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About Generalized Soft Quotient Groups

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Abstract: The aim of this paper is to introduce the notions of generalized soft group, generalized soft (normal) subgroup, generalized soft quotient group etc., generalizing the corresponding notions of a soft group over a group and show that several of the crisp group theoretic results naturally extended to these new objects too.

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1. Introduction

As yet another tool to model uncertainties, Molodtsov [2] introduced the new concept of a soft set over a universal set in the year 1999. A soft set is nothing but a parametrized family of subsets of a set. Mathematically speaking, if E is a set of parameters and U is a universal set, a soft set is any function from E into the set of all subsets of U, namely $P(U)$. The concept is not only seemingly simple but also natural and most appealing. For example, if E is a set of parameters, say $E = \{verytall, tall, average, short, veryshort\}$, under study for a group of people, say U, naturally the set of all very tall people is a subset of U. The same applies for all other parameters too. When a person is not easily classified as very tall or tall, it will appear in both sub collections associated with the parameters very tall and tall. Ever since the new notion of soft set appeared, mathematicians started studying parametrized families of various sub objects of algebraic, topological and topologically algebraic objects in mathematics. Noting that every soft set needs a universal set U, a parameter set E and a map $F : E \to P(U)$ and any three of them determine a soft set, Murthy-Maheswari [9] generalized the notion of soft set to that of a Generalized Soft Set as a triplet $(A, F, P(U))$ and the notion of soft map to that of a Generalized Soft Map, $(f, \phi) : (A, F, P(U)) \to (B, G, P(V))$ where $f : A \to B$ is any map and $\phi : P(U) \to P(V)$ is any complete homomorphism, deriving several of their mapping properties from the generalized fuzzy sets and generalized fuzzy maps developed in Murthy [7] of 1997. Now, our aim in this paper is to use the above notion of generalized soft (sub) set and introduce the notions of generalized soft group, generalized soft (normal) subgroup, generalized soft quotient group etc., generalizing the corresponding notions of a soft group over a group and show that several of the crisp theoretic results naturally extend to these new objects too.

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2. Preliminaries

In what follows we recall some basic notions and results from the theories of Groups, Soft sets, Generalized soft sets, which are used in due course.

2.1. Groups

(a). For any group G and for any pair of (normal) subgroups A, B of G, we have A is a (normal) subgroup of B iff A is a subset of B.

(b). (i). For any pair of subgroups H, K of G such that $H \subseteq K$ we have H is a subgroup of K iff H is a subgroup of G.

(ii). For any pair of subsets H, K of G such that $H \subseteq K$ and K is a subgroup of G we have H is a normal subgroup of G implies H is a normal subgroup of K but *not* conversely.

(iii). For any group G, for any subgroup A of G and for any normal subgroup B of G we have $A \cap B$ is a normal subgroup of A.

(c). For any group G, the following are true:

(i). for any subset H of G, H is a subgroup of G iff $HH = H$ and $H^{-1} = H$ iff $HH^{-1} = H$.

(ii). for any pair of subsets H, K of G, $(HK)^{-1} = K^{-1}H^{-1}$.

(iii). for any pair of subgroups H, K of G we have HK is a subgroup of G iff $HK = KH$

(iv). for any pair of subgroups H, K of G, if H or K is a normal subgroup of G then HK is a subgroup of G. Further, when both H and K are normal subgroups of G then HK is a normal subgroup of G iff $HK = KH$.

(v). for any normal subgroup K of G and for any (normal) subgroup H of G such that $K \subseteq H$ we have $\frac{H}{K}$ is a (normal) subgroup of $\frac{G}{K}$.

(vi). For any group G and for any normal subgroup N of G such that A is a subgroup of G, B is a subgroup of N and B is a normal subgroup of A, there is a *naturally induced* homomorphism $\phi : \frac{A}{B} \to \frac{G}{N}$ defined by $\phi(Ba) = Na$ for all $a \in A$ such that $\phi(\frac{A}{B}) = \frac{AN}{N}$. Later on we use this result to define the s-quotient group.

2.2. Soft Sets

In what follows we recall the following basic definitions from the Soft Set Theory which are used in due course:

(d). [2] Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F : E \longrightarrow P(U)$ is a mapping defined by for each $e \in E$, $F(e)$ is a subset of U. In other words, a soft set over U is a parametrized family of subsets of U.

For any soft set (F, E) over U, we let (F, E_r) be the *associated regular soft set* where $E_r = \{e \in E / Fe \neq \phi\}$.

Clearly, a soft set (F, E) over U is non-null (cf.(g) below) iff $E_r \neq \phi$.

Notice that a collective presentation of all the notions, soft sets and gs-sets raised some serious notational conflicts and to fix the same Murthy-Maheswari[10] deviated from the above notation for a soft set and adapted the following notation for convenience as follows:

Let U be a universal set. A typical *soft set* over U is an ordered pair (San Serif) $\mathsf{S} = (\sigma_S, S)$, where S is a set of *parameters*, called the *underlying parameter set* for S, $P(U)$ is the power set of U and $\sigma_S : S \longrightarrow P(U)$ is a map, called the *underlying set valued map* for S. Some times σ_S is also called the *soft structure* on S.

(e). [6] The *empty* soft set over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique.

(f). [5] A soft set S over U is said to be a *whole* soft set, denoted by U_S, iff $\sigma_S s = U$ for all $s \in S$.

(g). [6] A soft set S over U is said to be a *null* soft set, denoted by Φ_S, iff $\sigma_S s = \phi$, the empty set, for all $s \in S$. Notice that $\Phi_\phi = \phi$, the empty soft (sub) set.

For any pair of soft sets A, B over U.

(h). [3] A is a *soft subset* of B, denoted by $\mathsf{A} \subseteq \mathsf{B}$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$.

(i). The following are easy to see:

(1). Always the empty soft set Φ is a soft subset of every soft set A

(2). $\mathsf{A} = \mathsf{B}$ iff $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{B} \subseteq \mathsf{A}$ iff $A = B$ and $\sigma_A a = \sigma_B a$ for all $a \in A$.

(j). For any family of soft subsets $(\mathsf{A}_i)_{i\in I}$ of S,

(1). the *soft union* of $(\mathsf{A}_i)_{i\in I}$, denoted by $\cup_{i\in I}\mathsf{A}_i$, is defined by the soft set A, where (i). $A = \cup_{i\in I} A_i$ (ii). $\sigma_A a = \cup_{i\in I_a}\sigma_{A_i} a$, where $I_a = \{i \in I / a \in A_i\}$, for all $a \in A$.

(2). the *soft intersection* of $(\mathsf{A}_i)_{i\in I}$, denoted by $\cap_{i\in I}\mathsf{A}_i$, is defined by the soft set A, where (i). $A = \cap_{i\in I} A_i$, (ii). $\sigma_A a = \cap_{i\in I}\sigma_{A_i} a$ for all $a \in A$.

Notice that $\cap_{i\in I}\mathsf{A}_i$ can become empty soft set.

2.3. Generalized Soft Sets

In this section we recall the notions of generalized soft set, gs-set or s-set for short, gs-subset or s-subset, gs-union or s-union, gs-intersection or s-intersection etc. from Murthy-Maheswari [9]. From now on, the script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ etc. and the suffixed ones, denote s-sets and/or their subsets and any such script letter $\mathcal{Q}$ stands for the triplet $(Q, \overline{Q}, P(U_Q))$.

(k). A *generalized soft set* or an *s-set* in short, is any triplet $\mathcal{A}$, where A is the *underlying set of parameters for* U_A *or parameter set* in short, $P(U_A)$ is *the complete lattice of all subsets of* U_A *parametrized under* $\overline{A}$ *with parameters from* A and $\overline{A} : A \longrightarrow P(U_A)$ is the *underlying parametrizing map for* U_A.

(l). For any soft set (F, A) over a universal set U with the parameter set A, the *associated s-set for* (F, A), is defined by the s-set $(A, F, P(U))$, where A is the underlying parameter set, $P(U)$ is the power set of all subsets of U and $F : A \longrightarrow P(U)$ is the parametrizing map.

