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Radio Multiplicative Number of Certain Classes of Transformation Graphs

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Abstract: The concept of radio labeling motivated by the channel assignment problem is generalised herein to include various other types of radio labelings. Let $\mathbb{M}$ be a subset of non-negative integers and $(\mathbb{M}, \star)$ be a monoid with the identity e. We define a radio $\star$-labeling of graph $G(V, E)$ as a mapping $f : V \to \mathbb{M}$ such that $|f(u) - f(v)| \star\ d(u, v) \geq diam(G) + 1 - e$, for all $u, v \in V$. The radio $\star$-number $rn^{\star}(f)$ of a radio $\star$-labeling f of G is the maximum label assigned to a vertex of G. The *radio $\star$-number* of G denoted by $rn^{\star}(G)$ is $min\{rn^{\star}(f)\}$ taken over all radio $\star$-labeling f of G. In this paper we completely determine $rn^{\times}(G)$ of some transformation graphs of path and cycle.

MSC: 05C12, 05C15, 05C78.

Keywords: Radio labeling, radio multiplicative labeling, radio multiplicative number.

1. Introduction

All the graphs considered here are finite, simple, nontrivial, connected, and undirected. Let $G(V, E)$ be a graph on n vertices. The distance between any two vertices u, $v \in V$ in the graph G, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path between u and v. The eccentricity of a vertex $v \in V$, denoted by $e(v)$, is defined as $e(v) = \max\{d_G(v, u) : u \in V\}$. The radius and diameter of G are defined respectively as $rad(G) = \min\{e(v) : v \in V\}$ and $diam(G) = \max\{e(v) : v \in V\}$. For the terms not defined here we refer to [4, 6]. The main purpose of this paper is to introduce Radio $\star$-number and compute a particular type this new invariant for some classes of transformation graphs.

Let $\mathbb{M}$ be a subset of the set of non-negative integers and $(\mathbb{M}, \star)$ be a monoid with the identity e. We define a radio $\star$-labeling of G as a mapping $f : V \to \mathbb{M}/_{\{0\}}$ such that $|f(u) - f(v)| \star\ d(u, v) \geq diam(G) + 1 - e$, for all $u, v \in V$. The radio $\star$-number of a radio $\star$-labeling f, denoted by $rn^{\star}(f)$, is the maximum label assigned to a vertex of G. The *radio $\star$-number* of G denoted by $rn^{\star}(G)$ is $min\{rn^{\star}(f)\}$ taken over all radio $\star$-labeling f of G for all possible monoids $\mathbb{M}$ under the binary operation $\star$. It is easy to see that when the binary operation $\star$ is the usual addition, then the radio $\star$-labeling coincides with the radio labeling introduced by W. K. Hale et.al in [7], and studied by others in [1–3, 8–10, 12–15, 18]. For the similar work we refer to [11] and for entire survey work on radio labeling we refer [5]. The term *radio graceful* was introduced by Sooryanarayana and Raghunath in [14]. A graph is said to be radio graceful if it has a radio labeling with radionumber of G, $rn(G) = |V|$. Sooryanarayana and Ramya in [16, 17], characterized radio graceful graphs interms of its order and diameter. Throughout this chapter, we consider the binary operation $\star$ as usual multiplication $\times$ and call radio $\times$-labeling as radio multiplicative

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labeling or in short rm-labeling. A graph G is said to be radio multiplicative graceful or in short rm-graceful if it has a radio multiplicative labeling with $rn^{\times}(G) = |V|$. It is easy to see that the span of an rm-labeling f of a graph G is minimum if the label starts with the integer 1, that is $1 \in \mathbb{M}$ and $f(v) = 1$ for some $v \in V$. Further for any rm-labeling f of G if two vertices u and v are adjacent in a graph G, then $|f(u) - f(v)| \geq diam(G)$, also $|f(u) - f(v)| = 1$ only if u and v are diametrically opposite vertices. As a consequence of this, we have the following theorems:

Theorem 1.1. *For any graph G, if $diam(G) = 1$, then $rn^{\times}(G) = rn^{+}(G) = rn(G)$.*

Proof. When $diam(G) = 1$, the graph is a complete graph. For the complete graph, $d(u, v) = 1$ for every pair $u, v \in V$, so $|f(u) - f(v)| + d(u, v) \geq diam(G) + 1 \Leftrightarrow |f(u) - f(v)| \geq diam(G) \Leftrightarrow |f(u) - f(v)| \times d(u, v) \geq diam(G)$. □

From the above theorem, for any positive integer n, we get $rn^{\times}(K_n) = n$.

Remark 1.2. *From the definition of an rm-labeling it is clear that every rm-labeling is one-one and hence for every rm-labeling f of a graph G it follows that $rn^{\times}(G) \geq |V|$.*

2. Transformation Graphs and Radio Multiplicative Number

For each triplet abc, where $a, b, c \in \{+, -\}$ and a graph G, the transformation graph of G, denoted by G^{abc}, is the graph on $V \cup E$ such that two vertices u, v are adjacent in G^{abc} if and only if one of the following hold:

(1) $a = +$ and, u and v are adjacent vertices in G.

(2) $b = +$ and, u and v are adjacent edges in G.

(3) $c = +$ and, u and v are incident pair in G.

(4) $a = -$ and, u and v are non adjacent vertices in G.

(5) $b = -$ and, u and v are non adjacent edges in G.

(6) $c = -$ and, u and v are non incident pair in G.

Observation 2.1. *For any positive integer n,*

(1) $diam(P_n^{+-+}) = \begin{cases} n-1, & if \;\; n \in \{2,3\} \\ 3, & if \;\; n \geq 4 \end{cases}$

(2) $diam(P_n^{-++}) = \begin{cases} 2, & if \;\; 2 \leq n \leq 4 \\ 3, & if \;\; n \geq 5 \end{cases}$

(3) $diam(P_n^{+--}) = \begin{cases} 4, & if \;\; n = 3 \\ 2, & if \;\; n \geq 4 \end{cases}$

(4) $diam(P_n^{++-}) = diam(P_n^{--+}) = 2$

(5) $diam(P_n^{-+-}) = diam(P_n^{---}) = \begin{cases} 3, & if \;\; n = 4 \\ 2, & if \;\; n \geq 5 \end{cases}$

(6) $diam(C_n^{+-+}) = diam(C_n^{-++}) = \begin{cases} 2, & if \;\; n = 3,4,5 \\ 3, & if \;\; n \geq 6 \end{cases}$

(7) $diam(C_n^{++-}) = diam(C_n^{---}) = 2$

(8) $diam(C_n^{+--}) = diam(C_n^{--+}) = diam(C_n^{-+-}) = \begin{cases} 3, & if \;\; n = 3 \\ 2, & if \;\; n \geq 4 \end{cases}$

In the next sections of this paper, we prove the following theorems which computes the actual minimum span of an *rm*-labeling for each transformation graphs of a path and a cycle. In all the following theorems, the stated are the only possible values of $n \in \mathbb{Z}^+$ and the graph is disconnected or trivial otherwise.

Theorem 2.2. $rn^{\times}(P_n^{+-+}) = \begin{cases} 3n-3, & if \;\; n \in \{2,3\} \\ 3n, & if \;\; n \in \{4,5\} \\ 3n-1, & if \;\; n = 6 \\ 3n-2, & if \;\; n \geq 7 \end{cases}$

Theorem 2.3. $rn^{\times}(P_n^{-++}) = \begin{cases} 4, & if \;\; n = 2 \\ 2n-1, & if \;\; n \in \{3,4\} \\ 3n+1, & if \;\; n \in \{5,6\} \\ 3n, & if \;\; n = 7 \\ 3n-1, & if \;\; n \geq 8 \end{cases}$

Theorem 2.4. $rn^{\times}(P_n^{++-}) = 2n-1$ *if* $n \geq 3$.

Theorem 2.5. $rn^{\times}(P_n^{+--}) = \begin{cases} 9, & if \;\; n = 3 \\ 2n-1, & if \;\; n \geq 4 \end{cases}$

Theorem 2.6. $rn^{\times}(P_n^{--+}) = \begin{cases} 4, & if \;\; n = 2 \\ 2n-1, & if \;\; n \geq 3 \end{cases}$

Theorem 2.7. $rn^{\times}(P_n^{-+-}) = \begin{cases} 12, & if \;\; n = 4 \\ 2n-1, & if \;\; n \geq 5 \end{cases}$

Theorem 2.8. $rn^{\times}(P_n^{---}) = \begin{cases} 12, & if \;\; n = 4 \\ 2n-1, & if \;\; n \geq 5 \end{cases}$

Theorem 2.9. $rn^{\times}(C_n^{+-+}) = rn^{\times}(C_n^{-++}) = \begin{cases} 7, & if \;\; n = 3 \\ 2n, & if \;\; n \in \{4,5\} \\ 3n+2, & if \;\; n = 6 \\ 3n+3, & if \;\; n = 7 \\ 3n, & if \;\; n \geq 8 \end{cases}$

Theorem 2.10. $rn^{\times}(C_n^{++-}) = 2n$ *if* $n \geq 3$.

Theorem 2.11. $rn^{\times}(C_n^{+--}) = rn^{\times}(C_n^{-+-}) = \begin{cases} 10, & if \;\; n = 3 \\ 2n, & if \;\; n \geq 4 \end{cases}$

Theorem 2.12. $rn^{\times}(C_n^{--+}) = \begin{cases} 8, & if \;\; n = 3 \\ 2n, & if \;\; n \geq 4 \end{cases}$

Theorem 2.13. $rn^{\times}(C_n^{---}) = 2n$ *if* $n \geq 4$.

The proof of these theorems follows by the results in next four sections. First two sections prove results of transformation graphs of path and next two sections prove results of transformation graphs of cycle.

Throughout first two sections, we label the vertices of P_n^{xyz} as $v_0, v_1, \ldots, v_i, \ldots, v_{n-1}$ for those in $V(P_n)$ in order and $e_0, e_1, e_2, \ldots, e_{n-2}$ for those are the edges of the corresponding path such that $e_j = v_j v_{j+1}$ for each j, $0 \leq j \leq n-2$.

