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Fixed Points of a New Type of Contractive mappings in $G-$Metric Spaces

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Abstract: In this paper, using a mapping $F:\mathbb{R}^+ \to \mathbb{R}$, a new notion of contraction in G-metric space is introduced and related fixed point theorem which generalizes various known results in $G-$metric spaces. The paper includes an example which shows the validity of our results.

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1. Introduction and Preliminaries

Throughout the paper denoted by $\mathbb{R}$ is the set of real numbers, by $\mathbb{R}^+$ is the set of all positive real numbers and by $\mathbb{N}$ is the set of natural numbers. The Banach contraction principle [2] is the simplest and one of the most versatile elementary results in fixed point theory. After that many mathematicians generalized the Banach contraction principle in different directions. Some of these works may be noted in [1, 3–9, 15]. Mustafa and Sims [10]introduced the concept of $G-$metric space in the year 2006 as a generalization of the metric space. In this spaces a non-negative real number is assigned to every triplet of elements. Several other studies relevant to metric spaces are being extended to $G-$metric spaces for examples [11–14]. Consistent with Mustafa and Sims [10], the following definitions and results will be needed in the sequel.

Definition 1.1 ([10]). *Let X be a non empty set, $G: X \times X \times X \to R^+$ be a function satisfying the following properties:*

(G_1) $G(x,y,z) = 0$ *if* $x=y=z$,

(G_2) $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $x \neq y$,

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \dots$ *(symmetry in all three variables),*

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*

Then the function G is called a generalized metric, or, more specially, a G- metric on X, and the pair (X,G) is called a G-metric space.

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Definition 1.2 ([10]). *Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore, we say that $\{x_n\}$ is G- convergent to $x \in X$ if $lim_{n,m\to+\infty} = G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists $n \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $lim_{n\to+\infty} x_n = x$.*

Lemma 1.3 ([10]). *Let (X,G) be a G- metric space. The following statements are equivalent:*

(1). $\{x_n\}$ is G- convergent to x,

(2). $G(x_n, x_n, x) \to 0$ as $n \to +\infty$,

(3). $G(x_n, x, x) \to 0$ as $n \to +\infty$,

(4). $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$,

Recently, Wardowski [16] introduced a new concept of F-contraction and proved a fixed point theorem which generalizes Banach contraction principle in a different direction than in the known results from the literature in complete metric spaces. In present article, an attempt has been made to introduce a new type of contraction called $F-$contraction in G-metric space and prove a some fixed point theorem concerning $F-$contraction. The article includes the examples of $F-$contractions and an example showing that the obtained extension is significant.

2. Main Results

Consider, together with Wardowski, the following properties for a mapping

Definition 2.1. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:*

$(F1)$ *F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;*

$(F2)$ *For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*

$(F3)$ *There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

We denote the set of all functions satisfying properties (F1)(F3), by $\mathcal{F}$.

Following the concept of F-contraction due to Wardowski [16], we define F-contractions in G-metric space as follows:

Definition 2.2. *A mapping $T : X \to X$ is said to be $F-$contraction if there exists $\tau > 0$ such that*

$$G(Tx, Ty, Tz) > 0 \Rightarrow \tau + F(G(Tx, Ty, Tz)) \leq F(G(x, y, z) \quad \forall x, y, z \in X \tag{1}$$

We obtain the variety of contraction, when we consider the different types of the mapping F in 1. Some of them are of a type known in the literature. See the following examples:

Example 2.3. *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies $F(1) - (F3)$, $(F3)$ for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfying 1 is an $F-$contraction such that*

$$G(Tx, Ty, Tz) \leq e^{-\tau} G(x, y, z), \quad \forall x, y, z \in X, \quad Tx \neq Ty \neq Tz. \tag{2}$$

It is clear that for $x, y, z \in X$ such that $Tx = Ty = Tz$ the inequality $G(Tx, Ty, Tz) \leq e^{-\tau} G(x, y, z)$ is also holds.