(m). The s-set $\mathcal{A}$, where $A = \square$, the empty set with *no* elements, $P(U_A) = \{\square\}$, and $\overline{A} = \square$, the empty map, is called the *empty s-set* and is denoted by $\square$.

(n). An s-set $\mathcal{A}$ is said to be a *whole s-set* iff the parametrizing map $\overline{A} : A \longrightarrow P(U_A)$ is defined by $\overline{A}a = U_A$ for all $a \in A$.

(o). An s-set $\mathcal{A}$ is said to be a *null s-set* iff the parametrizing map $\overline{A} : A \longrightarrow P(U_A)$ is defined by $\overline{A}a = \square$, the empty set, for all $a \in A$.

(p). For any pair of s-sets $\mathcal{S}$ and $\mathcal{A}$.

$\mathcal{S}$ is an *s-subset* of $\mathcal{A}$, denoted by $\mathcal{S} \subseteq \mathcal{A}$, iff (i) $S \subseteq A$ (ii) $U_S \subseteq U_A$ or equivalently $P(U_S)$ is a complete ideal of $P(U_A)$ and (iii) $\overline{S}a \subseteq \overline{A}a$ for all $a \in S$. An s-subset of $\mathcal{A}$ which is also a (empty) null s-set is a *(empty) null s-subset* of $\mathcal{A}$. An s-subset $\mathcal{S}$ of $\mathcal{A}$ is *degenerated*, denoted by , iff $S = \square$ and $\overline{S} = \square$, the empty map.

Clearly, (i). the empty s-(sub) set is degenerated. (ii). for any pair of s-subsets $\mathcal{S}, \mathcal{T}$ of $\mathcal{A}$, $\mathcal{S} = \mathcal{T}$ iff $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{S}$

iff $S = T$, $U_S = U_T$ or $P(U_S) = P(U_T)$ and $\bar{S} = \bar{T}$. (iii). for any pair of soft sets (F, A) and (G, B) over U, (G, B) is a soft subset of (F, A) iff $(B, G, P(U))$ is an s-subset of $(A, F, P(U))$.

(q). $\mathcal{A}$ is a *c-total* s-subset of $\mathcal{B}$ iff $\mathcal{A}$ is a s-subset of $\mathcal{B}$ and $U_A = U_B$ or equivalently $P(U_A) = P(U_B)$.

(r). The following are easy to see:

(i). Always the empty s-set $\square$ is an s-subset of every s-set $\mathcal{A}$.

(ii). For any pair of s-sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$ iff $A = B$, $U_A = U_B$ and $\overline{A} = \overline{B}$.

(s). For any family of s-subsets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{B}$,

(1). the *s-union* of $(\mathcal{A}_i)_{i\in I}$, denoted by $\cup_{i\in I}\mathcal{A}_i$, is defined by the s-set $\mathcal{A}$, where

(i). $A = \cup_{i\in I} A_i$ is the usual set union of the collection $(A_i)_{i\in I}$ of subsets of B.

(ii). $P(U_A) = \vee_{i\in I} P(U_{A_i}) = P(\cup_{i\in I} U_{A_i})$, where $\vee_{i\in I} P(U_{A_i})$ is the complete ideal generated by $\cup_{i\in I} P(U_{A_i})$ in $P(U_A)$ which is the same as $P(\cup_{i\in I} U_{A_i})$.

(iii). $\overline{A} : A \longrightarrow P(U_A)$ is defined by $\overline{A}a = \cup_{i\in I_a} \overline{A}_i a$, where $I_a = \{i \in I | a \in A_i\}$.

(2). the *s-intersection* of $(\mathcal{A}_i)_{i\in I}$, denoted by $\cap_{i\in I}(\mathcal{A}_i)$, is defined by the s-set $\mathcal{A}$, where

(i). $A{=}\cap_{i\in I} A_i$ is the usual set intersection of the collection $(A_i)_{i\in I}$ of subsets of B.

(ii). $P(U_A) = \cap_{i\in I} P(U_{A_i}) = P(\cap_{i\in I} U_{A_i})$ is the usual intersection of the complete ideals of $P(U_{A_i})_{i\in I}$ in $P(U_A)$.

(iii). $\bar{A} : A \longrightarrow P(U_A)$ is defined by $\overline{A}a = \cap_{i\in I} \overline{A}_i a$.

(t). It is easy to see that (i) for any soft set (F, A) over a universal set U with parameter set A, the complete lattice $\mathbb{C}$ of all soft subsets of the soft set (F, A) is complete isomorphic to the complete sub lattice $\mathbb{D}$ of all c-total s-subsets of the associated s-set $(A, F, P(U))$ for the soft set (F, A) and (ii) *any s-subset $\mathcal{B}$ of $(A, F, P(U))$, where $B \subseteq A$, $\bar{B}b$ is subset of Fb for all b in B and $U_B \subsetneq U$ is such that, $\mathcal{B}$ is an s-subset of $(A, F, P(U))$ but not in $\mathbb{D}$, indicating that the notion of s-(sub) set is a proper generalization of the notion of soft (sub) set.*

2.4. Soft (normal) (sub) Groups

In this section we first recall the existing notions of a soft group, soft (normal) subgroup etc..

(u). According to Aktas-Cagman [4], if (F, A) is a soft set over a group G, then (F, A) is said to be a *soft group* over G if and only if $F(x) \leq G$ for all $x \in A$. Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a *soft subgroup* of (F, A), written as $(H, K) \leq (F, A)$, if (1). $K \subseteq A$ (2) $H(x) \leq F(x)$ for all $x \in K$. Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is a *normal soft subgroup* of (F, A), written $(H, B) \trianglelefteq (F, A)$, if $H(x)$ is a normal subgroup of $F(x)$ i.e., $H(x) \trianglelefteq F(x)$, for all $x \in B$.

(v). According to Sezgin-Atagun[1], if G is a group and (F, A) is a non-null soft set over G, then (F, A) is called a *normalistic soft group* over G if $F(x)$ is a normal subgroup of G for all $x \in \mathrm{Supp}(F, A)$.

3. s-Groups

In this section we introduce the notions of s-group, s-(normal) subgroup, strong s-normal subgroup, s-point, s-product, s-quotient group, s-(left) right coset etc, generalizing the corresponding existing notions and study their properties.

Definitions 3.1.

(a). An s-set $\mathcal{G}$ is said to be an s-(normal) group iff (i) U_G is a group (ii) $\bar{G}g$ is a (normal) subgroup of U_G for all $g \in G$. An s-group which is also a whole s-set is a whole s-group.

(b). For any s-group $\mathcal{G}$ and for any s-subset $\mathcal{A}$ of $\mathcal{G}$,

(1) $\mathcal{A}$ is an s-subgroup of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a subgroup of U_G (iii) $\bar{A}g$ is a subgroup of $\bar{G}g$ for all $g \in G$.

Clearly, (i) a degenerated s-subset $\mathcal{A}$ of $\mathcal{G}$ is an s-subgroup iff U_A is a subgroup of U_G. (ii) the whole s-group is not necessarily an s-subgroup of $\mathcal{G}$ and both the null s-subset and the empty s-subset of an s-group $\mathcal{G}$ are not s-subgroups of $\mathcal{G}$.

For any soft set (F, A) over a group U, we have (F, A) is a soft group over U iff the associated s-set $(A, F, P(U))$ of (F, A) (cf.2.3(l)) is an s-subgroup of the whole s-group $\mathcal{G}$ where $G = A$ and $U_G = U$, called the associated s-subgroup for (F, A).

Clearly, (i). for any pair of soft sets (F, A) and (H, C) over a group U, (H, C) is a soft subgroup of (F, A) iff $(C, H, P(U))$ is an s-subgroup of $(A, F, P(U))$.

(ii). for any s-group $\mathcal{G}$ and for any c-total s-subset $\mathcal{A}$ of $\mathcal{G}$, $(A, \bar{A})$ is a soft subgroup of the soft group $(G, \bar{G})$ over U_G iff $\mathcal{A}$ is an s-subgroup of $\mathcal{G}$.