3. Lower Bounds of P_n^{xyz}

In this section, let us first obtain a lower-bound for *spanf*, for any *rm*-labeling f. Throughout this section, let $\mathrm{G} = P_n^{xyz}$, f be an *rm*-labeling of G and $\omega_1, \omega_2, \ldots, \omega_{2n-1}$ be an rearrangement of the elements in $V(G)$ such that $f(\omega_i) < f(\omega_{i+1})$ for each i, $1 \leq i \leq 2n-2$ with $f(\omega_1) = 1$.

3.1. For $xyz = + - +$

Remark 3.1. *$d_{\mathrm{G}}(\alpha,\ \alpha') = 3$, for any $\alpha,\ \alpha' \in V(P_n^{+-+})$ if, and only if,*

(1) $\alpha,\ \alpha' \in V(P_n)$.

(2) $d_{P_n}(\alpha,\ \alpha') \geq 3$.

Lemma 3.2. *For any positive integer $n > 1$,*

$$rn^{\times}(P_n^{+-+}) \geq \begin{cases} 3n-3, & if \quad n \in \{2,3\} \\ 3n, & if \quad n \in \{4,5\} \\ 3n-1, & if \quad n = 6 \\ 3n-2, & if \quad n \geq 7 \end{cases}$$

Proof. Let $\mathrm{G} = P_n^{+-+}$. We now prove the lemma in different possibilities as follows.

Case 1: $n = 2$

In this case, $\mathrm{G} \cong K_3$ and hence $rn^{\times}(\mathrm{G}) = rn(K_3) = 3 = 3n-3$ (by Theorem 1.1).

Case 2: $n = 3$

In this case, the vertex v_1 of P_n is adjacent to every other vertex α in G and hence $|f(\alpha) - f(v_1)| \geq 2$ for all $\alpha \in \mathrm{G}$. Therefore, f should leave at least one integer before or after labeling v_1 and hence $rn^{\times}(\mathrm{G}) \geq |V(\mathrm{G})| + 1 = 6 = 3n-3$.

Case 3: $n = 4, 5, 6$

Let $S_L = \{v_i : 0 \leq i \leq n-4\}$ and $S_R = \{v_j : 3 \leq j \leq n-1\}$. Then $d_{\mathrm{G}}(x,\ y) \leq 2 < diam(\mathrm{G})$, for all $x,\ y \in S_L$ as well as in S_R and hence, by Remark 3.1, each diametric path of G contains one end in S_L and other in S_R. Thus, for the Case $n = 4$, we have $|S_L| = |S_R| = 1$ and hence $f(\omega_{i+1}) - f(\omega_i) > 1$ for every i except possible for at most one i. Hence $rn^{\times}(\mathrm{G}) = f(\omega_7) = \sum_{i=0}^{5}[f(\omega_{7-i}) - f(\omega_{7-i-1})] + f(\omega_1) \geq [5(2) + 1(1)] + 1 = 12 = 3n$.

For the case $n = 5$, if possible, let $f(\omega_i) = a, f(\omega_{i+1}) = a+1$, and $f(\omega_{i+2}) = a+2$ for some $i \in \mathbb{N}_{n-2}$. Then, by the way of construction of the sets S_L and S_R, the set $\{\omega_i, \omega_{i+2}\}$ is exactly one of the sets S_L and S_R, and ω_{i+1} is not in this set. Without loosing the generality, we take $\{\omega_i, \omega_{i+2}\} = S_L$. But then, $|f(\omega_i) - f(\omega_{i+2})| = 2$ and $d_{\mathrm{G}}(\omega_i,\ \omega_{i+2}) = 1$. Hence $|f(\omega_i) - f(\omega_{i+2})| \times d_{\mathrm{G}}(\omega_i, \omega_{i+2}) = 2 \ngeq 3 = diam(\mathrm{G})$, a contradiction. Thus, when $n = 5$, $f(\omega_{i+1}) - f(\omega_i) > 1$ for every i except possible for at most one i and one j with $|j - i| \geq 2$ (since $|S_L| = |S_R| = 2$). Hence $rn^{\times}(\mathrm{G}) = f(\omega_9) = \sum_{i=0}^{7}[f(\omega_{9-i}) - f(\omega_{9-i-1})] + f(\omega_1) \geq [6(2) + 2(1)] + 1 = 15 = 3n$.

We now consider the Case $n = 6$. Suppose that $f(\omega_{i+j}) = a + j$, for some $i \in \mathbb{N}_7$ and every $j \in \mathbb{N}_4$. Then, by Remark 3.1 and the construction of the sets S_L and S_R, the only possibility is $i = 1$ (or symmetrically $i + 4 = 6$), so $S_L = \{\omega_1, \omega_3, \omega_5\}$

(or symmetrically $S_R = \{\omega_2, \omega_4, \omega_6\}$). But then, $|f(\omega_1) - f(\omega_3)| \times d_G(\omega_1, \omega_3) \geq 3 \Rightarrow 2 \times d_G(\omega_1, \omega_3) \geq 3 \Rightarrow d_G(\omega_1, \omega_3) \geq 2$. Similarly, $d_G(\omega_3, \omega_5) \geq 2$, a contradiction, since S_L contains only one non-adjacent pair namely υ_0, υ_2. Thus, every *rm*-labeling f of G can assign at most 4 consecutive integers for four (3 pairs of) vertices (2 in S_L and 2 in S_R) of G and two consecutive integers for the remaining two vertices (1 pair) one each in S_L and S_R. Hence $rn^{\times}(G) = f(\omega_{11}) = \sum_{i=0}^{9}[f(\omega_{11-i}) - f(\omega_{11-i-1})] + f(\omega_1) \geq [6(2) + (3+1)(1)] + 1 = 17 = 3n - 1$.

Case 4: $n \geq 7$

To label these n vertices of G that corresponds to n vertices of P_n, f requires at least n integers. To label an edge of P_n in G after or before label a vertex or an edge of P_n in G, f should leave at least one integer (since $d_G(e_i, e_j) \leq 2$ and $d_G(e_i, \upsilon_j) \leq 2$, for all i, j and $diam(G) = 3 > 2$). Thus, all together f requires n integers for n elements in $V(P_n)$ in G and $2(n-1)$ integers for $(n-1)$ elements in $E(P_n)$ in G. Hence, span $f \geq n + 2(n-1) = 3n - 2 \Rightarrow rn^{\times}(G) = min\{rn^{\times}(f)\} \geq 3n - 2$. Hence the lemma. □

3.2. For $xyz = -++$

Remark 3.3. *$d_G(\alpha,\ \alpha') = 3$ for any $\alpha, \alpha' \in V(P_n^{-++})$ if, and only if,*

(1). $d_{L(P_n)}(\alpha,\ \alpha') \geq 3$, where $L(P_n)$ is the line graph of P_n.

(2). $\alpha,\ \alpha' \in E(P_n)$

Lemma 3.4. *For any positive integer $n > 1$,*

$$rn^{\times}(P_n^{-++}) \geq \begin{cases} 4, & if \quad n = 2 \\ 2n-1, & if \quad n \in \{3,4\} \\ 3n+1, & if \quad n \in \{5,6\} \\ 3n, & if \quad n = 7 \\ 3n-1, & if \quad n \geq 8 \end{cases}$$

Proof. Proof in various cases is as below.

Case 1: $2 \leq n \leq 4$

In this case, when $n = 2$, $e_0\alpha \in E(G)$ for every $\alpha \in V(G)$ and $\alpha \neq e_0$. Hence, f should leave atleast one integer before or after labeling e_0. Hence $rn^{\star}(G) \geq |V| + 1 = 4$. For $n = 3, 4$, $rn^{\times}(G) = rn(G) \geq |V(G)| = 2n - 1$ (by Remark 1.2).

Case 2: $n = 5, 6, 7$

Let $S_L = \{e_i : 0 \leq i \leq n-5\}$ and $S_R = \{e_j : 3 \leq j \leq n-2\}$. Then $d_G(e_i,\ e_j) \leq 2 < diam(G)$, whenever $e_i,\ e_j \in S_L$ as well as in S_R and hence, by Remark 3.3, each diametric path contains one end in S_L and other in S_R. Now, akin to the Case 3 of Lemma 3.2, with $n-1$ edges instead of n vertices of P_n, we see that for $n = 5$, $f(\omega_{i+1}) - f(\omega_i) > 1$ for every i except possible for at most one i. Hence $rn^{\times}(G) = f(\omega_9) = \sum_{i=0}^{7}[f(\omega_{9-i}) - f(\omega_{9-i-1})] + f(\omega_1) \geq [7(2) + 1(1)] + 1 = 16 = 3n + 1$. For the Case $n = 6$, $f(\omega_{i+1}) - f(\omega_i) > 1$ for every i except possible for at most one i and one j with $|j - i| \geq 3$. Hence $rn^{\times}(G) = f(\omega_{11}) = \sum_{i=0}^{9}[f(\omega_{11-i}) - f(\omega_{11-i-1})] + f(\omega_1) \geq [8(2) + 2(1)] + 1 = 19 = 3n + 1$.

Lastly when $n = 7$, every *rm-* labeling f of G can assign at most 4 consecutive integers for four (3 pairs of) vertices (2 in S_L and 2 in S_R) of G and two consecutive integers for the remaining two vertices (1 pair) one each in S_L and S_R. Hence $rn^{\times}(G) = f(\omega_{13}) = \sum_{i=0}^{11}[f(\omega_{13-i}) - f(\omega_{13-i-1})] + f(\omega_1) \geq [8(2) + (3+1)(1)] + 1 = 21 = 3n$.