Example 2.4. *If $F(\alpha) = \ln \alpha + \alpha, \alpha > 0$ then F satisfies $(F1) - (F3)$ and the condition 1 is of the form*

$$\frac{G(Tx,Ty,Tz)}{G(x,y,z)} e^{G(Tx,Ty,Tz)-G(x,y,z)} \leq e^{-\tau}, \quad \forall x,y,z \in X, \quad Tx \neq Ty \neq Tz. \tag{3}$$

Example 2.5. *Consider $F(\alpha) = \frac{-1}{\sqrt{\alpha}}, \alpha > 0$. Then F satisfies $(F1) - (F3)$, $(F3)$ for any $k \in (1/2, 1)$. . In this case, each $F-$contraction T satisfies,*

$$G(Tx,Ty,Tz) \leq \frac{1}{(1+\tau\sqrt{G(x,y,z)})^2} G(x,y,z), \quad \forall x,y,z \in X, \quad Tx \neq Ty \neq Tz. \tag{4}$$

Here, we obtain a special case of nonlinear contraction of the type

$$G(Tx,Ty,Tz) \leq \alpha(G(x,y,z))G(x,y,z).$$

Example 2.6. *Let $F(\alpha) = \ln(\alpha^2 + \alpha), \alpha > 0$. Obviously F satisfies $(F1) - (F3)$ and for $F-$contraction T, the following condition holds;*

$$\frac{G(Tx,Ty,Tz)(G(Tx,Ty,Tz)+1)}{G(x,y,z)(G(x,y,z)+1)} \leq e^{\tau}, \quad \forall x,y,z \in X, \quad Tx \neq Ty \neq Tz. \tag{5}$$

Let us observe that in Example 2.3- 2.6 the contractive conditions are satisfied for $x, y, z \in X$ such that $Tx = Ty = Tz$.

Remark 2.7. *From $(F1)$ and 1 it is easy to conclude that every $F-$contraction T is contractive mapping, i.e.*

$$G(Tx,Ty,Tz) \leq G(x,y,z), \quad \forall x,y,z \in X, Tx \neq Ty \neq Tz.$$

Then every $F-$contraction is continuous mapping.

Remark 2.8. *Let F_1 and F_2 be the mappings satisfying $(F1) - (F3)$. If $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and a mapping $P = F_2 - F_1$ is nondecreasing then every F_1-contraction T is F_2-contraction.*

Indeed, from Remark 2.7 we have $P(G(Tx,Ty,Tz) \leq P(G(x,y,z))$ for all $x, y, z \in X$, $Tx \neq Ty \neq Tz$. Thus, for all $x, y, z \in X$, $Tx \neq Ty \neq Tz$ we obtain

$$\begin{aligned} \tau + F_2(G(Tx,Ty,Tz)) &= F_1(G(Tx,Ty,Tz)) + P(G(Tx,Ty,Tz)) \\ &\leq F_1(G(x,y,z)) + P(G(x,y,z)) = F_2(G(x,y,z)). \end{aligned}$$

Theorem 2.9. *Let (X, G) be a complete $G-$complete metric space and let $T : X \to X$ be an $F-$contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Proof. First, we shall show the existence of fixed point of T. Let $x_0 \in X$ be an arbitrary point of X. We define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X, x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$ then x_{n_0} is a fixed point of T, that is $Tx_{n_0} = x_{n_0}$. Therefore, assume that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$. Denote $\gamma_n = G(x_n, x_{n+1}, x_{n+2})$. Then $\gamma_n > 0$ for all $n \in \mathbb{N}$ and the following holds for every $n \in \mathbb{N}$ by successive applications of using (1) that

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq ... \leq F(\gamma_0) - n\tau. \tag{6}$$

From 6, we obtain $\lim_{n\to\infty} F(\gamma_0) = -\infty$ that together with $F \in \mathcal{F}$ gives

$$\lim_{n\to\infty} \gamma_n = 0. \tag{7}$$

From $F \in \mathcal{F}$ there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{8}$$

From 6, the following hold for all $n \in \mathbb{N}$,

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq \gamma_n^k (F(\gamma_0) - n\tau) - \gamma_n^k F(\gamma_0) = -\gamma_n^k n\tau \leq 0. \tag{9}$$