(2). $\mathcal{A}$ is an s-normal subgroup of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a normal subgroup of U_G (iii). $\bar{A}g$ is a normal subgroup of $\bar{G}g$ for all $g \in G$.

Clearly, (i) every s-normal subgroup is an s-subgroup (ii) the whole s-group is an s-normal subgroup of $\mathcal{G}$ but both the null s-subset and the empty s-subset of an s-group $\mathcal{G}$ are not s-normal subgroups of $\mathcal{G}$. (iii) a degenerated s-subset $\mathcal{A}$ of $\mathcal{G}$ is an s-normal subgroup iff U_A is a normal subgroup of U_G and a degenerated s-subset which is also an s-(normal) subgroup is a degenerated s-(normal) subgroup.

For any soft normal subgroup (H, B) of a soft group (F, A), the associated s-set $(B, H, P(U))$ for (H, B) (cf.2.3(1)) is an s-normal subgroup of the associated s-set $(A, F, P(U))$, called the associated s-normal subgroup for (H, B).

Clearly, (i) for any soft group (F, A) over a group U, and for any soft subset (H, B) of (F, A), (H, B) is a normal soft subgroup of (F, A) iff $(B, H, P(U))$ is an s-normal subgroup of $(A, F, P(U))$

(ii) for any s-group $\mathcal{G}$ and for any c-total s-subset $\mathcal{A}$ of $\mathcal{G}$, $(A, \bar{A})$ is a soft normal subgroup of the soft group $(G, \bar{G})$ over U_G iff $\mathcal{A}$ is an s-normal subgroup of $\mathcal{G}$.

(3). $\mathcal{A}$ is a strong s-normal subgroup of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a normal subgroup of U_G (iii) $\bar{A}g$ is a normal subgroup of U_G for all $g \in G$.

Clearly, (i). for a non-null soft set (H, B) over a group G, (H, B) is a normalistic soft group over G iff $(B_r, H, P(G))$ is an s-normal group, where (H, B_r) is the associated regular soft set. (ii) for any whole s-group $\mathcal{G}$ and for any c-total s-subset $\mathcal{A}$ of $\mathcal{G}$, $(A, \bar{A})$ is a normalistic soft group over U_G iff $\mathcal{A}$ is an strong s-normal subgroup of $\mathcal{G}$. (iii). every strong s-normal subgroup is an s-normal subgroup, but not conversely.

The following example shows that the above 3.1(3)(iii) converse is *not* true.

Example 3.2. *Let $\mathcal{N} = (\{x\}, \{(x, H_2)\}, P(A_4))$, $\mathcal{G} = (\{x\}, \{(x, K_4)\}, P(A_4)))$ implies $\mathcal{N}$ is an s-normal subgroup of $\mathcal{G}$ because $N \subseteq G$, U_N is normal subgroup of U_G and $\bar{N}x$ is a normal subgroup of $\bar{G}x$ but $\mathcal{N}$ is not strong s-normal subgroup of $\mathcal{G}$ because $\bar{N}x$ is not normal subgroup of U_G.*

In what follows we show that the complete lattice of all soft (normal) subgroups of a soft group is a complete sublattice of certain s-(normal) subgroups of an s-group upto isomorphism, *indicating that the notions of s-substructures of an s-group are a proper generalization of the notions of soft substructures of a soft group.*

Theorem 3.3. *For any soft group (F, A) over a group U with parameter set A, the complete lattice of all soft (normal) subgroups of the soft group (F, A) is complete isomorphic to the complete sublattice of all c-total s-(normal) subgroups of the associated s-group $(A, F, P(U))$ for the soft group (F, A).*

Proof. Let $\mathbb{C}$ be the complete lattice of all soft (normal) subgroups of the soft group (F, A) and let $\mathbb{D}$ be the complete sub lattice of all c-total s-(normal) subgroups of the associated s-group $(A, F, P(U))$ for the soft group (F, A). Define $\phi : \mathbb{C} \to \mathbb{D}$ by $\phi(G, B) = (B, G, P(U))$. Then it easily to verify that ϕ is a bijective, intersection/meet homomorphism and hence complete isomorphism.

Further, ϕ is *not* onto the complete lattice of all s-(normal) subgroups of $(A, F, P(U))$ as any s-(normal) group $\mathcal{B}$ of $(A, F, P(U))$, where $B \subseteq A$, $\bar{B}b$ is a (normal) subgroup of Fb for all b in B and $U_B \subsetneqq U$ is such that, $\mathcal{B}$ is an s-(normal) subgroup of $(A, F, P(U))$ *not* in $\mathbb{D}$. $\square$

Proposition 3.4. *For any pair of s-(normal) subgroups $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{G}$, $\mathcal{A}$ is an s-(normal) subgroup of $\mathcal{B}$ iff $\mathcal{A}$ is an s-subset of $\mathcal{B}$.*

Proof. It follows from the corresponding crisp result 2.1(a) and the definition of s-(normal) subgroup. $\square$

Proposition 3.5. *For any s-group $\mathcal{G}$, for any s-subgroup $\mathcal{A}$ of $\mathcal{G}$ and for any s-normal subgroup $\mathcal{B}$ of $\mathcal{G}$ we have $\mathcal{A} \cap \mathcal{B}$ is an s-normal subgroup of $\mathcal{A}$.*

Proof. It follows from the corresponding crisp result 2.1(b)(iii) and the definition of s-normal subgroup. $\square$

Proposition 3.6. *For any s-group $\mathcal{G}$, for any s-subset $\mathcal{H}$ of $\mathcal{G}$ and for any s-subgroup $\mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{K}$, we have $\mathcal{H}$ is an s-subgroup of $\mathcal{K}$ iff $\mathcal{H}$ is an s-subgroup of $\mathcal{G}$.*

Proof. It follows from the corresponding crisp result 2.1(b)(i) and the definition of s-subgroup. $\square$

Proposition 3.7. *For any s-group $\mathcal{G}$ and for any pair of subsets $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $\mathcal{K}$ is an s-subgroup of $\mathcal{G}$ we have $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$ implies $\mathcal{H}$ is an s-normal subgroup of $\mathcal{K}$.*

Proof. ($\Rightarrow$): Let $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$. Then $H \subseteq G$, U_H is a normal subgroup of U_G and $\bar{H}h$ is a normal subgroup of $\bar{G}h$ for all $h \in H$. We show that $\mathcal{H}$ is an s-normal subgroup of $\mathcal{K}$ or (i). $H \subseteq K$ (ii). U_H is a normal subgroup of U_K and (iii). $\bar{H}h$ is a normal subgroup of $\bar{K}h$ for all $h \in H$.

(i): Clearly, $H \subseteq K$.

(ii): Since U_H, U_K are subsets of U_G such that $U_H \subseteq U_K$, U_H is a normal subgroup of U_G, U_K is a subgroup of U_G, we have U_H is a normal subgroup of U_K.