Case 3: $n \geq 8$

Since there are $n-1$ vertices in G corresponding to $n-1$ edges of P_n, to label these $(n-1)$ edges of G, f requires atleast $(n-1)$ integers. To label an vertex of P_n in G after or before label an edge or a vertex of P_n in G, f should leave atleast one integer (since $d_{\mathrm{G}}(v_i, v_j) \leq 2$ and $d_{\mathrm{G}}(e_i, v_j) \leq 2$, for all i, j and $diam(\mathrm{G}) = 3 > 2$). Thus, all together f requires $(n-1)$ integers for $(n-1)$ elements of $E(P_n)$ in G and $2n$ integers for n elements of $V(P_n)$ in G. Hence, $spanf \geq (n-1)+2n = 3n-1$ $\Rightarrow rn^{\times}(\mathrm{G}) = min\{rn^{\times}(f)\} \geq 3n-1$. Hence the lemma. □

3.3. For $xyz = ++-$

Lemma 3.5. *For any positive integer $n > 2$, $rn^{\times}(P_n^{++-}) \geq 2n - 1$.*

Proof. Follows immediately by Remark 1.2. □

3.4. For $xyz = +--$

Lemma 3.6. *For any positive integer $n > 2$,*

$$rn^{\times}(P_n^{+--}) \geq \begin{cases} 9, & if \quad n = 3 \\ 2n-1, & if \quad n \geq 4 \end{cases}$$

Proof. We first consider the case $n = 3$, In this case $\mathrm{G} \cong P_5$, so G has exactly one diametric path and hence $f(\omega_{i+1}) - f(\omega_i) = 1$ possible only for at most one i. This yields, $rn^{\times}(f) = f(\omega_5) = \sum_{i=1}^{4}[f(\omega_{i+1}) - f(\omega_i)] \geq [1 + 2(3)] + 1 = 8$. We now show that the strict inequality holds. If possible, suppose $f(\omega_5) = 8$ for some f. Then certainly $\sum_{i=1}^{4}[f(\omega_{i+1}) - f(\omega_i)] \geq \sum_{i=1}^{4} \frac{diam(\mathrm{G})}{d_{\mathrm{G}}(\omega_i, \omega_{i+1})} \Rightarrow f(\omega_5) - f(\omega_1) \geq diam(\mathrm{G}) \left(\sum_{i=1}^{4} \frac{1}{d_{\mathrm{G}}(\omega_i, \omega_{i+1})} \right) \Rightarrow 8 - 1 \geq 4(\sum_{i=1}^{4} \frac{1}{d_{\mathrm{G}}(\omega_i, \omega_{i+1})}) \Rightarrow \sum_{i=1}^{4} \frac{1}{d_{\mathrm{G}}(\omega_i, \omega_{i+1})} \leq \frac{7}{4}$ and $f(\omega_{i+1}) - f(\omega_i) = 1$ for exactly one i. But then,

Case 1: $i = 1$.

In this case, ordering of the arrangement $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ is isomorphic to one of the four possibilities $(v_0,\ v_4,\ v_1,\ v_3,\ v_2)$ or $(v_0,\ v_4,\ v_1,\ v_2,\ v_3)$ or $(v_0,\ v_4,\ v_2,\ v_3,\ v_1)$ or $(v_0,\ v_4,\ v_3,\ v_1,\ v_2)$. In all the above possibilities it is easy to see that $\sum_{i=1}^{4} \frac{1}{d_{\mathrm{G}}(\omega_i, \omega_{i+1})} > \frac{7}{4}$, a contradiction.

Case 2: $i = 2$.

In this case, arrangement $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ is isomorphic to $(v_2,\ v_0,\ v_4,\ v_1,\ v_3)$ or $(v_2,\ v_0,\ v_4,\ v_3,\ v_1)$ or $(v_1,\ v_0,\ v_4,\ v_2,\ v_3)$ or $(v_1,\ v_0,\ v_4,\ v_3,\ v_2)$ or $(v_3,\ v_0,\ v_4,\ v_2,\ v_1)$ or $(v_3,\ v_0,\ v_4,\ v_1,\ v_2)$. Out of all these pattern except the first, we again get $\sum_{i=1}^{4} \frac{1}{d_{\mathrm{G}}(\omega_i, \omega_{i+1})} > \frac{7}{4}$, a contradiction and for the first $f(\omega_1) = 1$, $f(\omega_3) - f(\omega_2) = 1$, $f(\omega_4) - f(\omega_3) = 1$, $f(\omega_5) - f(\omega_4) = 1$ and, $f(\omega_5) = 8 \Rightarrow f(\omega_4) = 6$, $f(\omega_3) = 4$, $f(\omega_2) = 3$. So $f(\omega_4) - f(\omega_2) \times d_{\mathrm{G}}(\omega_4, \omega_2) = (6-3) \times d_{\mathrm{G}}(v_1, v_2) = (6-3) \times 1 = 3 < diam(\mathrm{G})$, a contradiction ($\because f$ is an rm-labeling).

Similarly the cases $i = 3, 4$ follow by symmetry. Thus we conclude $rn^{\times}(f) \geq 9$. Finally for $n \geq 4$, the result follows immediately by Remark 1.2. □

3.5. For $xyz = --+$

Lemma 3.7. *For any positive integer $n > 1$,*

$$rn^{\times}(P_n^{--+}) \geq \begin{cases} 4, & if \quad n = 2 \\ 2n-1, & if \quad n \geq 3 \end{cases}$$

Proof. For $n \geq 3$, result follows immediately by Remark 1.2. When $n = 2$, the vertex e_0 is adjacent to every $\alpha \in V(\mathrm{G})$ and hence $|f(\alpha) - f(e_0)| \geq 2$. Therefore, we should leave at least one integer before or after labeling e_0 then, $rn^{\times}(\mathrm{G}) \geq |V(\mathrm{G})| + 1 = 4$. □

3.6. For $xyz = -+-$

Lemma 3.8. *For any positive integer $n > 3$,*

$$rn^{\times}(P_n^{-+-}) \geq \begin{cases} 12, & if \quad n = 4 \\ 2n-1, & otherwise \end{cases}$$

Proof. For $n \geq 5$, result follows immediately by Remark 1.2. Now when $n = 4$, only v_1 and v_2 are diametrically opposite. So for at most one i, $d_{\mathrm{G}}(\omega_i, \omega_{i+1}) = diam(\mathrm{G})$, and hence $f(\omega_{i+1}) - f(\omega_i) \geq 2$ except for one i, $1 < i < 6$. So, $rn^{\times}(\mathrm{G}) = f(\omega_7) = \sum_{i=0}^{5}[f(\omega_{7-i}) - f(\omega_{7-i-1})] + f(\omega_1) \geq [5(2) + 1(1)] + 1 = 12$. Hence the lemma. □

3.7. For $xyz = ---$

Lemma 3.9. *For any positive integer $n > 3$,*

$$rn^{\times}(P_n^{---}) \geq \begin{cases} 12, & if \quad n = 4 \\ 2n-1, & otherwise \end{cases}$$

Proof. Similar to that of Lemma 3.8. □

4. Upper Bound and an Optimal rm-labeling of P_n^{xyz}

Here, we actually show the lower limit, established in the earlier sections for each of the transformation graphs $\mathrm{G} = P_n^{xyz}$, is tight by executing a minimal rm-labeling.

4.1. For $xyz = +-+$

Lemma 4.1. *For any positive integer $n > 1$,*

$$rn^{\times}(P_n^{+-+}) \leq \begin{cases} 3n-3, & if \quad n \in \{2,3\} \\ 3n, & if \quad n \in \{4,5\} \\ 3n-1, & if \quad n = 6 \\ 3n-2, & if \quad n \geq 7 \end{cases}$$

Proof. The result follows by the rm-labeling shown in Figure 1 for each $n \leq 7$. When $n \geq 8$, for $l, k \in \mathbb{Z}^+$; $0 \leq l \leq 2$,

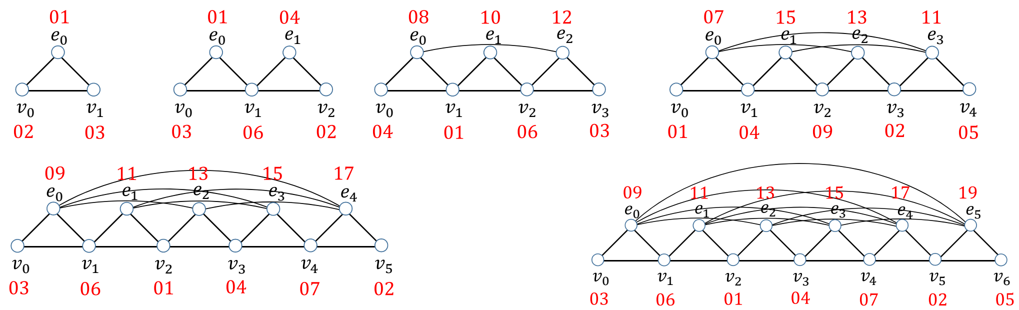

Figure 1. **A rm-labeling of P_n^{+-+} for $n = 2$ to 7.**

$0 \leq k \leq \lceil \frac{n}{3} \rceil - 1$, define $f : V \to Z^+$ as;

$$f(v_{3k+l}) = \begin{cases} k + \lceil \frac{n}{3} \rceil l & \text{if } l = 2 \text{ and } n \equiv 1 \pmod 3 \\ k + \lceil \frac{n}{3} \rceil l + 1 & \text{otherwise} \end{cases}$$

$$f(e_i) = (n+2) + 2i, \quad \text{for } i = 0, 1, \ldots, n-2.$$

Since $diam(\mathrm{G}) = 3$, to show f is an rm-labeling it is enough to take the elements in $V(P_n^{+-+})$ that are atmost distance two apart.

Let α and α' be any two vertices at a distance 2 apart in G.

Case 1: $\alpha \in V(P_n)$ and $\alpha' \in V(P_n)$.

Let $\alpha = v_i$ and $\alpha' = v_j$. where $i < j$ and $i, j \in \mathbb{Z}_n$.

Subcase 1: $i = 3k$, $j = 3k+1$

In this case, $f(\alpha) = f(v_{3k}) = k+1$ and $f(\alpha') = f(v_{3k+1}) = k + \lceil \frac{n}{3} \rceil + 1$.

