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Ideal Extension and Idempotents of a Rees Matrix Semigroup

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Abstract: This paper deals with the method of resolving two relvent problems of Rees matrix semigroups, the ideal extension problem and the structure of sandwich set of idempotents. To resolve the first problem let us consider the ideal extension of Rees matrix semigroup over the multiplicative group $U(n)$. The study is done using the Rees matrix semigroup containing only the zero element and a semigroup which has more than one element. The idempotents of Rees matrix semigroup is studied using Biordered set and finally we arrive at a conclusion about the structure of sandwich set of idempotents.

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Keywords: Rees matrix semigroups, Ideal Extension, Biordered Set.

1. Introduction

The semigroup theory gains importance now a days not only from the mathematical point of view where it is investigated as an abstract algebraic structure but also from the theoretical computer science point of view where it has tremendous applications in significant areas like Automata, Neural networks etc. Rees matrix semigroup is a special class of semigroup introduced by David Rees in 1940, which has fundamental importance in semigroup theory because they are used to classify certain class of simple semigroups. In this paper the concept of ideal extension of semigroup introduced by Clifford [1] has been studied in Rees matrix semigroups over the multiplicative group $U(n)$. In the last section, we studied the idempotents of a rees matrix semigroup using Biordered sets. In 1970 K.S.S Nambooripad [4] introduced the concept of a Biordered set, which is very useful to explain the structure of idempotents in a semigroup. He identified that collection of all idempotents in a semigroup forms a Biordered set.

2. Preliminaries

The results derived out of this sequel makes use of the definitions mentioned below. We follow the notations and definitions as in [2] and [3]. A semigroup is a set S together with an associative binary operation on S. An element $e \in S$ such that $e.e = e$ is called an idempotent and the set of all idempotents in S will be denoted by $E(S)$. An element z of S is said to be a zero element if $xz = zx = z$ for all $x \in S$. If G is group, $G^0 = G \cup \{0\}$ is a semigroup. Semigroup formed in this way is called a 0-group, or a group with zero. A non-empty subset I of S is called left ideal if $SI \subseteq I$ a right ideal if $IS \subseteq I$, and

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an ideal if it is both a left and a right ideal. An element a in a semigroup S is said to be regular if there exist an element x in S such that $axa = a$, if every element of S is regular then S is a regular semigroup.

Definition 2.1. *Let S be a semigroup and I be an ideal in S. Define a congruence relation ρ on S as $a\ \rho\ b \Longleftrightarrow a = b$ or $a, b \in I$. Then ρ is called as the Rees congruence modulo I on S. We denote this as S/I or S/ρ and call this as the Rees factor semigroup S modulo I. Let S be a semigroup and I be an ideal in S. Thus a set, S/I may be identified with $S \setminus I$ with an element θ (zero element) adjoined.*

Definition 2.2. *Let G be a group with identity element e, and let I, Λ be non-empty sets. Let $P = (p_{\lambda i})$ be $\Lambda \times I$ matrices with entries in the 0-group $G^0 = G \cup \{0\}$ and suppose that P is regular, in the sense that no row or column of P consists entirely of zeros. The Rees matrix semigroup $S = M^0(G, I, \Lambda, P)$ is the set of all triples of the form $(I \times G \times \Lambda)$ with a zero element 0 adjoined to S. In matrix terminology any element (i, g, λ) be the matrix with g in (i, λ)th position and zero elsewhere.*

If P contains no zero entry, then there are no proper divisors of zero in $M^0(G, I, \Lambda, P)$. The semigroup $M^0(G, I, \Lambda, P)\setminus\{0\}$ is called the Rees $I \times \Lambda$ matrix semigroup without zero over the group G with sandwich matrix P, and denote by $M(G, I, \Lambda, P)$. The multiplication on a Rees matrix semigroup is defined by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & if\ p_{\lambda j} \neq 0 \\ 0 & if\ p_{\lambda j} = 0 \end{cases}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0$$

where (i, a, λ),$(j, b, \mu) \in M^0(G, I, \Lambda, P)$.