(iii): Let $h \in H \subseteq K$. Then $\bar{H}h, \bar{K}h$ are subsets of $\bar{G}h$ such that $\bar{H}h \subseteq \bar{K}h$, $\bar{H}h$ is a normal subgroup of $\bar{G}h$, $\bar{K}h$ is a subgroup of $\bar{G}h$ we have $\bar{H}h$ is a normal subgroup of $\bar{K}h$ or $\mathcal{H}$ is an s-normal subgroup of $\mathcal{K}$. $\square$

Definition 3.8. *For any s-group $\mathcal{G}$ and for any s-subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{G}$, the s-product of $\mathcal{A}$ by $\mathcal{B}$, denoted by $\mathcal{AB}$, is defined by $\mathcal{C}$, where $C = A \cap B$, $U_C = U_A U_B$ and $\bar{C}c = \bar{A}c\bar{B}c$ for all $c \in C$.*

Definition 3.9. *For any s-group $\mathcal{G}$ and for any s-subset $\mathcal{A}$ of $\mathcal{G}$, the s-inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{-1}$, is defined by $\mathcal{C}$, where $C = A$, $U_C = U_A^{-1}$ and $\bar{C}c = (\bar{A}c)^{-1}$ for all $c \in C$.*

Proposition 3.10. *For any s-subset $\mathcal{H}$ of $\mathcal{G}$, the following are equivalent: $\mathcal{H}$ is an s-subgroup of $\mathcal{G}$ iff $\mathcal{HH} = \mathcal{H}$ and $\mathcal{H}^{-1} = \mathcal{H}$ iff $\mathcal{HH}^{-1} = \mathcal{H}$.*

Proof. It follows from the definition of s-subgroup, 3.8, 3.9 and 2.1(c)(i). □

Proposition 3.11. *For any s-group $\mathcal{G}$ and for any pair of s-subsets $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$, $(\mathcal{H}\mathcal{K})^{-1} = \mathcal{K}^{-1}\mathcal{H}^{-1}$.*

Proof. It follows from the definitions 3.8, 3.9 and 2.1(c)(ii). □

Proposition 3.12. *For any s-group $\mathcal{G}$ and for any pair of s-subgroups $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$, $\mathcal{H}\mathcal{K}$ is an s-subgroup of $\mathcal{G}$ iff $\mathcal{H}\mathcal{K} = \mathcal{K}\mathcal{H}$.*

Proof. $(\Rightarrow)$: Let $\mathcal{H}\mathcal{K} = \mathcal{C}$. Then $C = H \cap K$, $U_C = U_H U_K$ and $\bar{C}c = \bar{H}c\bar{K}c$ for all $c \in C$. Since $\mathcal{C}$ is an s-subgroup of $\mathcal{G}$, $C \subseteq G$, U_C is a subgroup of U_G and $\bar{C}c$ is a subgroup of $\bar{G}c$ for all $c \in C$. Let $\mathcal{K}\mathcal{H} = \mathcal{B}$. Then $B = K \cap H$, $U_B = U_K U_H$ and $\bar{B}b = \bar{K}b\bar{H}b$ for all $b \in B$. We show that $\mathcal{C} = \mathcal{B}$ or (i) $C = B$ (ii) $U_C = U_B$ and (iii) $\bar{C}c = \bar{B}c$ for all $c \in C$. If one of C or B is ϕ then the other one is ϕ, $\bar{C} = \phi = \bar{B}$, $U_H U_K = U_C$ is a subgroup of U_G which implies $U_H U_K = U_K U_H$ or $U_C = U_B$, in total, $\mathcal{C} = \mathcal{B}$. If $C \neq \phi$ then

(i): $C = H \cap K = K \cap H = B$

(ii): Since U_H, U_K are subgroups of U_G, $U_C = U_H U_K$ is a subgroup of U_G we have $U_C = U_H U_K = U_K U_H = U_B$

(iii): Let $c \in C = B$. Then $\bar{H}c, \bar{K}c$ are subgroups of $\bar{G}c$, $\bar{C}c = \bar{H}c\bar{K}c$ is a subgroup of $\bar{G}c$ implies $\bar{C}c = \bar{H}c\bar{K}c = \bar{K}c\bar{H}c = \bar{B}c$ or $\mathcal{C} = \mathcal{B}$ or $\mathcal{H}\mathcal{K} = \mathcal{K}\mathcal{H}$.

$(\Leftarrow)$: Let $\mathcal{H}\mathcal{K} = \mathcal{C}$ and $\mathcal{K}\mathcal{H} = \mathcal{B}$ be as above and $\mathcal{C} = \mathcal{B}$ then $C = H \cap K = K \cap H = B$, $U_C = U_H U_K = U_K U_H = U_B$ and $\bar{C}c = \bar{H}c\bar{K}c = \bar{K}c\bar{H}c = \bar{B}c$ for all $c \in C$. We show that $\mathcal{C}$ is an s-subgroup of $\mathcal{G}$ or (i) $C \subseteq G$ (ii) U_C is a subgroup of U_G and (iii) $\bar{C}c$ is a subgroup of $\bar{G}c$ for all $c \in C$. If C is empty then $\overline{C}$ is empty. Since $U_H U_K = U_C = U_B = U_K U_H$, we have $U_C = U_H U_K$ is a subgroup of U_G and so $\mathcal{C}$ is an s-subgroup of $\mathcal{G}$. So, let C be non-empty.

(i): $C = H \cap K \subseteq G$.

(ii): Since U_H, U_K are subgroups of U_G, $U_H U_K = U_K U_H$ implies $U_C = U_H U_K$ is a subgroup of U_G.

(iii): Since $\bar{H}c, \bar{K}c$ are subgroups of $\bar{G}c$, $\bar{H}c\bar{K}c = \bar{K}c\bar{H}c$ implies $\bar{C}c = \bar{H}c\bar{K}c$ is a subgroup of $\bar{G}c$ for all $c \in C$ or $\mathcal{C}$ is an s-subgroup of $\mathcal{G}$ or $\mathcal{H}\mathcal{K}$ is an s-subgroup of $\mathcal{G}$. □

Proposition 3.13. *For any s-group $\mathcal{G}$ and for any pair of s-normal subgroups $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$ we have $\mathcal{H}\mathcal{K}$ is an s-normal subgroup of $\mathcal{G}$ iff $\mathcal{H}\mathcal{K} = \mathcal{K}\mathcal{H}$.*

Proof. $(\Rightarrow)$: Since $\mathcal{H}\mathcal{K}$ is an s-normal subgroup, we have $\mathcal{H}\mathcal{K}$ is an s-subgroup and by previous proposition $\mathcal{H}\mathcal{K} = \mathcal{K}\mathcal{H}$.

$(\Leftarrow)$: Let $\mathcal{H}\mathcal{K} = \mathcal{C}$ and $\mathcal{K}\mathcal{H} = \mathcal{B}$ be as in the previous proposition. Then $\mathcal{C} = \mathcal{B}$ implies $C = H \cap K = K \cap H = B$, $U_C = U_H U_K = U_K U_H = U_B$ and $\bar{C}c = \bar{H}c\bar{K}c = \bar{K}c\bar{H}c = \bar{B}c$ for all $c \in C$. We show that $\mathcal{C}$ is an s-normal subgroup of $\mathcal{G}$ or (i) $C \subseteq G$ (ii) U_C is a normal subgroup of U_G and (iii) $\bar{C}c$ is a normal subgroup of $\bar{G}c$ for all $c \in C$. If C is empty then $\overline{C}$ is empty. Since $U_H U_K = U_C = U_B = U_K U_H$ and U_H, U_K are normal subgroups of U_G, we have $U_C = U_H U_K$ is a normal subgroup of U_G and so $\mathcal{C}$ is an s-normal subgroup of $\mathcal{G}$. So, let C be non-empty.

(i): $C = H \cap K \subseteq G$.

(ii): Since U_H, U_K are normal subgroups of U_G, $U_H U_K = U_K U_H$ we have $U_C = U_H U_K$ is a normal subgroup of U_G.