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \lceil \frac{n}{3} \rceil \times 1 \geq 3.$

Subcase 2: $i = 3k+1$, $j = 3k+2$

In this case, $f(\alpha) = k + \lceil \frac{n}{3} \rceil + 1$ and $f(\alpha') = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$

$$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \begin{cases} \lceil \frac{n}{3} \rceil - 1 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil \geq 3 & \text{otherwise} \end{cases}$$

Subcase 3: $i = 3k+2$, $j = 3k+3 = 3(k+1)$

In this case, $f(\alpha) = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$ and $f(\alpha') = (k+1) + 1$.

$$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \begin{cases} 2\lceil \frac{n}{3} \rceil - 2 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ 2\lceil \frac{n}{3} \rceil - 1 \geq 3 & \text{otherwise} \end{cases}$$

Subcase 4: $i = 3k$, $j = 3k+2$

In this case, $f(\alpha) = (k+1)$ and $f(\alpha') = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$

$$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \begin{cases} 4\lceil \frac{n}{3} \rceil - 2 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ 4\lceil \frac{n}{3} \rceil \geq 3 & \text{otherwise} \end{cases}$$

Subcase 5: $i = 3k+1$, $j = 3k+3 = 3(k+1)$

In this case, $f(\alpha) = k + \lceil \frac{n}{3} \rceil + 1$; $f(\alpha') = (k+1) + 1$.

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = (\lceil \frac{n}{3} \rceil - 1) \times 2 \geq 3.$

Subcase 6: $i = 3k+2$, $j = 3k+4 = 3(k+1)+1$

In this case, $f(\alpha) = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$ and $f(\alpha') = (k+1) + \lceil \frac{n}{3} \rceil + 1$.

$$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \begin{cases} (\lceil \frac{n}{3} \rceil - 2) \times 2 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ (\lceil \frac{n}{3} \rceil - 1) \times 2 \geq 3 & \text{otherwise} \end{cases}$$

Case 2: $\alpha \in E(P_n)$ and $\alpha^{'} \in E(P_n)$

Let $\alpha = e_i$ and $\alpha^{'} = e_j$ with $i < j$ and $i, j \in \mathbb{Z}_{n-1}$. Then $|f(\alpha) - f(\alpha^{'})| = |f(e_i) - f(e_j)| = 2(j-i)$ is equal to 2 if $d_G(\alpha,\ \alpha^{'}) = 2$ (i.e $j = i+1$) and, at least 4 if $d_G(\alpha,\ \alpha^{'}) = 1$ (i.e $j > i+1$). In each of these cases $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) \geq 4 > diam(G)$.

Case 3: $\alpha \in V(P_n)$ and $\alpha^{'} \in E(P_n)$

Let $\alpha = \upsilon_{3k+l}$ with $0 \leq l \leq 2$, $0 \leq k \leq \lceil \frac{n}{3} \rceil - 1$ and $\alpha^{'} = e_j$ for $i, j \in \mathrm{Z}_{n-1}, i \neq j$.

Subcase 1: $j = (3k+l) - 1$.

In this case, $d_G(\alpha,\ \alpha^{'}) = 1$, Therefore

$$\begin{aligned} &|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) = f(e_{(3k+l)-1}) - f(\upsilon_{3k+l}) \\ = &\begin{cases} (n+2) + (6k+2l-2) - k - \lceil \frac{n}{3} \rceil \times l & \text{if } n \equiv 1 \pmod 3 \text{ and } l = 2 \\ (n+2) + (6k+2l-2) - k - \lceil \frac{n}{3} \rceil \times l - 1 & \text{otherwise} \end{cases} \\ \geq &\ \frac{n}{3} + 5k + 2 \geq \frac{8}{3} + 2 > 3, \text{ for all } l, 0 \leq l \leq 2 \end{aligned}$$

Subcase 2: $j = (3k+l)$

In this case, $d_G(\alpha,\ \alpha^{'}) = 1$ and $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) = f(e_{3k+l}) - f(\upsilon_{3k+l}) = (n+2) + 2(3k+l) - f(\upsilon_{3k+l}) = (n+2) + (6k+2l-2) - f(\upsilon_{3k+l}) + 2 = f(e_{(3k+l)-1}) - f(\upsilon_{3k+l}) + 2 > 3 + 2 > 3$ (by Subcase1 of Case 3)

Subcase 3: $j \notin \{(3k+l) - 1, (3k+l)\}$.

In this case, $d_G(\alpha,\ \alpha^{'}) = 2$ and hence it suffices to show $|f(\alpha) - f(\alpha^{'})| > 1$. In fact $|f(\alpha) - f(\alpha^{'})| = f(\alpha^{'}) - f(\alpha) = (n+2) + 2j - f(\alpha) \geq n + 2 + 2j - n = 2(j+1) > 1$

Thus, from all the above subcases, f is an rm-labeling and $span f = 3n-2$. Therefore $rn^{\times}(P_n^{+-+}) \leq 3n-2$ for all $n \geq 8$. $\square$

4.2. For $xyz = -++$

Lemma 4.2. *For any positive integer $n > 1$,*

$$rn^{\times}(P_n^{-++}) \leq \begin{cases} 4, & if \quad n = 2 \\ 2n-1, & if \quad n \in \{3,4\} \\ 3n+1, & if \quad n \in \{5,6\} \\ 3n, & if \quad n = 7 \\ 3n-1, & if \quad n \geq 8 \end{cases}$$

Proof. Let $G = P_n^{-++}$. For $n \leq 8$, result follows by the rm-labeling showed in Figure 2.

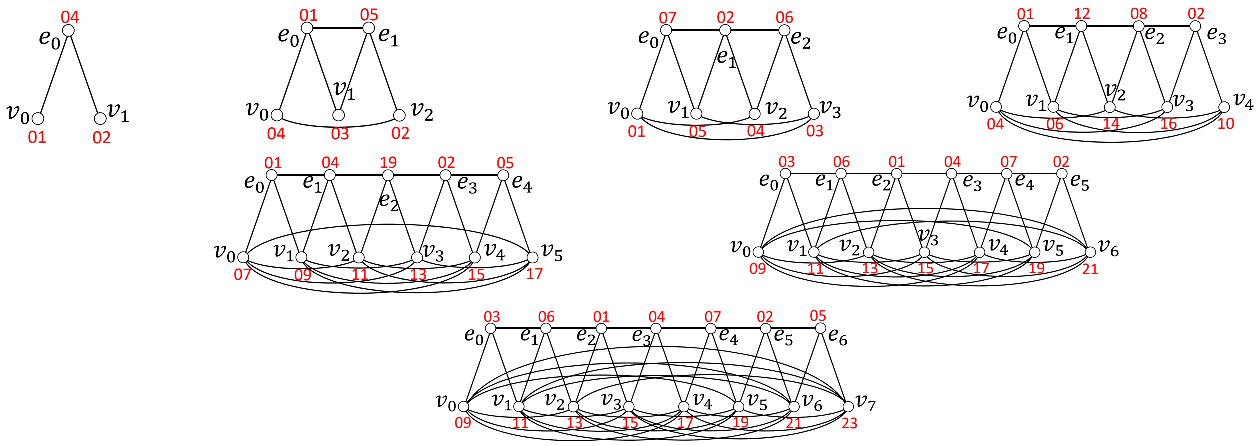

Figure 2. **A rm-labeling of P_n^{-++} with $n \le 8$.**

When $n \ge 9$, for each integer l, k; $0 \le l \le 2$, $0 \le k \le \lceil \frac{n-1}{3} \rceil - 1$, define a function $f : V \to Z^+$ by

$$f(e_{3k+l}) = \begin{cases} k + \lceil \frac{n-1}{3} \rceil l & \text{if } n \equiv 1 \pmod 3 \text{ and } l = 2 \\ k + \lceil \frac{n-1}{3} \rceil l + 1 & \text{otherwise} \end{cases}$$

$$f(v_i) = (n+1) + 2i, \text{with } \mathbb{Z}_n.$$

Since $diam(\mathrm{G}) = 3$, now to show f is an rm-labeling it is enough take the vertices α and α' that are at most distance two apart.

Case 1: $\alpha \in V(P_n)$ and $\alpha' \in V(P_n)$

Let $\alpha = v_i$ and $\alpha' = v_j$ with $i, j \in \mathbb{Z}_n, i < j$. Then $|f(\alpha) - f(\alpha')| = |f(v_i) - f(v_j)| = 2(j-i)$ is equal to 2 if $d_{\mathrm{G}}(\alpha,\ \alpha') = 2$ (i.e $j = i+1$) and, at least 4 if $d_{\mathrm{G}}(\alpha,\ \alpha') = 1$ (i.e $j > i+1$). In each of these cases $|f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') \ge 4 > diam(\mathrm{G})$.

Case 2: $\alpha \in E(P_n)$ and $\alpha' \in E(P_n)$.

Let $\alpha = e_i$ and $\alpha' = e_j$, $i < j$ and $i, j \in \mathbb{Z}_{n-1}$.

Subcase 1: $i = 3k$, $j = 3k+1$.

In this case, $f(\alpha) = f(v_{3k}) = k+1$ and $f(\alpha') = f(v_{3k+1}) = k + \lceil \frac{n-1}{3} \rceil + 1$.

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \lceil \frac{n-1}{3} \rceil \times 1 \ge 3$.

Subcase 2: $i = 3k+1$, $j = 3k+2$.

In this case, $f(\alpha) = k + \lceil \frac{n-1}{3} \rceil + 1$ and $f(\alpha') = \begin{cases} 2\lceil \frac{n-1}{3} \rceil + k & \text{if } n \equiv 2 \pmod 3 \\ 2\lceil \frac{n-1}{3} \rceil + k + 1 & \text{otherwise} \end{cases}$

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \begin{cases} \lceil \frac{n-1}{3} \rceil - 1 \ge 3 & \text{if } n \equiv 2 \pmod 3 \\ \lceil \frac{n-1}{3} \rceil \ge 3 & \text{otherwise} \end{cases}$

Subcase 3: $i = 3k+2$, $j = 3k+3 = 3(k+1)$

In this case, $f(\alpha) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil + k & \text{if } n \equiv 2 \pmod 3 \\ 2\lceil \frac{n-1}{3} \rceil + k + 1 & \text{otherwise} \end{cases}$ and $f(\alpha') = (k+1) + 1$.

