. Thus there exists some positive integers k and t such that $km-tn=1$, so we obtain $a^{km}=a^{1+tn}=aa^{tn}=a$.

Claim 3. Let $\bar{x},\bar{a}\in\bar{G}$, $f\in\Omega$, where the integer k is the same as Claim 2, then

(1). $[\bar{x}, \bar{a}; f] \in H$;

(2). $F(\bar{x}) = \prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{a}; f]^k$ is a function from $\bar{G}$ to H;

(3). $\hat{f} = f \cdot F \in \Omega$.

Since

$$\pi[\bar{x}, \bar{a}; f] = \pi(f(\bar{x})^{-1} f(\bar{a})^{-1} f(\bar{x}\bar{a})) = \pi(f(\bar{x}\bar{a}))\pi(f(\bar{a})^{-1})\pi(f(\bar{x})^{-1}) = \pi \circ f(\bar{x}\bar{a})\pi \circ f(\bar{a})^{-1}\pi \circ f(\bar{x})^{-1} = \bar{x}\bar{a}\bar{a}^{-1}\bar{x}^{-1} = 1,$$

then $[\bar{x}, \bar{a}; f] \in Ker(\pi) = H$. Note that H is an Abelian group, we obtain $\prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{a}; f]^k \in H$ and it follows that F is a function from $\bar{G}$ to H. By $[1, \bar{a}; f] = f(1)^{-1} f(\bar{a})^{-1} f(\bar{a}) = 1$, it implies that $F(1) = 1$ and $\hat{f}(1) = f(1)F(1) = 1$. Since

$$\pi \circ \hat{f}(\bar{x}) = \pi(\hat{f}(\bar{x})) = \pi(f(\bar{x})F(\bar{x})) = \pi(f(\bar{x}) \prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{a}; f]^k) = \pi \circ \prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{a}; f]^k \pi \circ f(\bar{x}) = 1 \cdot \pi \circ f(\bar{x}) = \bar{x}.$$

Hence $\hat{f} \in \Omega$.

Claim 4. With the notation above, then $\hat{f}$ is a anti-homomorphism from $\bar{G}$ to G.

Let $\bar{a}, \bar{b}, \bar{x} \in \bar{G}$, By Lemma 2.10, we have equation

$$\hat{f}^{\bar{b}}(\bar{x}) = (f \cdot F)^{\bar{b}}(\bar{x}) = f^{\bar{b}}(\bar{x})^{F(\bar{b})} F^{\bar{b}}(\bar{x}) = F(\bar{b})^{-1} f^{\bar{b}}(\bar{x}) F(\bar{x}\bar{b}). \tag{1}$$

By Lemma 2.6, we get $[\bar{x}\bar{b}, \bar{a}; f]^k = [\bar{x}, \bar{b}; f]^{-k}([\bar{b}, \bar{a}; f]^k)^{f(\bar{x})}[\bar{x}, \bar{b}\bar{a}; f]^k$. Note that H is abelian, then

$$\prod_{\bar{a} \in \bar{G}} [\bar{x}\bar{b}, \bar{a}; f]^k = \prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{b}; f]^{-k} \prod_{\bar{a} \in \bar{G}} ([\bar{b}, \bar{a}; f]^k)^{f(\bar{x})} \prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{b}\bar{a}; f]^k. \tag{2}$$

By Claim 2, it follows that $\prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{b}; f]^{-k} = [\bar{x}, \bar{b}; f]^{-km} = [\bar{x}, \bar{b}; f]^{-1}$. Thus the equation (2) becomes equation

$$F(\bar{x}\bar{b}) = [\bar{x}, \bar{b}; f]^{-1} f(\bar{x})^{-1} F(\bar{b}) f(\bar{x}) F(\bar{x}). \tag{3}$$

From (1) and (3), we get

$$\hat{f}^{\bar{b}}(\bar{x}) = F(\bar{b})^{-1} f^{\bar{b}}(\bar{x}) [\bar{x}, \bar{b}; f]^{-1} f(\bar{x})^{-1} F(\bar{b}) f(\bar{x}) F(\bar{x}) = f(\bar{x}) F(\bar{x}) = \hat{f}(\bar{x}).$$

By Lemma 2.4, $\hat{f}$ is a anti-homomorphism from $\bar{G}$ to G.