(iii): Since $\bar{H}c, \bar{K}c$ are normal subgroups of $\bar{G}c$, $\bar{H}c\bar{K}c = \bar{K}c\bar{H}c$ we have $\bar{C}c = \bar{H}c\bar{K}c$ is a normal subgroup of $\bar{G}c$ for all $c \in C$ or $\mathcal{C}$ is an s-normal subgroup of $\mathcal{G}$ or $\mathcal{H}\mathcal{K}$ is an s-normal subgroup of $\mathcal{G}$. □

Proposition 3.14. *For any s-group $\mathcal{G}$ and for any pair of s-subgroups $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H}$ or $\mathcal{K}$ is an s-normal subgroup of $\mathcal{G}$ we have $\mathcal{H}\mathcal{K}$ is an s-subgroup of $\mathcal{G}$.*

Proof. Let $\mathcal{H}\mathcal{K} = \mathcal{C}$. Then $C = H \cap K$, $U_C = U_H U_K$ and $\bar{C}c = \bar{H}c\bar{K}c$ for all $c \in C$. Let $\mathcal{H}$ be an s-normal subgroup of $\mathcal{G}$. We show that $\mathcal{C}$ is an s-subgroup of $\mathcal{G}$ or (i) $C \subseteq G$ (ii) U_C is a subgroup of U_G and (iii) $\bar{C}c$ is a subgroup of $\bar{G}c$ for all

$c \in C$. If $C = \phi$ then $\bar{C} = \phi$. $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$ implies U_H is a normal subgroup of U_G which implies $U_C = U_H U_K$ is a subgroup of U_G or $(\phi, \phi, P(U_C)) = \mathcal{C}$ is an s-subgroup of $\mathcal{G}$. If $C \neq \phi$ then it is enough to show that $\mathcal{HK} = \mathcal{KH}$. Let $\mathcal{KH} = \mathcal{D}$. Then $D = K \cap H$, $U_D = U_K U_H$ and $\bar{D}d = \bar{K}d\bar{H}d$ for all $d \in D$. We show that $\mathcal{C} = \mathcal{D}$ or (i) $C = D$ (ii) $U_C = U_D$ and (iii) $\bar{C}c = \bar{D}c$ for all $c \in C$.

(i): $C = H \cap K = K \cap H = D$.

(ii): Since $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$, U_H is a normal subgroup of U_G which implies $U_H U_K$ is a subgroup of U_G which in turn implies $U_H U_K = U_K U_H$ or $U_C = U_H U_K = U_K U_H = U_D$.

(iii): Again since $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$, $\bar{H}h$ is a normal subgroup of $\bar{G}h$ which implies $\bar{H}h\bar{K}h$ is a subgroup of U_G which in turn implies $\bar{H}h\bar{K}h = \bar{B}h\bar{K}h$ or $\bar{C}h = \bar{H}h\bar{K}h = \bar{K}h\bar{H}h = \bar{D}h$ for all $h \in C$.

Similarly, if $\mathcal{K}$ is an s-normal subgroup of $\mathcal{G}$ then $\mathcal{HK}$ is an s-normal subgroup of $\mathcal{G}$. □

In what follows we want to introduce the notion of s-quotient of an s-group $\mathcal{G}$ by an s-normal subgroup $\mathcal{N}$ of $\mathcal{G}$. Let $\mathcal{N}$ be an s-normal subgroup of an s-group $\mathcal{G}$. Then U_N is a normal subgroup of U_G and $\bar{N}n$ is a normal subgroup of $\bar{G}n$ for all $n \in N$. However, $\frac{\bar{G}n}{\bar{N}n}$ need *not* be a subgroup of $\frac{U_G}{U_N}$ unless $\bar{N}n = U_N$. On the other hand, there is a *natural* induced homomorphism $\phi_n : \frac{\bar{G}n}{\bar{N}n} \to \frac{U_G}{U_N}$ defined by $\phi_n(\bar{N}n\ a) = U_N\ a$ for all $a \in \bar{G}n$ and for all $n \in N$ such that $\phi_n(\frac{\bar{G}n}{\bar{N}n}) = \frac{\bar{G}n\ U_N}{U_N}$ for all $n \in N$. Hence the parameter set for the s-quotient group is defined to be N and for each parameter n in N the image under the parameter map for the s-quotient group is defined to be the image, $\frac{\bar{G}n\ U_N}{U_N}$, of $\frac{\bar{G}n}{\bar{N}n}$ under the induced natural map ϕ_n. Now since image of a subgroup under a homomorphism is a subgroup, $\frac{\bar{G}n\ U_N}{U_N}$ is a subgroup of $\frac{U_G}{U_N}$.

Definition 3.15. *For any s-group $\mathcal{G}$, and for any s-normal subgroup $\mathcal{N}$ of $\mathcal{G}$, the s- quotient group, denoted by $\frac{\mathcal{G}}{\mathcal{N}}$, is defined by $\mathcal{C}$, where $C = N \cap G = N$, $U_C = \frac{U_G}{U_N}$ and $\bar{C}c = \frac{\bar{G}cU_N}{U_N}$ for all $c \in C$.*
Clearly, in the above, if C is empty then $\bar{C}$ is also empty but U_C equals $\frac{U_G}{U_N}$ and $\frac{\mathcal{G}}{\mathcal{N}} = (\phi, \phi, P(\frac{U_G}{U_N}))$.

Proposition 3.16. *For any s-group $\mathcal{G}$, for any s-normal subgroup $\mathcal{A}$ of $\mathcal{G}$ and for any s-subgroup $\mathcal{B}$ of $\mathcal{G}$ such that $\mathcal{A} \subseteq \mathcal{B}$, $\frac{\mathcal{B}}{\mathcal{A}}$ is an s-subgroup of $\frac{\mathcal{G}}{\mathcal{A}}$.*

Proof. Let $\frac{\mathcal{B}}{\mathcal{A}} = \mathcal{C}$. Then $C = B \cap A = A$, $U_C = \frac{U_B}{U_A}$ and $\bar{C}c = \frac{\bar{B}c\ U_A}{U_A}$ for all $c \in C$. Let $\frac{\mathcal{G}}{\mathcal{A}} = \mathcal{D}$. Then $D = G \cap A = A$, $U_D = \frac{U_G}{U_A}$ and $\bar{D}d = \frac{\bar{G}d\ U_A}{U_A}$ for all $d \in D$.

We show that $\mathcal{C}$ is an s-subgroup of $\mathcal{D}$ or (i). $C \subseteq D$ (ii). U_C is a subgroup of U_D and (iii). $\bar{C}c$ is a subgroup of $\bar{D}c$ for all $c \in C$.

(i): Clearly $C = D$.

(ii): Since U_B is a subgroup of U_G, U_A is a normal subgroup of U_G we have U_A is a normal subgroup of U_B, $\frac{U_G}{U_A}$ is a quotient group, $\frac{U_B}{U_A}$ is a quotient group which implies $\frac{U_B}{U_A}$ is a subgroup of $\frac{U_G}{U_A}$ or U_C is a subgroup of U_D.

(iii): Again since U_A is a normal subgroup of U_G, $\bar{B}c, \bar{G}c$ are subgroups of U_G implies $U_A\ \bar{B}c$, $U_A\ \bar{G}c$ are subgroups. U_A is a normal subgroup of U_G, $\bar{B}cU_A \supseteq U_A$, $\bar{G}cU_A \supseteq U_A$, imply U_A is a normal subgroup of both $\bar{B}c\ U_A$, U_A and $\bar{G}c\ U_A$ which implies $\frac{\bar{B}cU_A}{U_A}$ and $\frac{\bar{G}cU_A}{U_A}$ are quotient groups.

Therefore, $\bar{B}c$ is a subgroup of $\bar{G}c$ which implies $\bar{B}c\ U_A$ is a subgroup of $\bar{G}c\ U_A$ which in turn implies $\frac{\bar{B}c\ U_A}{U_A}$ is a subgroup of $\frac{\bar{G}c\ U_A}{U_A}$ or $\bar{C}c$ is a subgroup of $\bar{D}c$ or $\mathcal{C}$ is an s-subgroup of $\mathcal{D}$. □

Proposition 3.17. *For any s-group $\mathcal{G}$ and for any pair of s-normal subgroup $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{G}$ such that $\mathcal{A} \subseteq \mathcal{B}$, $\frac{\mathcal{B}}{\mathcal{A}}$ is an s-normal subgroup of $\frac{\mathcal{G}}{\mathcal{A}}$.*

Proof. Let $\frac{\mathcal{B}}{\mathcal{A}} = \mathcal{C}$. Then $C = B \cap A = A$, $U_C = \frac{U_B}{U_A}$ and $\bar{C}c = \frac{\bar{B}c\ U_A}{U_A}$ for all $c \in C$. Let $\frac{\mathcal{G}}{\mathcal{A}} = \mathcal{D}$. Then $D = G \cap A = A$, $U_D = \frac{U_G}{U_A}$ and $\bar{D}d = \frac{\bar{G}d\ U_A}{U_A}$ for all $d \in D$.