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha,\ \alpha') = \begin{cases} 2\lceil \frac{n-1}{3} \rceil - 2 \ge 3 & \text{if } n \equiv 2 \pmod 3 \\ 2\lceil \frac{n-1}{3} \rceil - 1 \ge 3 & \text{otherwise} \end{cases}$

Subcase 4: $i = 3k$, $j = 3k + 2$

In this case, $f(\alpha) = k + 1$ and $f(\alpha^{'}) = \begin{cases} 2\lceil\frac{n-1}{3}\rceil + k & \text{if } n \equiv 2 \pmod 3 \\ 2\lceil\frac{n-1}{3}\rceil + k + 1 & \text{otherwise} \end{cases}$

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} 4\lceil\frac{n-1}{3}\rceil - 2 \geq 3 & \text{if } n \equiv 2 \pmod 3 \\ 4\lceil\frac{n-1}{3}\rceil \geq 3 & \text{otherwise} \end{cases}$

Subcase 5: $i = 3k + 1$, $j = 3k + 3$.

In this case, $f(\alpha) = k + \lceil\frac{n-1}{3}\rceil + 1$ and $f(\alpha^{'}) = (k + 1) + 1$.

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = (\lceil\frac{n-1}{3}\rceil - 1) \times 2 \geq 3.$

Subcase 6: $i = 3k + 2$, $j = 3k + 4$.

In this case, $f(\alpha) = \begin{cases} 2\lceil\frac{n-1}{3}\rceil + k & \text{if } n \equiv 2 \pmod 3 \\ 2\lceil\frac{n-1}{3}\rceil + k + 1 & \text{otherwise} \end{cases}$

and $f(\alpha^{'}) = (k + 1) + \lceil\frac{n-1}{3}\rceil + 1$.

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} 2\lceil\frac{n-1}{3}\rceil - 4 \geq 3 & \text{if } n \equiv 2 \pmod 3 \\ 2\lceil\frac{n-1}{3}\rceil - 2 \geq 3 & \text{otherwise} \end{cases}$

Case 3: $\alpha \in V(P_n)$ and $\alpha^{'} \in E(P_n)$

Let $\alpha = v_i$ with $j > i$ and $i, j \in \mathbb{Z}_n$ and $\alpha^{'} = e_{3k+l}$ with $0 \leq l \leq 2$, $0 \leq k \leq \lceil\frac{n-1}{3}\rceil - 1$.

Subcase 1: $j = (3k + l)$

In this case, $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 1$, Therefore

$$\begin{aligned} & |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) \\ = \ & f(v_{3k+l}) - f(e_{3k+l}) \\ = \ & \begin{cases} (n+1) + (6k+2) - k - \lceil\frac{n-1}{3}\rceil \times l & \text{if } l = 2 \text{ and } n \equiv 2 \pmod 3 \\ (n+1) + (6k+2l) - k - \lceil\frac{n-1}{3}\rceil \times l - 1 & \text{otherwise} \end{cases} \\ \geq \ & \frac{n}{3} + 5k \geq \frac{9}{3} \geq 3, \text{for all } l, 0 \leq l \leq 2. \end{aligned}$$

Subcase 2: $j = (3k + l + 1)$

In this case, $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 1$. Therefore, $|f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = f(v_{3k+l}) - f(e_{(3k+l)+1}) = (n+1) + (6k + 2l + 2) - f(e_{3k+l}) = (n+1) + 2(3k+l) - f(e_{3k+l}) + 2 = f(v_{3k+l}) - f(e_{3k+l}) + 2 \geq 3 + 2 > 3$ (by Subcase 1 of Case 3).

Subcase 3: $j \notin \{(3k + l), (3k + l) + 1\}$

In this case, $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 2$ and hence it required to show $|f(\alpha) - f(\alpha^{'})| > 1$. In fact $|f(\alpha) - f(\alpha^{'})| = f(\alpha) - f(\alpha^{'}) = (n + 1) + 2i - f(\alpha) \geq (n + 1) + 2i - (n - 1) = 2(i + 1) > 1$.

Thus, from all the above subcases, f is an rm-labeling and $span f = 3n - 1$.

Thus, $rn^{\times}(\mathrm{G})$ is at the most $3n - 1$ for every $n \geq 9$. □

4.3. For $xyz = + + -$

Lemma 4.3. *For any positive integer $n > 2$, $rn^{\times}(P_n^{++-}) \leq 2n - 1$.*

Proof. Let $f : V \to Z^{+}$ defined by $f(v_i) = 2i + 1$ for $i \in \mathbb{Z}_n$, and $f(e_i) = 2(i+1)$ for $\mathbb{Z}_{n-1}$. Since $diam(\mathrm{G}) = 2$, now to show f is an rm-labeling it is enough to consider the vertices that are adjacent. Let $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 1$. Then

Case 1: $\alpha \in V(P_n)$ and $\alpha' \in V(P_n)$.

Let $\alpha = v_i$ and $\alpha' = v_j$ with $i, j \in \mathbb{Z}_n, i < j$. Then $|f(\alpha) - f(\alpha')| = |f(v_i) - f(v_j)| = 2(j-i)$ is at least 2 if $d_G(\alpha, \alpha') = 1$. In this case $|f(\alpha) - f(\alpha')| \times d_G(\alpha, \alpha') \geq 2$.

Case 2: $\alpha \in E(P_n)$, $\alpha' \in E(P_n)$.

Let $\alpha = e_i$ and $\alpha' = e_j$ with $i, j \in \mathbb{Z}_{n-1}, i < j$. Then $|f(\alpha) - f(\alpha')| = |f(e_i) - f(e_j)| = 2(j-i)$ is at least 2 if $d_G(\alpha, \alpha') = 1$. In this case $|f(\alpha) - f(\alpha')| \times d_G(\alpha, \alpha') \geq 2$.

Case 3: $\alpha \in V(P_n)$, $\alpha' \in E(P_n)$.

Let $\alpha = v_i$ with $\mathbb{Z}_n$ and $\alpha' = e_j$ with $0 \leq j \leq n-2$. Then $|f(\alpha) - f(\alpha')| = |f(v_i) - f(e_j)| = |2(i-j) - 1|$ is at least 3 if $d_G(\alpha, \alpha') = 1$ (i.e $|i-j| \geq 2$). In this case $|f(\alpha) - f(\alpha')| \times d_G(\alpha, \alpha') \geq 3$.

Thus, from all the above cases, f is an rm-labeling and $span f = 2n - 1$.

Therefore $rn^{\times}(G) \leq 2n - 1$. □

4.4. For $xyz = + - -$

Lemma 4.4. *For any positive integer $n > 2$,*

$$rn^{\times}(P_n^{+--}) \leq \begin{cases} 9, & if \quad n = 3 \\ 2n - 1, & if \quad n \geq 4 \end{cases}$$

Proof. For $n = 3$, result follows by the rm-labeling shown in Figure 3. For $n \geq 4$, proof is same as in Lemma 4.3. □

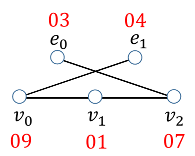

Figure 3. **A rm-labeling of P_3^{+--}**

4.5. For $xyz = - - +$

Lemma 4.5. *For any positive integer $n > 1$,*

$$rn^{\times}(P_n^{--+}) \leq \begin{cases} 4, & if \quad n = 2 \\ 2n - 1, & otherwise \end{cases}$$

Proof. For $n = 2$, result follows by the rm-labeling shown in Figure 4.

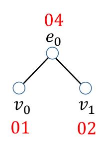

Figure 4. **A rm-labeling of P_2^{--+}**

For $n \geq 3$, define a function $f : V \to Z^+$ by $f(v_i) = i + 1$ for each $i \in \mathbb{Z}_n$.; $f(e_i) = (n+1) + i$ for each $i \in \mathbb{Z}_{n-1}$. Since $diam(G) = 2$, now to show f is an rm-labeling it is enough to check the cases when $d_G(\alpha, \alpha') = 1$.

Case 1: $\alpha \in V(P_n)$, $\alpha^{'} \in V(P_n)$.

Let $\alpha = v_i$ and $\alpha^{'} = v_j$ with $i, j \in \mathbb{Z}_n, i < j$. Then $|f(\alpha) - f(\alpha^{'})| = |f(v_i) - f(v_j)| = (j - i)$ is at least 2 if $d_G(\alpha, \ \alpha^{'}) = 1$ (i.e $j \geq i + 2$). In this case $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha, \ \alpha^{'}) \geq 2$.

Case 2: $\alpha \in E(P_n)$, $\alpha^{'} \in E(P_n)$

Let $\alpha = e_i$ and $\alpha^{'} = e_j$ with $i, j \in \mathbb{Z}_{n-1}, i < j$. Then $|f(\alpha) - f(\alpha^{'})| = |f(e_i) - f(e_j)| = (j - i)$ is at least 2 if $d_G(\alpha, \ \alpha^{'}) = 1$ (i.e $j \geq i + 2$). In this case $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha, \ \alpha^{'}) \geq 2$.

Case 3: $\alpha \in V(P_n)$ and $\alpha^{'} \in E(P_n)$

Let $\alpha = v_i$ for some $i \in \mathbb{Z}_n$ and $\alpha^{'} = e_j$ for some $j \in \mathbb{Z}_{n-1}$.

Subcase 1: $i = j$.

In this case $|f(\alpha) - f(\alpha^{'})| = |f(v_i) - f(e_j)| = |i - j - n|$ is at least 3 (since $n \geq 3$).

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_G(\alpha, \ \alpha^{'}) \geq 3$.

Subcase 2: $i = j + 1$.

In this case $|f(\alpha) - f(\alpha^{'})| = |f(v_i) - f(e_j)| = |i - j - n|$ is at least 2 (since $n \geq 3$).

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_G(\alpha, \ \alpha^{'}) \geq 2$.

Thus, from all the above cases, f is an rm-labeling and $span f = 2n - 1$.