Remark 2.3 ([3])**.** *A non-zero element (i, a, λ) of a Rees matrix semigroup is an idempotent iff $p_{\lambda i} \neq 0$ and $a = p_{\lambda i}^{-1}$.*

Proof. Let (i, a, λ) be a non-zero idempotent element in a Rees matrix semigroup. Then $(i, a, \lambda)(i, a, \lambda) = (i, a, \lambda)$. But the multiplication in Rees matrix semigroup implies $(i, ap_{\lambda i}a, \lambda) = (i, a, \lambda)$, that is $ap_{\lambda i}a = a$. Which happends only if $p_{\lambda i} \neq 0$ and $a = p_{\lambda i}^{-1}$.
Conversly let $p_{\lambda i} \neq 0$ and $a = p_{\lambda i}^{-1}$. Then

$$(i, a, \lambda)(i, a, \lambda) = (i, ap_{\lambda i}a, \lambda) = (i, a, \lambda)$$

. □

Definition 2.4 ([2])**.** *Let S be a non-empty semigroup and Q be a semigroup with zero disjoint from S. An ideal extension of S by Q is a semigroup E such that S is an ideal of E and the rees quotient E/S is isomorphic to Q. Such a semigroup doesn't always exist. If it does, E is as a set, disjoint union of S and $Q^* = Q \setminus 0$. Constructing the operation $*$ on E is from semigroup S and Q, is the ideal extension problem.*

In an ideal extension E of S by Q there exist five types of products.

- products $x * y$ with $x, y \in S$; then $x * y = xy \in S$.
- products $a * x$ with $x \in S$, $a \in Q^*$; then $a * x \in S$.
- products $x * a$ with $x \in S, a \in Q^*$; then $x * a \in S$.

- products $a * b$ with $a, b \in Q^*$, $ab = 0$ in Q; then $a * b \in S$
- products $a * b$ with $a, b \in Q^*, ab \neq 0$ in Q; then $a * b = ab \in Q^*$.

We must determine all these products in order to solve the ideal extension problem.

2.1. Biordered Sets

Definition 2.5. *A partial algebra E is a set together with a partial binary operation on E. On E we define*

$$\omega^r = \{(e, f) : fe = e\};\ \omega^l = \{(e, f) : ef = e\}$$

and

$$\mathcal{R} = (\omega^r) \cap (\omega^r)^{-1}, \mathcal{L} = (\omega^l) \cap (\omega^l)^{-1}, \omega = \omega^r \cap \omega^l.$$

We denote $e\,\omega^r\,f$ for $fe = e$ and $e\,\omega^l\,f$ for $ef = e$.

Definition 2.6. *Let E be the partial algebra. Then E is a biordered set if the following axioms and their duals hold:*

(1). ω^r and ω^l are quasi orders on E and $\mathcal{D}_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$

(2). $f \in \omega^r(e) \implies f\mathcal{R}fe\,\omega\,e$

(3). $g\,\omega^l\,f$ and $f, g \in \omega^r(e) \implies ge\,\omega^l\,fe$

(4). $g\,\omega^r\,f\omega^r e \implies gf = (ge)f$

(5). $g\,\omega^l\,f$ and $f, g \in\omega^r\,(e) \implies (fg)e = (fe)(ge)$

Definition 2.7. *Let $\mathcal{M}(e, f)$ denote quasi ordered set $(\omega^l(e) \cap \omega^r(f), <)$ where $<$ is defined by*

$$g < h \iff eg\omega^r eh \text{ and } gf\omega^l hf$$

Then the set

$$S(e, f) = \{h \in \mathcal{M}(e, f) : g < h\ \forall g \in \mathcal{M}(e, f)\}$$

is called sandwich set of e and f.

(6). $f, g \in \omega^r(e) \implies S(f, g)e = S(fe, ge)$.

The biordered set E is said to be regular if $S(e, f) \neq \phi$ for every $e, f \in E$.

Example 2.8. *The idempotents of a semigroup $E(S)$ is a Biordered set with the partial algebra consisting of the set $E = E(S)$ and the multiplication restricted to*

$$\mathcal{D}_E = \{(e, f) \in E \times E \mid ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}$$

Thus the product of two idempotents is defined in the partial algebra if and only if one is a right or left zero of the other.

3. Ideal Extension of a Rees matrix Semigroup

In this section we study the ideal extension for a Rees matrix semigroup over the multiplicative group $U(n)$ by two different semigroups. Here $U(n)$ denotes the set with numbers which are less than n and relatively prime to n. Clearly $U(n)$ is a group with respect to $\times_n$.