We show that $\mathcal{C}$ is an s-normal subgroup of $\mathcal{D}$ or (i). $C \subseteq D$ (ii). U_C is a normal subgroup of U_D and (iii). $\bar{C}c$ is a normal subgroup of $\bar{D}c$ for all $c \in C$.

(i): Clearly $C = D$.

(ii): Since U_B is a normal subgroup of U_G, $U_A \subseteq U_B$ we have U_A is a normal subgroup of U_B, $\frac{U_G}{U_A}$, $\frac{U_B}{U_A}$ are quotient groups which imply $\frac{U_B}{U_A}$ is a normal subgroup of $\frac{U_G}{U_A}$ or U_C is a normal subgroup of U_D.

(iii): Again since U_A is a normal subgroup of U_G and both $\bar{B}c$ and $\bar{G}c$ are subgroups of U_G we have $U_A\bar{B}c, U_A\bar{G}c$ are subgroups. Now U_A is a normal subgroup of U_G, $\bar{B}c\,U_A \supseteq U_A$, $\bar{G}c\,U_A \supseteq U_A$ imply U_A is a normal subgroup of both $\bar{B}c\,U_A$, U_A and $\bar{G}c\,U_A$ which implies $\frac{\bar{B}c\,U_A}{U_A}$ and $\frac{\bar{G}c\,U_A}{U_A}$ are quotient groups.

Therefore, $\bar{B}c$ is a normal subgroup of $\bar{G}c$ and U_A is a normal subgroup of U_G imply $\bar{B}c\,U_A$ is a normal subgroup of $\bar{G}c\,U_A$ which in turn implies $\frac{\bar{B}c\,U_A}{U_A}$ is a normal subgroup of $\frac{\bar{G}c\,U_A}{U_A}$ or $\bar{C}c$ is a normal subgroup of $\bar{D}c$ or in total $\mathcal{C}$ is an s-normal subgroup of $\mathcal{D}$. □

In what follows we introduce the notion of s-point of an s-group, s-(left) right coset and restriction of an s-set to a parameter subset and use them to obtain equivalent conditions for s-normality.

Definitions and Statements 3.18.

(a). For any s-subset $\mathcal{X}$ of an s-set $\mathcal{A}$, $\mathcal{X}$ is an s-point iff $X = \{x\}$, $U_X = \{\alpha\}$ and $\bar{X}x = \alpha$ and it is denoted by (x, α). In other words, $(x, \alpha) = (\{x\}, \{(x, \alpha)\}, \{\phi, \{\alpha\}\})$.

(b). For any s-subgroup $\mathcal{H}$ of an s-group $\mathcal{G}$ and for any s-point (x, α) in $\mathcal{G}$, the s-left coset of $\mathcal{H}$ by (x, α) denoted by (x, α) $\mathcal{H}$, is defined by $\mathcal{K} = (\{x\}, \{(x, \alpha\ \bar{H}x)\}, P(\alpha U_H))$ if $x \in H$ and $\mathcal{K} = (\phi, \phi, P(\alpha U_H))$ if $x \notin H$.

(c). For any s-subgroup $\mathcal{H}$ of an s-group $\mathcal{G}$ and for any s-point (x, α) in $\mathcal{G}$, the s-right coset of $\mathcal{H}$ by (x, α) denoted by $\mathcal{H}(x, \alpha)$, is defined by $\mathcal{K} = (\{x\}, \{(x, \bar{H}x\ \alpha)\}, P(U_H\alpha))$ if $x \in H$ and $\mathcal{K} = (\phi, \phi, P(U_H\alpha))$ if $x \notin H$.

(d). For any s-set $\mathcal{A}$ and for all $B \subseteq A$, the restriction of $\mathcal{A}$ to B, denoted by $\mathcal{A}|B$, is defined by $\mathcal{C}$ iff $C = B$, $U_C = U_A$, $\bar{C} : C \to P(U_C)$ is given by $\bar{C}b = \bar{A}b$ for all $b \in B$.

Clearly, whenever $\mathcal{K} = (x, \alpha)$ is an s-point, $\mathcal{HK}$ is the same as the s-right coset $\mathcal{H}(x, \alpha)$ and whenever $\mathcal{K} = (x, \alpha)$ is an s-point, $\mathcal{KH}$ is same as the s-left coset $(x, \alpha)\mathcal{H}$.

Lemma 3.19. *For any s-group $\mathcal{G}$, for any $x \in G$ arbitrary but fixed and for any s-point $(x, \alpha) \in \mathcal{G}$, there is a unique $(x, \beta) \in \mathcal{G}$ with respect to (x, e) such that $(x, \alpha)(x, \beta) = (x, e) = (x, \beta)(x, \alpha)$.*

Proof. (1). First observe that, $(x, \alpha)(x, \beta) = (x, (\alpha\beta))$, $(x, \beta)(x, \alpha) = (x, (\beta\alpha))$ and $(x, \alpha) = (x, \beta)$ implies $\alpha = \beta$. Next, $(x, \alpha)(x, \beta) = (x, e) = (x, \beta)(x, \alpha)$ and $(x, \alpha)(x, \gamma) = (x, e) = (x, \gamma)(x, \alpha)$ implies $(x, (\alpha\beta)) = (x, e) = (x, (\beta\gamma))$ and $(x, (\alpha\gamma))$ $= (x, e) = (x, (\gamma\alpha))$ implies $\alpha\beta = e = \beta\alpha$ and $\alpha\gamma = e = \gamma\alpha$ implies $\beta = (\beta e) = \beta(\alpha\gamma) = (\beta\alpha)\gamma = e\gamma = \gamma$ or $\beta = \gamma$. Hence, there is a unique (x, β) for the s-point (x, α) such that $(x, \alpha)(x, \beta) = (x, e) = (x, \beta)(x, \alpha)$.

(2). Second observe that $(x, \alpha)(x, \alpha^{-1}) = (x, \alpha\alpha^{-1}) = (x, e) = (x, \alpha^{-1})(x, \alpha)$ or $(x, \beta) = (x, \alpha^{-1})$. □

Definition 3.20. *For any s-point (x, α) in an s-group $\mathcal{G}$, the unique point (x, β) such that $(x, \alpha)(x, \beta) = (x, e) = (x, \beta)(x, \alpha)$, is said to be the s-inverse of (x, α) with respect to (x, e) and is denoted by (x, α^{-1}).*

Notice that $(x, \alpha)^{-1} = (x, \alpha^{-1})$, by (2) of the above lemma.

Proposition 3.21. *For any s-subset $\mathcal{H}$ of an s-group $\mathcal{G}$, $\mathcal{H}$ is an s-subgroup of $\mathcal{G}$ iff $(x, \alpha)\mathcal{H}(x, \alpha)^{-1}$ is an s-subgroup of $\mathcal{G}$ for all $(x, \alpha) \in \mathcal{G}$.*

Proof. $(\Rightarrow)$: Let $(x,\alpha)\mathcal{H}(x,\alpha)^{-1} = \mathcal{K}$. Then $\mathcal{K} = (\{x\}, \{(x, \alpha\bar{H}x\alpha^{-1})\}, P(\alpha U_H \alpha^{-1}))$ if $x \in H$, $\mathcal{K} = (\phi, \phi, P(\alpha U_N \alpha^{-1}))$ if $x \notin H$. We show that $\mathcal{K}$ is an s-subgroup of $\mathcal{G}$ or (i) $K \subseteq G$ (ii) U_K is a subgroup of U_G and (iii) $\bar{K}x$ is a subgroup of $\bar{G}x$ for all $x \in K$.

Case 1: If $x \in H$. Then

(i): $K = \{x\} \subseteq G$.

(ii): Since $\alpha \in U_G$, U_G is a group, U_H is a subgroup of U_G we have $U_K = \alpha U_H \alpha^{-1}$ is a subgroup of U_G.