Therefore $rn^{\times}(G) \leq 2n - 1$, $\forall \ n \geq 3$. □

4.6. For $xyz = - + -$

Lemma 4.6. *For any positive integer* $n > 3$,

$$rn^{\times}(P_n^{-+-}) \leq \begin{cases} 12, & if \quad n = 4 \\ 2n - 1, & otherwise \end{cases}$$

Proof. For $n = 4$, result follows by the rm-labeling shown in Figure 5 and for $n \geq 5$, proof is similar to Lemma 4.3. □

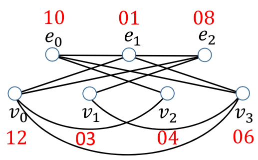

Figure 5. **A rm-labeling of P_4^{-+-}**

4.7. For $xyz = - - -$

Lemma 4.7. *For any positive integer* $n > 3$,

$$rn^{\times}(P_n^{---}) \leq \begin{cases} 12, & if \quad n = 4 \\ 2n - 1, & otherwise \end{cases}$$

Proof. For $n = 4$, result follows by the rm-labeling shown in Figure 6 and for $n \geq 5$, proof is same as Lemma 4.3. □

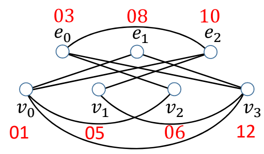

Figure 6. **A rm-labeling of P_4^{---}**

5. Lower Bounds of C_n^{xyz}

Here, we find a lower-bound for the span of its rm-labeling. Throughout this section, let $\mathrm{G} = C_n^{xyz}$, f be an rm-labeling of the graph G and $\omega_1, \omega_2, \ldots, \omega_{2n}$ be the rearrangement of the elements of $V(\mathrm{G})$ such that $f(\omega_1) = 1$, $f(\omega_i) < f(\omega_{i+1})$ for all $i \in \mathbb{Z}_{2n}$. We label the elements in $V(C_n^{xyz})$ that are in $V(C_n)$ as $v_0, v_1, v_2, \ldots, v_{n-1}$; those are in $E(C_n)$ as $e_0, e_1, e_2, \ldots, e_{n-1}$, with $e_j = v_j v_{j+1(mod n)}$ for each $j \in \mathbb{Z}_n$.

5.1. For $xyz = +-+$ or $xyz = -++$

Now let $xyz = +-+$. The graph $C_n^{+-+} \cong C_n^{-++}$ and hence the proof follows immediately for the Case $xyz = -++$.

Lemma 5.1. *For any positive integer $n > 2$,*

$$rn^{\times}(C_n^{+-+}) \geq \begin{cases} 7, & if \quad n = 3 \\ 2n, & if \quad n \in \{4, 5\} \\ 3n+2, & if \quad n = 6 \\ 3n+3, & if \quad n = 7 \\ 3n, & if \quad n \geq 8 \end{cases}$$

Proof. We prove in case-wise as below.

Case 1: $n = 3$.

In this case for at least one i with $0 \leq i \leq 5$, $d_{\mathrm{G}}(\omega_i, \omega_{i+1}) < diam(\mathrm{G})$ (because, any vertex with degree four is adjacent all other vertices except one and it is the only option to be diametrically opposite, so that to label with consecutive integers we have to keep them as starting or ending vertex. But In this case we have three vertices with degree four so at least one of them should be labeled in middle)and hence $f(\omega_{i+1}) - f(\omega_i) \geq 2$ at least for one i then, $rn^{\times}(\mathrm{G}) \geq |V(\mathrm{G})| + 1 = 6 + 1 = 7$

Case 2: $n = 4, 5$.

Follows immediately by Remark 1.2.

Case 3: $n = 6, 7$.

If possible, let f assigns three consecutive integers for the vertices v_i, v_j and v_k of G that corresponds to any three vertices of C_n. But then $d_{\mathrm{G}}(v_i, v_k) \geq 2$ (since $|f(v_i) - f(v_k)| = 2$). But, In this case for any such f we get $d_{\mathrm{G}}(v_i, v_k) \leq 1$ (since $diam(\mathrm{G}) = diam(C_n)$), a contradiction. Hence if $n = 6$ or $n = 7$, $f(\omega_{i+1}) - f(\omega_i) \geq 2$ for every i except possible for at most three i. Then $f(\omega_{2n}) = \sum_{i=1}^{2n-1} [f(\omega_{i+1}) - f(\omega_i)] + f(\omega_1) \geq 1 \times l(\text{pairs}) + 2 \times (2n - 1 - l)(\text{pairs}) + f(\omega_1)$, where $l = 3$ if $n = 6, 7$. This yields with $f(\omega_1) = 1$ that;

$$rn^{\times}(C_n^{xyz}) = min\{f(\omega_{2n})\} \geq \begin{cases} 1(3) + 2(8) + 1 = 20 = 3n+2, & if \quad n = 6 \\ 1(3) + 2(10) + 1 = 24 = 3n+3, & if \quad n = 7 \end{cases}$$

Case 4: $n \geq 8$.

To label these n vertices of G that corresponds to n vertices of C_n, f requires at least n integers. To label an edge of C_n in G after or before label a vertex or an edge of C_n in G, f should leave at least one integer (since $d_{\mathrm{G}}(e_i, e_j) \leq 2$ and $d_{\mathrm{G}}(e_i, \upsilon_j) \leq 2$, for all i, j and $diam(\mathrm{G}) = 3 > 2$). Thus, all together f requires n integers for n vertices of C_n in G and $2n$ integers for n edges of C_n in G. Hence, span $f \geq n + 2n = 3n \Rightarrow rn^{\times}(\mathrm{G}) = min\{rn^{\times}(f)\} \geq 3n$.

Hence the lemma. □

5.2. For $xyz = ++-$

Lemma 5.2. *For any positive integer $n > 2$, $rn^{\times}(C_n^{++-}) \geq 2n$.*

Proof. Result follows immediately by Remark 1.2. □

5.3. For $xyz = +--$ or $xyz = -+-$

Now let $xyz = +--$. As $C_n^{+--} \cong C_n^{-+-}$ the proof follows immeadiately for $xyz = -+-$.

Lemma 5.3. *For any positive integer $n > 2$,*

$$rn^{\times}(C_n^{+--}) \geq \begin{cases} 10, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Proof. When $n = 3$, let us relabel the elements in $V(\mathrm{G})$ as $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ so that $f(\omega_i) < f(\omega_{i+1})$ for each $i, 1 \leq i \leq 5$. Now for at most two i, $d_{\mathrm{G}}(\omega_i,\ \omega_{i+1}) = diam(\mathrm{G})$ (since only three vertices e_0, e_1, e_2 are with eccentricity equal to diameter), and hence $|f(\omega_{i+1}) - f(\omega_i)| \geq 2$ except for two i, $i \leq 5$ also for at least one i, $i \leq 5$, $d_{\mathrm{G}}(\omega_i, \omega_{i+1}) = 1$ (since here three vertices of C_n are mutually adjacent and to avoid successive labeling of these an edge must be labeled in between, But to give three consecutive numbers, edges must be labeled continuously so one pair vertices of C_n labeled successively), so $|f(\omega_{i+1}) - f(\omega_i)| \geq 3$. Or to choose one pair with $d_{\mathrm{G}}(\omega_i, \omega_{i+1}) = 3$, (so $|f(\omega_{i+1}) - f(\omega_i)| \geq 1$) and other all pairs with $d_{\mathrm{G}}(\omega_i, \omega_{i+1}) = 2$, (so $|f(\omega_{i+1}) - f(\omega_i)| \geq 2$). This yields with $f(\omega_1) = 1$ that;

$$\begin{aligned} rn^{\times}(C_n^{xyz}) = min\{f(\omega_6)\} &= \sum_{i=1}^{5}[f(\omega_{i+1}) - f(\omega_i)] + f(\omega_1) \\ &\geq \begin{cases} 2(1) + 2(2) + 1(3) + 1 = 10 \\ 1(1) + 4(2) + 1 = 10 \end{cases} \end{aligned}$$

In the second Case, when $n \geq 4$, the result follows immediately by Remark 1.2. □

5.4. For $xyz = --+$

Lemma 5.4. *For any positive integer $n > 2$,*

$$rn^{\times}(C_n^{--+}) \geq \begin{cases} 8, & if \quad n = 3 \\ 2n, & \quad otherwise \end{cases}$$

Proof. For $n = 3$, if possible, let f assigns three consecutive integers for the vertices υ_i, υ_j and υ_k of G. But then $d_{\mathrm{G}}(\upsilon_i, \upsilon_k) \geq 2$ (since $|f(\upsilon_i) - f(\upsilon_k)| = 2$). In this case for any such f we get $d_{\mathrm{G}}(\upsilon_i, \upsilon_k) \leq 1$, a contradiction. Hence, there are exactly 3 pairs of vertices can be assigned by consecutive integers. Let us relabel the elements in $V(G)$ as $\omega_1, \omega_2, \omega_3, \ldots, \omega_6$ so that $f(\omega_i) < f(\omega_{i+1})$, $\forall i \leq 5$. Then $|f(\omega_{i+1}) - f(\omega_i)| \geq 2$ except for three i, $i \leq 5$. This yields with $f(\omega_1) = 1$ that; $rn^{\times}(C_n^{--+}) = min\{f(\omega_6)\} = \sum_{i=1}^{5}[f(\omega_{i+1}) - f(\omega_i)] + f(\omega_1) \geq 3 \times 1 + 2 \times 2 + 1 = 8$. The case $n \geq 2$ follows immediately by Remark 1.2. $\square$

5.5. For $xyz = ---$

Lemma 5.5. *For any positive integer $n > 3$, $rn^{\times}(C_n^{---}) \geq 2n$.*

Proof. Result follows immediately by Remark 1.2. $\square$

6. Upper Bound and an Optimal rm-labeling of C_n^{xyz}.

Here, we actually show the lower limit, established in the previously, for each of the transformation graphs $G = C_n^{xyz}$, is tight by executing a minimal rm-labeling.

6.1. For $xyz = +-+$ or $xyz = -++$

Now let $xyz = +-+$. As $C_n^{+-+} \cong C_n^{-++}$ proof follows immeadiately for $xyz = -++$.