Theorem 3.1. *Let $S = M(G, I, \Lambda, P)$ be the Rees matrix semigroup over the multiplicative group $U(n)$ and $|I| = |\Lambda| = k$ for any positive integer k. Also let Q be another Rees matrix semigroup which contains only the zero element of S. Then the ideal extension of S by Q is the semigroup E under the operation $*$ which is same as operation defined on S.*

Proof. Let

$$E = \left\{ \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & a \end{bmatrix} \right\}$$

where $a \in U(n)$. Then $E = S \cup Q^*$. Since Q has only the zero element of S, $Q^* = \phi$. Then E is the set $S = M(G, I, \Lambda, P)$ itself. Define an operation $*$ on E as follows

$$(i, a, \lambda) * (j, b, \mu) = (i, ap_{\lambda j}b, \mu)$$

where $(i, a, \lambda), (j, b, \mu) \in M(G, I, \Lambda, P)$. Clearly $*$ is a well-defined operation on E. Since S is a semigroup under this operation, E is also a semigroup with the same operation. Since E is same as the semigroup S, S is an ideal of E. Also the product of any two elements in S is same as the product in E. The remaining four products in E trivially holds since $Q^* = \phi$. Now it remains to establish an isomorphism from E/S to Q. But the only possible map is to assign S to the zero element in Q. Clearly this map is an isomorphism. Hence E is an ideal extension of S by Q. ☐

Theorem 3.2. *Let $S = M(G, I, \Lambda, P)$ be the Rees matrix semigroup over the multiplicative group $U(n)$ with $|I| = |\Lambda| = 2$ and $Q = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ be a semigroup under matrix multiplication. Then an ideal extension of S by Q is the semigroup E under the operation $*$ defined as follows. For $A, B \in E$,*

$$A * B = \begin{cases} APB & if\ A, B \in S, where\ P\ is\ the\ regular\ matrix\ over\ U(n) \\ APB & otherwise,\ where\ P\ is\ the\ identity\ matrix \end{cases} \tag{1}$$

Proof. Let $E = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, where $a \in U(n)$. Then clearly $E = S \cup Q^*$, where S and Q^* are disjoint and $Q^* = Q \setminus \{0\}$. Define an operation $*$ on E as follows. For $A, B \in E$,

$$A * B = \begin{cases} APB & if\ A, B \in S, \text{where } P\ is\ the\ regular\ matrix\ over\ U(n) \\ APB & otherwise,\ where\ P\ is\ the\ identity\ matrix \end{cases}$$

Clearly the operation $*$ is well-defined on E and $*$ is an associative binary operation on E. Then E is a semigroup under the operation $*$. Now we show that E is an ideal extension of S by Q. In order to prove this, we first verify that S is an ideal of E. Let $p \in S$ and $q \in E$. Then there are two possibilities $q \in S$ or $q \in Q^*$. Suppose $p \in S$ and $q \in Q^*$. Then

$$p \in \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right\},$$

where a is an element in $U(n)$. If $q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e$ then $pq = pe = p \in S$. If $q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $pq \in S$ for all choices of P. If $p \in S$ and $q \in S$. Since S is a Rees matrix semigroup using closure propety of S $pq \in S$. Hence $SE \subseteq S$. Similarly we can show that $ES \subseteq S$. Therefore S is an ideal of E. Consider the Rees quotient E/S. Then

$$E/S = \{S\} \cup \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \left\{ S, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\},$$

where S is the zero element of E/S. Define a map ϕ from E/S to Q by

$$\phi(A) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & if\ A = S \\ A & otherwise \end{cases}$$

Then $\phi(A * B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, if either A or B is in S or both $A = B = S$ and $\phi(A).\phi(B)$ is equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in either cases. Otherwise

$$\phi(A * B) = A * B = AB \ and \ \phi(A).\phi(B) = AB$$

Hence the map ϕ preserves the operation as defined in (1). In order to show ϕ is one-one, let $\phi(A) = \phi(B)$

Case 1: If $\phi(A) = \phi(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then from the definition of ϕ this is possible only if $A = S$ and $B = S$. Therefore $A = B$.

Case 2: If $\phi(A)$ and $\phi(B)$ are non-zero elements in Q. Then by the definition of ϕ it is clear that $A = B$. Hence in either cases we get ϕ is one-one. Thus we get there is an isomorphism from $E \setminus S$ to Q. Also all the five products defined in an ideal extension holds from the definition of $*$ on E. Therefore $E = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, is an ideal extension of S by Q. □

Corollary 3.3. *Let $S = M(G, I, \Lambda, P)$ be a Rees matrix semigroup over the multiplicative group $U(n)$ with $|I| = |\Lambda| = n$ and* $Q = \left\{ \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \right\}$ *be a semigroup under matrix multiplication. Then an ideal extension of S by Q is the semigroup E under the operation $*$ defined as follows.*

For $A, B \in E$

$$A * B = \begin{cases} APB & if\ A, B \in S, where\ P\ is\ the\ regular\ matrix\ over\ U(n) \\ APB & otherwise,\ where\ P\ is\ the\ identity\ matrix \end{cases}$$

4. Idempotents of a Rees Matrix Semigroup

In this section we study the idempotents of a Rees matrix semigroup and evaluate the Sandwich set of idempotents of a Rees matrix semigroup.