Claim 5. With the notation above, then $Im(\hat{f})$ is complement to H in G.

Let $x \in G$. Since $\pi(x) = xH = \bar{x} \in \bar{G}$, then $\hat{f}(\pi(x)) = \hat{f}(\bar{x}) = a \in Im(\hat{f})$. We also have $xH = \bar{x} = \pi(\hat{f}(\bar{x})) = \pi(a) = aH$, thus there exists $h \in H$ such that $x = ah$, which implies that $G \subseteq Im(\hat{f})H$, Hence $Im(\hat{f})H = G$.

Next we need to show that $Im(\hat{f}) \bigcap H = 1$. For any $x \in Im(\hat{f}) \bigcap H$, since $x \in Im(\hat{f})$, then there exists $\bar{a} \in \bar{G}$ such that $\hat{f}(\bar{a}) = x$, and we have $\pi(x) = xH = H$, so $\bar{a} = \pi(\hat{f}(\bar{a})) = \pi(x) = H$. Note that $\hat{f} \in \Omega$ is a anti-homomorphism, which implies $x = \hat{f}(\bar{a}) = \hat{f}(H) = 1$. Therefore $Im(\hat{f})$ is a complement to H in G.

Claim 6. Let $f_1, f_2 \in \Omega$, and $\bar{x}, \bar{a} \in \bar{G}$. Write $h(\bar{x}) = f_1(\bar{x})^{-1} f_2(\bar{x})$, $L(\bar{x}) = \prod_{\bar{a} \in \bar{G}} [\bar{x}, \bar{a}; h]^k$, $\bar{h} = h \cdot L$, where the integer k is the same as Claim 2, then $\bar{h}$ is trivial anti-homomorphism from $\bar{G}$ to H.

Since $\pi(h(\bar{x})) = \pi(f_1(\bar{x})^{-1} f_2(\bar{x})) = \pi \circ f_2(\bar{x}) \pi \circ f_1(\bar{x})^{-1} = \bar{x}\bar{x}^{-1} = 1$, then $h(\bar{x}) \in Ker(\pi) = H$, it follows that h is a function from $\bar{G}$ to H, which imply

$$\pi([\bar{x}, \bar{a}; h]) = \pi(h(\bar{x})^{-1} h(\bar{a})^{-1} h(\bar{x}\bar{a})) = \pi(h(\bar{x}\bar{a}))\pi(h(\bar{a})^{-1})\pi(h(\bar{x})^{-1}) = \bar{x}\bar{a}\bar{a}^{-1}\bar{x}^{-1} = 1,$$

so $[\bar{x}, \bar{a}; h] \in H$. Since H is an abelian group, then $\prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; h]^k \in H$, it follows that L is a function from $\bar{G}$ to H. Next the proof is the same as Claim 4, we deduce that $\bar{h} = h \cdot L$ is a anti-homomorphism from $\bar{G}$ to H. By the fundamental theorem of anti-homomorphism, this is Definition 2.3, it follows that $\bar{G}/Ker(\bar{h}) \cong Im(\bar{h})$ implies that $|Im(\bar{h})|||\bar{G}|$; Furthermore, note that $Im(\bar{h})$ is the subgroup of H. we conclude that $|Im(\bar{h})|||H|$. Now $(|H|, |\bar{G}|) = 1$ implies that $|Im(\bar{h})| = 1$. Hence $\bar{h}$ is the trivial anti-homomorphism.

Claim 7. With the notation above, then there exists $\mu \in G$ such that $\hat{f}_2(\bar{x}) = \mu^{-1}\hat{f}_1(\bar{x})\bar{h}(\bar{x})\mu = (\hat{f}_1(\bar{x}))^{\mu}$, for any $\bar{x} \in \bar{G}$. Let $\bar{X}, \bar{A} \in \bar{G}$, by Claim 6, we have $f_2(\bar{x}) = f_1(\bar{x})h(\bar{x})$, so $[\bar{x}, \bar{a}; f_2] = [\bar{x}, \bar{a}; f_1 h]$. By Lemma 2.7, we get

$$[\bar{x}, \bar{a}; f_2] = (h(\bar{a})^{-1})^{f_1(\bar{x})} h(\bar{a})[\bar{x}, \bar{a}; f_1][\bar{x}, \bar{a}; h],$$