Lemma 6.1. *For any positive integer $n > 2$,*

$$rn^{\times}(C_n^{+-+}) \leq \begin{cases} 7, & if \quad n = 3 \\ 2n, & if \quad n \in \{4, 5\} \\ 3n + 2, & if \quad n = 6 \\ 3n + 3, & if \quad n = 7 \\ 3n, & if \quad n \geq 8 \end{cases}$$

Proof. For $n \leq 7$, result follows by the rm-labeling shown in Figure 7.

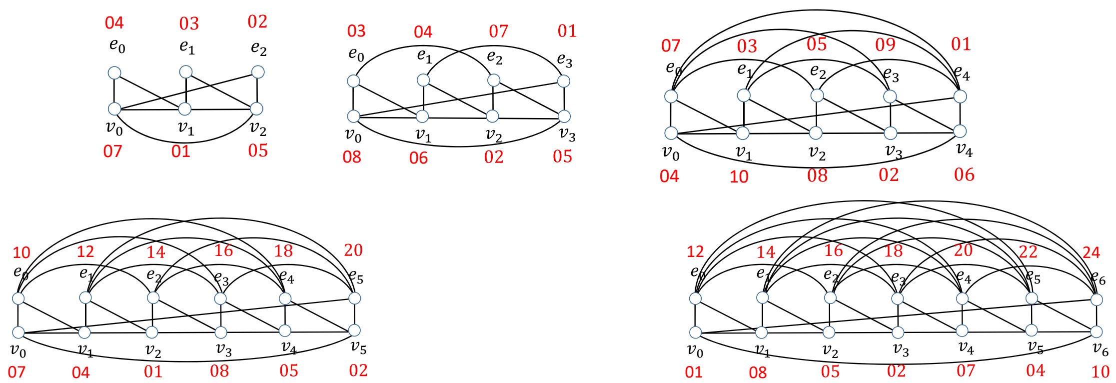

Figure 7. An rm-labeling of C_n^{+-+} with $n \leq 7$.

When $n \geq 8$, for each integer l, k; $0 \leq l \leq 2$, $0 \leq k \leq \lceil \frac{n}{3} \rceil - 1$, define a function $f : V \to Z^+$ by

$$f(v_{3k+l}) = \begin{cases} \lceil \frac{n}{3} \rceil (l+1) + k & \text{if } n \equiv 1 \pmod 3 \text{ and } l = 1 \\ \lceil \frac{n}{3} \rceil (l-1) + k + 1 & \text{if } n \equiv 1 \pmod 3 \text{ and } l = 2 \\ \lceil \frac{n}{3} \rceil l + k + 1 & \text{otherwise} \end{cases}$$

$$f(e_i) = (n+2) + 2i, \text{with } \ \mathbb{Z}_n.$$

Since $diam(\mathrm{G}) = 3$, now to show f is a rm-labeling it is sufficient to take $\alpha, \alpha^{'} \in V(\mathrm{G})$ with $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) \leq 2$.

Case 1: $\alpha \in V(C_n)$ and $\alpha^{'} \in V(C_n)$.

Let $\alpha = v_i$ and $\alpha^{'} = v_j$ with $i < j$ and $i, j \in \mathbb{Z}_n$.

Subcase 1: $i = 3k,\ j = 3k+1$

In this case, $f(\alpha) = k+1$ and $f(\alpha^{'}) = \begin{cases} 2\lceil \frac{n}{3} \rceil + k & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil + k + 1 & \text{otherwise} \end{cases}$

$$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} 2\lceil \frac{n}{3} \rceil - 1 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil \geq 3 & \text{otherwise} \end{cases}$$

Subcase 2: $i = 3k+1,\ j = 3k+2$

In this case, $f(\alpha) = \begin{cases} 2\lceil \frac{n}{3} \rceil + k & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil + k + 1 & \text{otherwise} \end{cases}$

and $f(\alpha^{'}) = \begin{cases} k + \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$

$$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} \lceil \frac{n}{3} \rceil - 1 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil \geq 3 & \text{otherwise} \end{cases}$$

Subcase 3: $i = 3k+2,\ j = 3k+3 = 3(k+1)$.

In this case, $f(\alpha) = \begin{cases} k + \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$ and $f(\alpha^{'}) = (k+1) + 1$.

$$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} \lceil \frac{n}{3} \rceil - 1 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ 2\lceil \frac{n}{3} \rceil \geq 3 & \text{otherwise} \end{cases}$$

Subcase 4: $i = 3k,\ j = 3k+2$.

In this case, $f(\alpha) = k+1$ and $f(\alpha^{'}) = \begin{cases} k + \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$

$$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} (\lceil \frac{n}{3} \rceil) \times 2 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ (2\lceil \frac{n}{3} \rceil) \times 2 \geq 3 & \text{otherwise} \end{cases}$$

Subcase 5: $i = 3k+1,\ j = 3k+3 = 3(k+1)$.

In this case, $f(\alpha) = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ k + \lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$ and $f(\alpha^{'}) = (k+1) + 1$.

$$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} (2\lceil \frac{n}{3} \rceil - 2) \times 2 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ (\lceil \frac{n}{3} \rceil - 1) \times 2 \geq 3 & \text{otherwise} \end{cases}$$

Subcase 6: $i = 3k+2,\ j = 3k+4 = 3(k+1)+1$.

In this case, $f(\alpha) = \begin{cases} k + \lceil \frac{n}{3} \rceil + 1 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ k + 2\lceil \frac{n}{3} \rceil + 1 \geq 3 & \text{otherwise} \end{cases}$

and $f(\alpha^{'}) = \begin{cases} (k\ +\ 1) + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ (k\ +\ 1) + \lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$

So, $|f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} (\lceil \frac{n}{3} \rceil) \times 2 \geq 3 & \text{if } n \equiv 1 \pmod 3 \\ (\lceil \frac{n}{3} \rceil - 1) \times 2 \geq 3 & \text{otherwise} \end{cases}$

Subcase 7: $i = 0,\ j = n-1 = 3k+l$

In this case, $f(\alpha) = 1$ and $f(\alpha^{'}) = \begin{cases} k+1 & \text{if } l=0 \\ k+\lceil \frac{n}{3} \rceil + 1 & \text{if } l=1 \\ k+2\lceil \frac{n}{3} \rceil + 1 & \text{if } l=2 \end{cases}$

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} k \geq 3 & \text{if } l=0\ (n \geq 10) \\ k+\lceil \frac{n}{3} \rceil \geq 3 & \text{if } l=1\ (n \geq 8) \\ k+2\lceil \frac{n}{3} \rceil \geq 3 & \text{if } l=2\ (n \geq 9) \end{cases}$

Subcase 8: $i = 0,\ j = n-2 = 3k+l$

In this case, $f(\alpha) = 1$ and $f(\alpha^{'}) = \begin{cases} k+1 & \text{if } l=0 \\ k+\lceil \frac{n}{3} \rceil + 1 & \text{if } l=1 \\ k+\lceil \frac{n}{3} \rceil + 1 & \text{if } l=2 \end{cases}$

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} (k) \times 2 \geq 3 & \text{if } l=0\ (n \geq 8) \\ (k+\lceil \frac{n}{3} \rceil) \times 2 \geq 3 & \text{if } l=1\ (n \geq 9) \\ (k+\lceil \frac{n}{3} \rceil) \times 2 \geq 3 & \text{if } l=2\ (n \geq 10) \end{cases}$

Subcase 9: $i = 1,\ j = n-1 = 3k+l$

In this case, $f(\alpha) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \pmod 3 \\ \lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$ and $f(\alpha^{'}) = \begin{cases} 1+k & \text{if } l=0 \\ \lceil \frac{n}{3} \rceil + k + 1 & \text{if } l=1 \\ 2\lceil \frac{n}{3} \rceil + k + 1 & \text{if } l=2 \end{cases}$

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = \begin{cases} |k+1-2\lceil \frac{n}{3} \rceil| \times 2 \geq 3 & \text{if } l=0\ (n \geq 10) \\ (k) \times 2 \geq 3 & \text{if } l=1\ (n \geq 8) \\ (k+\lceil \frac{n}{3} \rceil + 1) \times 2 \geq 3 & \text{if } l=2\ (n \geq 9) \end{cases}$

Case 2: $\alpha,\ \alpha^{'} \in E(C_n)$

Let $\alpha = e_i$ and $\alpha^{'} = e_j$ with $i, j \in \mathbb{Z}_{n-1}, i < j$. Then $|f(\alpha) - f(\alpha^{'})| = |f(e_i) - f(e_j)| = 2(j-i)$ is equal to 2 if $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 2$ (i.e $j = i+1$) and, at least 4 if $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 1$ (i.e $j > i+1$). In each of these cases $|f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) \geq 4 > diam(\mathrm{G})$.

Case 3: $\alpha \in V(C_n)$ and $\alpha^{'} \in E(C_n)$.

Let $\alpha = v_{3k+l}$ with $0 \leq l \leq 2$, $0 \leq k \leq \lceil \frac{n}{3} \rceil - 1$ and $\alpha^{'} = e_j$ with $j > i$ and $i, j \in \mathbb{Z}_n$.

Subcase 1: $j = (3k+l) - 1$

In this case, $d_{\mathrm{G}}(\alpha,\ \alpha^{'}) = 1$, Therefore

a) when $l = 1$ and $n \equiv 1 \pmod 3$

$|f(\alpha) - f(\alpha^{'})| + d_{\mathrm{G}}(\alpha,\ \alpha^{'}) - 1 = f(e_{(3k+l)-1}) - f(v_{3k+l}) = (n+2) + 2(3k+l-1) - k - \lceil \frac{n}{3} \rceil \times (l+1) \geq n + 2 - 2\lceil \frac{n}{3} \rceil > 3$ for all $n \geq 10$

b) when $l = 2$ and $n \equiv 1 \pmod 3$.

$|f(\alpha) - f(\alpha^{'})| + d_{\mathrm{G}}(\alpha,\ \alpha^{'}) - 1 = f(e_{(3k+l)-1}) - f(v_{3k+l} = (n+2) + 2(3k+l-1) - k - \lceil \frac{n}{3} \rceil \times (l-1) - 1 \geq n - \lceil \frac{n}{3} \rceil + 3 > 3$ for all $n \geq 10$

c) when $l \neq 1, 2$ or $n \not\equiv 1 \pmod 3$.