Lemma 4.1. *Let $e = (i, a, \lambda)$ and $f = (j, b, \mu)$ be two non zero idempotents in a Rees matrix semigroup, then*

(1). $e\,\omega^r\, f$ if and only if $j = i$ and $p_{\mu i} = b^{-1}$.

(2). $e\,\omega^l\, f$ if and only if $\lambda = \mu$ and $p_{\lambda j} = b^{-1}$.

Proof. Let $e = (i, a, \lambda)$ and $f = (j, b, \mu)$ be two idempotents in a Rees matrix semigroup. Assume $e\,\omega^r\, f$, that is $fe = e$. Then $(j, b, \mu)(i, a, \lambda) = (j, bp_{\mu i}a, \lambda)$. But by the definition $(j, bp_{\mu i}a, \lambda) = (i, a, \lambda)$, that is $j = i$ and $bp_{\mu i}a = a$. Since f is an idempotent, $p_{\mu j} \neq 0$ by Remark 2.2 and since $j = i$, $p_{\mu i} \neq 0$. Thus $bp_{\mu i} = e$ and hence $p_{\mu i} = b^{-1}$. Conversely if $j = i$ and $p_{\mu i} = b^{-1}$ then $e\,\omega^r\, f$ by the definition of the quasi order ω^r.

Similarly let $e\,\omega^l\, f$ that is $ef = e$. Then $(i, a, \lambda)(j, b, \mu) = (i, a, \lambda)$. So that $(i, ap_{\lambda j}b, \mu) = (i, a, \lambda)$. Hence $\lambda = \mu$ and $ap_{\lambda j}b = a$. Since f is an idempotent $p_{\mu j} \neq 0$. We have $\mu = \lambda$, therefore $p_{\lambda j} \neq 0$ and so $p_{\lambda j}b = e$. Hence $p_{\lambda j} = b^{-1}$. Conversely if $\lambda = \mu$ and $p_{\lambda j} = b^{-1}$ then $e\,\omega^l\, f$ by the definition of quasi order ω^l. □

Theorem 4.2. *Let S be a Rees matix semigroup $M^0(G, I, \Lambda, P)$ for a regular matrix P over G. Then for a fixed λ and j the sandwich set $S(e, f)$ of idempotents e, f where $e = (i, a, \lambda)$ and $f = (j, b, \mu)$ is given by $S(e, f) = \{(j, g, \lambda)\}$ where $g = p_{\lambda j}^{-1}$.*

Proof. Let $e = (i, a, \lambda)$ and $f = (j, b, \mu)$ be two idempotents in a Rees matrix semigroup. Here each entry in the regular matrix P is non-zero. Using Remark 2.2 corresponding to each entry in the regular matrix there exist only one idempotent in $E(S)$. From Lemma 3.2 we have $e\,\omega^l\, f$ if and only if $\lambda = \mu$ and $p_{\lambda j} = b^{-1}$. Hence $\omega^l(e)$ contains idempotents of the form (i, g, λ) for every $i \in I$ and $g = p_{\lambda i}^{-1}$. Hence in particular $(j, g, \lambda) \in \omega^l(e)$, where $g = p_{\lambda j}^{-1}$.

Similarly $\omega^r(f)$ contains idempotents is of the form (j, g, μ) where $\mu \in \Lambda$ and $g = p_{\mu j}^{-1}$. Therefore $(j, g, \lambda) \in \omega^r(f)$ when $\lambda = \mu$ and $g = p_{\lambda j}^{-1}$. Thus $\omega^l(e) \cap \omega^r(f)$ contains idempotents of the form(j, g, λ) where $g = p_{\lambda j}^{-1}$. Corresponding to $(\lambda, j)^{th}$ entry in P there exist only one idempotent in $\omega^l(e) \cap \omega^r(f)$. Then $\mathcal{M}(e, f)$ contains only the element (j, g, λ) and hence $S(e, f) = \{(j, g, \lambda)\}$ □

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