where the integer k is the same as Claim 2, and note that $\bar{H}$ is abelian , then

$$[\bar{x}, \bar{a}; f_2]^k = (h(\bar{a})^{-k})^{f_1(\bar{x})} h(\bar{a})^k [\bar{x}, \bar{a}; f_1]^k [\bar{x}, \bar{a}; h]^k,$$

thus

$$\prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; f_2]^k = f_1(\bar{x})^{-1}(\prod_{\bar{a}\in\bar{G}} h(\bar{a})^{-k}) f_1(\bar{x}) \prod_{\bar{a}\in\bar{G}} h(\bar{a})^k \prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; f_1]^k \prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; h]^k,$$

Furthermore, we have $\prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; f_2]^k = [f_1(\bar{x}), \prod_{\bar{a}\in\bar{G}} h(\bar{a})^k] \prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; f_1]^k \prod_{\bar{a}\in\bar{G}} [\bar{x}, \bar{a}; h]^k$ and write $\mu = \prod_{\bar{a}\in\bar{G}} h(\bar{a})^k$ implying $\mu \in G$, then $F_2(\bar{x}) = [f_1(\bar{x}), \mu] F_1(\bar{x}) L(\bar{x})$, thus we obtain

$$\hat{f}_2(\bar{x}) = f_2(\bar{x})F_2(\bar{x}) = f_1(\bar{x})h(\bar{x})f_1(\bar{x})^{-1}\mu^{-1}f_1(\bar{x})\mu F_1(\bar{x})L(\bar{x}).$$

Since $\bar{h}$ is trivial anti-homomorphism from $\bar{G}$ to H, hence $\hat{f}_2(\bar{x}) = \mu^{-1}\hat{f}_1(\bar{x})\bar{h}(\bar{x})\mu = (\hat{f}_1(\bar{x}))^{\mu}$.

Claim 8. Let K_1 and K_2 are any two complements of H in G, then there exits $\hat{f}_1, \hat{f}_2 \in \Omega$ such that $K_1 = Im(\hat{f}_1)$ and $K_2 = Im(\hat{f}_2)$.

In fact, $G/H \cong K_1 \leq G$, i.e., $\bar{G} \cong K_1$. Let $\hat{f}_1 : \bar{G} \to K_1$ $(gH \mapsto k_1)$, where $g = k_1 h$, $k_1 \in K_1, h \in H$, it is obvious that $\hat{f}_1 : \hat{G} \to K_1$ is an anti-isomorphism, it follows that $\hat{f}_1(1) = 1$. For any $\bar{g} \in \bar{G}$, let $\bar{g} = gH$, then we have $\pi(\hat{f}_1(\bar{g})) = \pi(\hat{f}_1(gH)) = \pi(k_1) = k_1 H = k_1 h H = gH = \bar{g}$, hence $\hat{f}_1 \in \Omega$. Note that $\hat{f}_1$ is an anti-isomorphism and by Claim 5, we obtain that $Im(\hat{f}_1) = K_1$. Similarly, $Im(\hat{f}_2) = K_2$. By Claim 7, then there exists $\mu \in G$ such that $\hat{f}_2(\bar{x}) = (\hat{f}_1(\bar{x}))^{\mu}$, for any $\bar{x} \in \bar{G}$. So $K_2 = K_1^{\mu}$. This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let H be a normal subgroup of G such that $(|H/H'|, |G/H|) = 1$, then*

(1). There is a relative complement of H over H';

(2). Any two relative complements of H over H' are conjugate.

References

[1] C. Hobby, *A characteristic subgroup of a p-group*, Pacific Journal of Mathematics, 10(1960), 853-858.

[2] I. Hawthorn and Y. Guo, *Arbitary functions in group theory*, New Zealand Journal of Mathematics, 45(2013), 1-9.

[3] Zh. Q. Wang, *Group inverse action on function set and its applications*,Qingdao:Qingdao University, (2018).

[4] L. K. Hua, *On the automorphisms of a sfield*, Proceedings of the National Academy of Sciences. USA, 35(1949), 386-389.

[5] H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Spinger-Verlag, Berlin, (2004).

[6] D. T. S. Robinson, *A course in the theory of groups*, Spinger-Verlag, 2nd Edition, New York, (1995).