(iii): Again since $\alpha \in U_G$, U_G is a group, $\bar{H}x$ is a subgroup of $\bar{G}x$ for all $x \in H$ we have $\bar{K}x = \alpha\bar{H}x\alpha^{-1}$ is a subgroup of $\bar{G}x$ or in total $\mathcal{K}$ is an s-subgroup of $\mathcal{G}$.

Case 2: If $x \notin H$. Then

(i): $K = \phi \subseteq G$

(ii): Again as in case1(ii) above $U_K = \alpha U_H \alpha^{-1}$ is a subgroup of U_G

(iii): As $K = \phi$, $\bar{K} = \phi$ or in total $\mathcal{K}$ is an s-subgroup of $\mathcal{G}$.

$(\Leftarrow)$: Observe that, (i) $\mathcal{H} \subseteq \mathcal{G}$ implies $H \subseteq G$ (ii) for all $x \in H$, $(x,e)\mathcal{H}(x,e)^{-1} = (x,e)\mathcal{H}(x,e^{-1})$ or $(\{x\}, \{(x, \bar{H}x)\}, P(U_H))$ is a subgroup of $\mathcal{G}$ implies U_H is a subgroup of U_G and $\bar{H}x$ is a subgroup of $\bar{G}x$ or $\mathcal{H}$ is an s-subgroup of $\mathcal{G}$. $\square$

Proposition 3.22. *For any s-subset $\mathcal{N}$ of an s-group $\mathcal{G}$, $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$ iff for all $(x,\alpha) \in \mathcal{G}$, $(x,\alpha)\mathcal{N}(x,\alpha)^{-1} = \mathcal{N}|\{x\}$.*

Proof. $(\Rightarrow)$: Let $\mathcal{N}|\{x\} = \mathcal{A}$. Then $\mathcal{A} = (\{x\}, \{(x, \bar{N}x)\}, P(U_N))$ if $x \in N$, $\mathcal{A} = (\phi, \phi, P(U_N))$ if $x \notin N$. Let $(x,\alpha)\mathcal{N} = \mathcal{B}$. Then $\mathcal{B} = (\{x\}, \{(x, \alpha\bar{N}x)\}, P(\alpha U_N))$ if $x \in N$, $\mathcal{B} = (\phi, \phi, P(\alpha U_N))$ if $x \notin N$. Let $\mathcal{B}(x,\alpha)^{-1} = \mathcal{B}(x,\alpha^{-1}) = \mathcal{C}$. Then $\mathcal{C} = (\{x\}, \{(x, \bar{B}x\alpha^{-1})\}, P(U_B\alpha^{-1}))$ if $x \in B$, $\mathcal{C} = (\phi, \phi, P(U_B\alpha^{-1}))$ if $x \notin B$.

We show that $\mathcal{A} = \mathcal{C}$ or (i). $C = A$ (ii). $U_C = U_A$ and (iii). $\bar{C}x = \bar{A}x$.

If $x \notin N$ then $C = \phi = A$, $\bar{C} = \phi = \bar{A}$. Further, $U_C = U_B\alpha^{-1} = \alpha U_N \alpha^{-1} = U_N = U_A$, where the third equality is due to U_N being a normal subgroup of U_G and $\alpha \in U_G$, which implies $\mathcal{A} = \mathcal{C}$. So, let $x \in N$.

(i): $C = \{x\} = A$

(ii): $U_C = U_A$ as in the case when $x \notin N$ above.

(iii): Let $x \in C = A$. Then $\bar{C}x = \bar{B}x\alpha^{-1} = \alpha\bar{N}x\alpha^{-1} = \bar{N}x = \bar{A}x$, where the third equality is due to $\bar{N}x$ being a normal subgroup of U_G and $\alpha \in U_G$ or in total $\mathcal{A} = \mathcal{C}$.

$(\Leftarrow)$: We show that $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$ or (i) $N \subseteq G$ (ii) U_N is a normal subgroup of U_G and (iii) $\bar{N}x$ is a normal subgroup of U_G for all $x \in N$.

Since $\mathcal{N} \subseteq \mathcal{G}$ always we have $N \subseteq G$ for all $x \in N$ and for all $\alpha \in U_G$, $\mathcal{A} = \mathcal{C}$ as above implies $\bar{N}x = \alpha\bar{N}x\alpha^{-1}$ and $U_N = \alpha U_N \alpha^{-1}$ which imply $\bar{N}x$ is a normal subgroup of U_G and U_N is a normal subgroup of U_G which in turn implies $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$. $\square$

In what follows we show that if the condition on the RHS is *not* true for all $(x,\alpha) \in \mathcal{G}$ then the conclusion on the LHS is *not* true. Conversely, if $\mathcal{N}$ *not* a strong s-normal subgroup then there is an $(x,\alpha) \in \mathcal{G}$ such that the condition on the RHS is *not* true.

Example 3.23. *Let $\mathcal{N} = (\{x\}, \{(x, H_2)\}, P(A_4))$, $\mathcal{G} = (\{x\}, \{(x, K_4)\}, P(A_4)))$. Then $\mathcal{N}$ is an s-normal subgroup of $\mathcal{G}$ because $N \subseteq G$, U_N is normal subgroup of U_G and $\bar{N}x$ is a normal subgroup of $\bar{G}x$ but $\mathcal{N}$ is not strong s-normal subgroup of $\mathcal{G}$ because $\bar{N}x$ is not normal subgroup of U_G. Let $\alpha \in A_4$ be (123) then $\alpha H_2 \alpha^{-1} = \{(123), (12)(34), (321), e\} = \{e, (14)(23)\} \neq H_2$. Clearly, $(x,\alpha)\mathcal{N}(x,\alpha)^{-1} \neq \mathcal{N}|\{x\}$.*

Proposition 3.24. *For any s-subset $\mathcal{N}$ of an s-group $\mathcal{G}$, $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$ iff for all $(x, \alpha) \in \mathcal{G}$, $\mathcal{N}(x, \alpha)$ $= (x, \alpha)\mathcal{N}$.*

Proof. $(\Rightarrow)$: Let $\mathcal{N}(x, \alpha) = \mathcal{A}$. Then $\mathcal{A} = (\{x\}, \{(x, \bar{N}x\alpha)\}, P(U_N\alpha))$ if $x \in N$, $\mathcal{A} = (\phi, \phi, P(U_N\alpha))$ if $x \notin N$. Let $(x, \alpha)\mathcal{N}$ $= \mathcal{B}$. Then $\mathcal{B} = (\{x\}, \{(x, \alpha\bar{N}x)\}, P(\alpha U_N))$ if $x \in N$, $\mathcal{B} = (\phi, \phi, P(\alpha U_N))$ if $x \notin N$.

We show that $\mathcal{A} = \mathcal{B}$ or (i). $A = B$ (ii). $U_A = U_B$ and (iii). $\bar{A}x = \bar{B}x$ for all $x \in N$. If $x \notin N$ then $A = \phi = B$, $\bar{A} = \phi = \bar{B}$. Further, $U_A = \alpha U_N = U_N\alpha = U_B$, where the second equality is due to U_N being a normal subgroup of U_G and $\alpha \in U_G$, which implies $\mathcal{A} = \mathcal{B}$. So, let $x \in N$.

(i): $A = \{x\} = B$.

(ii): $U_A = U_B$ as in the case when $x \notin N$ above.

(iii): Let $x \in A = B$. Then $\bar{A}x = \bar{N}x\alpha = \alpha\bar{N}x = \bar{B}x$, where the second equality is due to $\bar{N}x$ being a normal subgroup of U_G and $\alpha \in U_G$ as $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$ or in total $\mathcal{A} = \mathcal{B}$.