$|f(\alpha) - f(\alpha^{'})| + d_{\mathrm{G}}(\alpha,\ \alpha^{'}) - 1 = f(e_{(3k+l)-1}) - f(v_{3k+l}) = (n\ +\ 2) + 2(3k\ +\ l\ -\ 1)\ -\ k\ - \lceil \frac{n}{3} \rceil \times l\ -\ 1 \geq n\ -\ 2\lceil \frac{n}{3} \rceil\ +\ 3 > 3$.

Subcase 2: $j = (3k + l)$.

In this case, $d_G(\alpha,\ \alpha^{'}) = 1$ and $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) = f(e_{3k+l}) - f(v_{3k+l}) = (n+2) + 2(3k+l) - f(v_{3k+l}) = (n+2) + 2(3k + l - 1) - f(v_{3k+l}) + 2 = f(e_{(3k+l)-1}) - f(v_{3k+l}) + 2 > 3 + 2 > 3$ (by subcase1 of Case3).

Subcase 3: $j \notin \{(3k + l) - 1, (3k + l)\}$

In this case, $d_G(\alpha,\ \alpha^{'}) = 2$ and hence it suffices to show $|f(\alpha) - f(\alpha^{'})| > 1$. In fact$|f(\alpha) - f(\alpha^{'})| = f(\alpha^{'}) - f(\alpha) = (n + 2) + 2j - f(\alpha) \geq (n + 2) + 2j - n = 2(j + 1) > 1$.

Thus, from all the above subcases, f is an rm-labeling and $span f = 3n$.

Therefore $rn^{\times}(G) \leq 3n$ for all $n \geq 8$. □

6.2. For $xyz = + + -$

Lemma 6.2. *For any positive integer $n > 2$, $rn^{\times}(C_n^{++-}) \leq 2n$.*

Proof. Consider a function $f : V \to Z^{+}$ defined by $f(v_i) = 2i + 1$, for all $i \in \mathbb{Z}_n$ and $f(e_j) = 2(j + 1)$ for all $j \in \mathbb{Z}_n$. Since $diam(G) = 2$, now to show f is a rm-labeling it is enough to take $\alpha, \alpha^{'} \in V(G)$ with $d_G(\alpha,\ \alpha^{'}) = 1$.

Case 1: $\alpha \in V(C_n)$ and $\alpha^{'} \in V(C_n)$.

Let $\alpha = v_i$ and $\alpha^{'} = v_j$ with $i, j \in \mathbb{Z}_n, i < j$. Then $|f(\alpha) - f(\alpha^{'})| = |f(v_i) - f(v_j)| = 2(j - i)$ is at least 2 if $d_G(\alpha,\ \alpha^{'}) = 1$. In this case $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) \geq 2$.

Case 2: $\alpha \in E(C_n)$ and $\alpha^{'} \in E(C_n)$.

Let $\alpha = e_i$ and $\alpha^{'} = e_j$ with $i, j \in \mathbb{Z}_n, i < j$. Then $|f(\alpha) - f(\alpha^{'})| = |f(e_i) - f(e_j)| = 2(j - i)$ is at least 2 if $d_G(\alpha,\ \alpha^{'}) = 1$. In this case $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) \geq 2$.

Case 3: $\alpha \in V(C_n)$ and $\alpha^{'} \in E(C_n)$.

Let $\alpha = v_i$ with $\mathbb{Z}_n$ and $\alpha^{'} = e_j$ with $0 \leq j \leq n - 1$. Then $|f(\alpha) - f(\alpha^{'})| = |f(v_i) - f(e_j)| = |2(i - j) - 1|$ is at least 3 if $d_G(\alpha,\ \alpha^{'}) = 1$ (i.e $|i - j| \geq 2$). In this case $|f(\alpha) - f(\alpha^{'})| \times d_G(\alpha,\ \alpha^{'}) \geq 3$.

Thus, from all the above cases, f is an rm-labeling and $span f = 2n$.

Therefore $rn^{\times}(G) \leq 2n$. □

6.3. For $xyz = + - -$ or For $xyz = - + -$

Now let $xyz = + - -$. As $C_n^{-+-} \cong C_n^{+--}$ proof follows immediately for $xyz = - + -$.

Lemma 6.3. *For any positive integer $n > 2$,*

$$rn^{\times}(C_n^{+--}) \leq \begin{cases} 10, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Proof. For $n = 3$, result follows by the rm-labeling f for Figure 8 defined by $f(v_0) = 1$, $f(v_1) = 10$, $f(v_2) = 7$, $f(e_0) = 3$, $f(e_1) = 4$ and $f(e_2) = 5$.

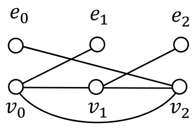

Figure 8. C_n^{+--} **with** $n = 3$

For $n \geq 4$, proof is similar to that of Lemma 6.2. □

6.4. For $xyz = --+$

Lemma 6.4. *For any positive integer* $n > 2$,

$$rn^{\times}(C_n^{--+}) \leq \begin{cases} 8, & if \;\; n = 3 \\ 2n, & if \;\; n \geq 4 \end{cases}$$

Proof. For $n = 3$, result follows by the rm-labeling shown in Figure 9.

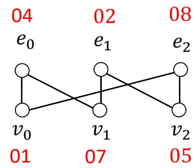

Figure 9. **An** rm**-labeling of** C_3^{--+}

For $n \geq 4$, define a function $f : V \to Z^+$ by $f(v_i) = i + 1$ for $i \in \mathbb{Z}_n$; $f(e_j) = (n + 1) + j$, for $j \in \mathbb{Z}_n$. Since $diam(\mathrm{G}) = 2$, now to show f is a rm-labeling it is enough to take $\alpha, \alpha' \in V(\mathrm{G})$ with $d_{\mathrm{G}}(\alpha, \alpha') = 1$.

Case 1: $\alpha \in V(C_n)$ and $\alpha' \in V(C_n)$.

Let $\alpha = v_i$ and $\alpha' = v_j$ with $i, j \in \mathbb{Z}_n, i < j$. Then $|f(\alpha) - f(\alpha')| = |f(v_i) - f(v_j)| = (j - i)$ is at least 2 if $d_{\mathrm{G}}(\alpha, \alpha') = 1$ (i.e $j \geq i + 2$). In this case $|f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha, \alpha') \geq 2$.

Case 2: $\alpha \in E(C_n)$ and $\alpha' \in E(C_n)$.

Let $\alpha = e_i$ and $\alpha' = e_j$ with $i, j \in \mathbb{Z}_{n-1}, i < j$. Then $|f(\alpha) - f(\alpha')| = |f(e_i) - f(e_j)| = (j - i)$ is at least 2 if $d_{\mathrm{G}}(\alpha, \alpha') = 1$ (i.e $j \geq i + 2$). In this case $|f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha, \alpha') \geq 2$.

Case 3: $\alpha \in V(C_n)$ and $\alpha' \in E(C_n)$

Let $\alpha = v_i$ with $\mathbb{Z}_n$ and $\alpha' = e_j$ with $0 \leq j \leq n - 1$.

Subcase 1: $i = j$

In this case $|f(\alpha) - f(\alpha')| = |f(v_i) - f(e_j)| = |i - j - n|$ is at least 4 (since $n \geq 4$).

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha, \alpha') \geq 3$.

Subcase 2: $i = j + 1$

In this case $|f(\alpha) - f(\alpha')| = |f(v_i) - f(e_j)| = |i - j - n|$ is at least 3 (since $n \geq 4$).

$\therefore \quad |f(\alpha) - f(\alpha')| \times d_{\mathrm{G}}(\alpha, \alpha') \geq 3$.

Subcase 3: $i = 0$ and $j = n - 1$

In this case $|f(\alpha) - f(\alpha^{'})| = |f(v_i) - f(e_j)| = |0 + 1 - 2n|$ is at least 7 (since $n \geq 4$).

$\therefore \quad |f(\alpha) - f(\alpha^{'})| \times d_{\mathrm{G}}(\alpha,\ \alpha^{'}) \geq 3.$

Thus, from all the above subcases, f is an rm-labeling and $span f = 2n$.

Therefore $rn^{\times}(\mathrm{G}) \leq 2n$ for all $n \geq 4$. □

6.5. For $xyz = - - -$

Lemma 6.5. *For any positive integer $n > 3$, $rn^{\times}(C_n^{---}) \leq 2n$.*

Proof. Proof is similar to that of Lemma 6.2. □

7. Conclusion

In wireless networks, an important task is the assignment of radio frequencies to transmitters in a way that avoids interference of their signals. The objective is to minimize range (or span) of used frequencies. Radio multiplicative labeling will serve this objective and a tool to solve many real world problems. Further, transformation graphs are graphs which use vertices, edges, adjacency and incidence of original graphs and recently more work is going on in such graphs. If at all we get a relation of a property with graph and its transformation graph then this paper will be very useful as diameter is constant unlike original graph. We completely determined $rn^{\times}(G)$ for certain graph families derived from path and cycle. Graceful graphs serves minimum span so such graphs is centre of attraction. Also investigating the graceful graph is an interesting and challenging task as well. Required minimum span is obtained in many graphs discussed in this paper except a few.

We now conclude this paper with the Table 1. This table shows the cases where the transformation graphs of paths and cycles are rm-graceful using the results of previous sections.

The graph G	value of x	value of y	Value of z	rm-graceful if
P_n	-	-	+	$n \geq 3$
	+	+	-	$n \geq 3$
	+	-	-	$n \geq 4$
	-	+	-	$n \geq 5$
	-	-	-	$n \geq 5$
C_n	+	+	-	$n \geq 3$
	+	-	-	$n \geq 4$
	-	+	-	$n \geq 4$
	-	-	+	$n \geq 4$
	-	-	-	$n \geq 4$

Table 1. **rm-gracefulness of transformation graphs of paths and cycles.**

8. Open Problems

We end up with the following problems.

Open Problem 1: For any positive integer n, determine $rn^{\times}(P_n)$.

Open Problem 2: For any positive integer n, determine $rn^{\times}(C_n)$.

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