Letting $n \to \infty$ in 9, and using 7 and 8, we obtain

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{10}$$

It follows from 10 that there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. Consequently we have

$$\gamma_n \leq \frac{1}{n^{1/k}}, \tag{11}$$

for all $n \geq n_1$. Further we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For this consider $m, n \in \mathbb{N}$ with $m > n \geq n_1$. From the definition of the $G-$metric space and from 11 we get

$$G(x_m, x_n, x_n) \leq \gamma_{m-1} + \gamma_{m-2} + ... + \gamma_n < \Sigma_{i=n}^{\infty} \gamma_i \leq \Sigma_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

As $k \in (0,1)$, the series $\Sigma_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges and so $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. From the completeness of X there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Finally from the continuity of T, we have

$$G(Tx^*, x^*, x^*) = \lim_{n\to\infty} G(Tx_n, x_n, x_n) = \lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = 0,$$

which completes the proof. □

Note that for the mappings $F_1(\alpha) = \ln(\alpha), \alpha > 0$, $F_2(\alpha) = \ln(\alpha) + \alpha, \alpha > 0$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, by Remark 2.8, we obtain that every contraction 2 satisfies the contraction condition 3. On the other side in next example, we show that a mapping T which is not F_1- contraction, but still F_2-contraction. Consequently, Theorem 2.9 gives the family of contractions which is general are not equivalent.

Example 2.10. *Consider a sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows;*

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 1+2, \\ S_3 &= 1+2+3, \\ &\dots \\ S_n &= 1+2+3+...+n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}. \end{aligned}$$

Let $X = \{S_n : n \in \mathbb{N}\}$ *and* $G(x,y,z) = \mid x-y \mid + \mid y-z \mid + \mid z-x \mid, x,y,z \in X$. *Then* (X,G) *is complete metric space. Define mapping* $T : X \to X$ *by the formulae;* $F(S_n) = S_{n-1}, \quad for \quad n > 1$ *and* $TS_1 = S_1$. *First, let us consider the mapping* F_1 *defined in Example 2.3. The mapping* T *is not the* F_1-*contraction in this case. Indeed, we have*

$$\begin{aligned}\lim_{n\to\infty} \frac{G(T(S_n),T(S_{n-1}),T(S_1))}{G(S_n,S_{n-1},S_1)} &= \lim_{n\to\infty} \frac{G(S_{n-1},S_{n-2},1)}{G(S_n,S_{n-1},1)} \\ &= \lim_{n\to\infty} \frac{\mid S_{n-1}-S_{n-2} \mid + \mid S_{n-2}-1 \mid + \mid 1-S_{n-1} \mid}{\mid S_n - S_{n-1} \mid + \mid S_{n-1}-1 \mid + \mid 1-S_n \mid} \\ &= 1.\end{aligned}$$

On the other side taking F_2 *as in Example 2.4, we obtain that* T *is* F_2-*contraction with* $\tau = 1$. *To see this, let us consider the following calculations;*

$$T(S_m) \neq T(S_n) \Leftrightarrow ((m > 2 \wedge n = 1) \vee (m > n > 1)) \quad \forall m,n \in \mathbb{N}$$

For every $m \in \mathbb{N}$, $m > 2$ *we have*

$$\begin{aligned}\frac{G(T(S_m),T(S_m),T(S_1))}{G(S_m,S_m,S_1)} e^{G(T(S_m),T(S_m),T(S_1))-G(S_m,S_m,S_1)} &= \frac{S_{m-1}-1}{S_m-1} e^{S_{m-1}-S_m} \\ &= \frac{m^2-m-2}{m^2+m-2} e^{-m} < e^{-m} < e^{-1}\end{aligned}$$

For every $m,n \in \mathbb{N}$, $m > n > 1$ *the following is holds,*

$$\begin{aligned}\frac{G(T(S_m),T(S_n),T(S_n))}{G(S_m,S_n,S_n)} e^{G(T(S_m),T(S_n),T(S_n))-G(S_m,S_n,S_n)} &= \frac{S_{m-1}-S_{n-1}}{S_m-S_n} e^{S_n-S_{n-1}+S_{m-1}-S_m} \\ &= \frac{m+n-1}{m+n+1} e^{n-m} < e^{n-m} < e^{-1}.\end{aligned}$$

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