$(\Leftarrow)$: Let $\mathcal{N}(x, \alpha) = (x, \alpha)\mathcal{N}$ for all $(x, \alpha) \in \mathcal{G}$ or $\mathcal{A} = \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are as above. Any way $N \subseteq G$ as $\mathcal{N} \subseteq \mathcal{G}$. Let $x \in N$ and $\alpha \in U_G$ be arbitrary. Then $\mathcal{A} = \mathcal{B}$ implies $(\{x\}, \{(x, \bar{N}x\alpha)\}, P(U_N\alpha)) = (\{x\}, \{(x, \alpha\bar{N}x)\}, P(\alpha U_N))$ for all $(x, \alpha) \in \mathcal{G}$ which implies $P(U_N\alpha) = P(\alpha U_N)$ or $U_N\alpha = \alpha U_N$ and $\bar{N}x\alpha = \alpha\bar{N}x$ which in turn implies U_N is a normal subgroup of U_G and $\bar{N}x$ is a normal subgroup of U_G or $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$. $\square$

In what follows we show that if the condition on the RHS is *not* true for all $(x, \alpha) \in \mathcal{G}$ then the conclusion on the LHS is *not* true. Conversely, if $\mathcal{N}$ *not* a strong s-normal subgroup then there is an $(x, \alpha) \in \mathcal{G}$ such that the condition on the RHS is *not* true.

Example 3.25. *Let $\mathcal{N}$, $\mathcal{G}$ be as in the example preceding this theorem. Then $\mathcal{N}$ is an s-normal subgroup but not a strong s-normal subgroup of $\mathcal{G}$. Let $\alpha \in A_4$ be (123). Then $(x, \alpha)\mathcal{N} = (\{x\}, \{(x, \alpha H_2)\}, P(\alpha A_4))$, $\mathcal{N}(x, \alpha)$ $= (\{x\}, \{(x, H_2\alpha)\}, P(A_4\alpha))$. Now $\alpha H_2 = (123)H_2 = \{(123), (123)(12)(34)\} = \{(123), (413)\}$, $H_2\alpha = H_2(123) =$ $\{(123), (12)(34)(123)\} = \{(123), (243)\}$. Clearly $(x, \alpha)\mathcal{N} \neq \mathcal{N}(x, \alpha)$.*

For any s-subset $\mathcal{N}$ of an s-group $\mathcal{G}$, clearly, (a). When N is ϕ, (i) $\mathcal{N}$ is a (strong) s-normal subgroup of $\mathcal{G}$ iff U_N is a normal subgroup of U_G (ii) $\mathcal{N}$ is an s-subgroup of $\mathcal{G}$ iff U_N is a subgroup of U_G.

(b). For a whole subset $\mathcal{N}$ of an s-group $\mathcal{G}$, $\mathcal{N}$ is strong s-normal iff it is s-normal.

Proposition 3.26. *For any s-subgroup $\mathcal{N}$ of an s-group $\mathcal{G}$, $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$ iff for all $(x, \alpha)(y, \beta) \in \mathcal{G}$, $\mathcal{N}(x, \alpha)(y, \beta) = \mathcal{N}(x, \alpha)\mathcal{N}(y, \beta)$.*

Proof. $(\Rightarrow)$: Let $\mathcal{N}(x, \alpha) = \mathcal{A}$. Then $\mathcal{A} = (\{x\}, \{(x, \bar{N}x\alpha)\}, P(U_N\alpha))$ if $x \in N$, $\mathcal{A} = (\phi, \phi, P(U_N\alpha))$ if $x \notin N$. Let $\mathcal{A}(y, \beta) = \mathcal{B}$. Then $\mathcal{B} = (\{x\}, \{(x, \bar{A}x\beta)\}, P(U_A\beta))$ if $y = x$ and $\mathcal{B} = (\phi, \phi, P(U_A\beta))$ if $y \neq x$. Let $\mathcal{N}(y, \beta) = \mathcal{C}$. Then $\mathcal{C} =$ $(\{y\}, \{(y, \bar{N}y\beta)\}, P(U_N\beta))$ if $y \in N$ and $\mathcal{C} = (\phi, \phi, P(U_N\beta))$ if $y \notin N$. Let $\mathcal{AC} = \mathcal{D}$. Then $\mathcal{D} = (\{x\}, \{(x, \bar{A}x\bar{C}x)\}, P(U_AU_C))$ if $y = x$ and $\mathcal{D} = (\phi, \phi, P(U_AU_C))$ if $y \neq x$.

We show that $\mathcal{B} = \mathcal{D}$ or (i). $D = B$ (ii). $U_D = U_B$ and (iii). $\bar{D}x = \bar{B}x$ for all $x \in N$.

If $x = y$ and $x \in N$ then

(i): $D = A \cap C = \{x\} = B$.

(ii): $U_D = U_AU_C = (U_N\alpha)(U_N\beta) = U_N(\alpha\beta) = (U_N\alpha)\beta = U_A\beta = U_B$, where the third equality is due to U_N being a normal subgroup of U_G and $\alpha, \beta \in U_G$

(iii): Let $x \in D = B$. Then $\bar{D}x = \bar{A}x\bar{C}x = (\bar{N}x\alpha)(\bar{N}y\beta) = \bar{N}x(\alpha\bar{N}x)\beta = \bar{N}x(\bar{N}x\alpha)\beta = \bar{N}x\bar{N}x\alpha\beta = \bar{N}x\alpha\beta = (\bar{N}x\alpha)\beta$ $= \bar{A}x\beta = \bar{B}x$ where the fourth equality is due to $\bar{N}x$ being a normal subgroup of U_G and the sixth equality is due to $\bar{N}x$ being a subgroup of U_G and $\alpha, \beta \in U_G$ or $\mathcal{B} = \mathcal{D}$.

In all other cases $D = \phi = B$, $U_D = U_B$, as above $\bar{D} = \phi = \bar{B}$ or $\mathcal{B} = \mathcal{D}$.

($\Leftarrow$:) We show that $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$ or $N \subseteq G$, both U_N and $\bar{N}x$ are normal subgroups of U_G for all $x \in N$. If $N = \phi$ then U_N is a normal subgroup of U_G follows from (a)(i) of remarks preceding this theorem and the fact that U_N is a subgroup of U_G. So, let $N \neq \phi$. Any way $N \subseteq G$ as $\mathcal{N} \subseteq \mathcal{G}$ and U_N is a normal subgroup of U_G follows as in the case $N = \phi$ above. Let $x \in N$, $\alpha, \beta \in U_G$ arbitrary and $\mathcal{B}$, $\mathcal{D}$ be as above. Then $\mathcal{B} = \mathcal{D}$ implies $\bar{B}x = \bar{D}x$ which implies $\bar{N}x(\alpha\beta) = (\bar{N}x\alpha)(\bar{N}x\beta)$ which in turn implies $\bar{N}x$ is a normal subgroup of U_G or $\mathcal{N}$ is a strong s-normal subgroup of $\mathcal{G}$. $\square$

In what follows we show that if the condition on the RHS is *not* true for all $(x, \alpha) \in \mathcal{G}$ then the conclusion on the LHS is *not* true. Conversely, if $\mathcal{N}$ *not* a strong s-normal subgroup then there is an $(x, \alpha) \in \mathcal{G}$ such that the condition on the RHS is *not* true.

Example 3.27. *Let $\mathcal{N}$, $\mathcal{G}$ be as in the example preceding this theorem. Then $\mathcal{N}$ is an s-normal subgroup but not a strong s-normal subgroup of $\mathcal{G}$.*

Let $\alpha \in A_4$ be (123) then $\mathcal{N}(x,\alpha)\mathcal{N}(x,\beta) = (\{x\}, \{(x, \{(243), (123)\})\}, P(A_4))$, $\mathcal{N}(x,\alpha)(x,\beta) = (\{x\}, \{(x, \{(21)(34), e\})\}, P(A_4))$. Clearly, $\mathcal{N}(x,\alpha)\mathcal{N}(x,\beta) \neq \mathcal{N}(x,\alpha)(x,\beta)$.

4. Conclusion

In this paper we introduced the notions of generalized soft group, generalized soft (normal) subgroup, generalized soft quotient group etc., generalizing the existing corresponding notions of a soft group over a group and showed that several of the crisp theoretic results naturally extended to these new objects too.

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