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Non-self Mapping in Metric Space of Hyperbolic Type

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Abstract: In this paper, we prove the fixed point theorem in a metric space of hyperbolic type for a pair of weakly compatible non-self mappings satisfying the generalized contraction.

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1. Introduction and Preliminaries

Many results have been derived in fixed point theory for self-mappings in metric space and Banach spaces. But in non-self mappings results are minimum. The author Kirk [1] had replaced Kranoselskii's result by extending metric space to metric space of hyperbolic type. The authors Assad [2] and Assad kirk [3] only first initiated non-self mappings in a fixed point theory for multivalued non-self mappings in metric space.

In cone metric space many authors [4–7] have derived results in non-self mappings in fixed point theory. But very few authors [8, 9] have derived fixed point results in metric space of hyperbolic type. Here we proved the fixed point theorem in a metric space of hyperbolic type for a pair of weakly compatible non-self mappings satisfying the generalized contraction.

Definition 1.1 ([14])**.** *Let (X, d) be a metric space that contains the metric segments of family L such that*

(1). $Seg[x, y]$ of L has two points $x, y \in X$ which are the end points.

(2). If $r \in Seg[x, y]$ and $p, q, t \in X$ satisfying $d(x, r) = \gamma d(x, y)$ for some $\gamma \in [0, 1]$ then

$$d(p, r) \leq (1-\gamma)d\,(p, x) + \gamma d(p, y) \tag{1}$$

This type of space is called metric space of hyperbolic type.

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2. Main Result

Theorem 2.1. *If (X, d) is a metric space of hyperbolic type and M a non-empty closed subset of X such that for each ∂M be nonempty and let the two non-self mappings be $T : M \to X$ and $f : M \cap T(M) \to X$ satisfying the condition*

$$d(fa, fb) \leq \gamma \left\{ d(Ta, Tb), \frac{d(Ta, fa)}{2}, \frac{d(Tb, fb)}{2} \right\} + \mu \{ d(fa, Tb) + d(fb, Ta) \} \tag{2}$$

for every a,b in M and γ. μ are positive real numbers such that $(\gamma + 2\mu) < 1$. If

(1). $\partial M \subseteq TM$, $fM \cap M \subseteq TM$,

(2). $Ta \in \partial M$ implies $fa \in M$,

(3). $fM \cap M$ is complete.

Then there exist a coincidence point z of f and T. Moreover if f and T are weakly compatible, then z is the unique common fixed point in M.

Proof. We construct the sequence $\{a_n\}$ and $\{c_n\}$ in M and a sequence $\{b_n\}$ in $fM \subset X$. Let c_0=x. Because $c_0 \in \partial M$ there exist $a_0 \in M$ such that $c_0 = Ta_0 \in \partial M$. From second condition $fa_0 \in M$. Let us choose $b_1 = fa_0$ with $b_1 \in fM \subset X$ which implies that $fa_0 \in fM \cap M \subset TM$. We set here $b_1 = fa_0$ and we choose $a_1 \in M$ such that $Ta_1 = fa_0$. Therefore we get $c_1 = Ta_1 = fa_0 = b_1$. So, we get $b_2 = fa_1$. Since $b_2 \in fM \cap M$, we get $b_2 \in TM$ from second condition. Let $a_1 \in M$ with $c_1 = Ta_1 \in \partial M$ such that $c_2 = Ta_2 = fa_1 = b_2$. If $fa_1 = b_2 \notin M$, then there exist $c_2 \in \partial M$ such that $c_2 \in seg[b_1, b_2]$. From first condition since $a_2 \in M$ we have $Ta_2 = c_2$. Therefore $c_2 \in \partial M \cap seg[b_1, b_2]$. Similarly on choosing $b_3 \in fM \cap M$, and from second condition $b_3 \in TM$ and let $a_2 \in M$ such that $Ta_3 = b_3 = fa_2$. In the manner of continuing this process, we can construct three sequences $\{a_n\} \subseteq M, \{c_n\} \subseteq M$ and $\{b_n\} \subseteq fM \subset X$ such that

(a). $b_n = fa_{n-1}$.

(b). c_n= Ta_n.

(c). c_n= b_n if and only if $b_n \in M$

(d). If $c_n \neq b_n$, whenever $b_n \notin M$ and then from equation (2), $c_n \in \partial M$ such that $c_n \in \partial M \cap seg(fa_{n-2}, fa_{n-1})$.

If $c_n \neq b_n$, then $c_n \in \partial M$ and we get $c_{n+1} = b_{n+1}$ and $c_{n-1} = b_{n-1} \in M$ from the above conditions. If $c_{n-1} \in \partial M$, then we get $c_n = b_n \in M$. From all above conditions we get three possibilities.

(P.1) $c_n = b_n \in M$ and $c_{n+1} = b_{n+1}$.

(P.2) $c_n = b_n \in M$ and $c_{n+1} \neq b_{n+1}$.

(P.3) If $c_n \neq b_n \in M$ then we get $c_n \in \partial M \cap seg[fx_{n-2}, fx_{n-1})$.

Now we discuss three cases.

Case 1: Let $c_n = b_n \in M$ and $c_{n+1} = b_{n+1}$. From (2) we get

$$\begin{aligned} d(c_n, c_{n+1}) &= d(b_n, b_{n+1}) \\ &= d(fa_{n-1}, fa_n) \end{aligned}$$

$$\begin{aligned}
&\le \gamma \left\{ d\left(Ta_{n-1}, Ta_n\right), \frac{d(Ta_{n-1}, fa_{n-1})}{2}, \frac{d(Ta_n, fa_n)}{2} \right\} + \mu \left\{ d\left(fa_{n-1}, Ta_n\right) + d(fa_n, Ta_{n-1})\right)\} \\
&= \gamma \left\{ d(c_{n-1}, c_n), \frac{d(c_{n-1}, b_n)}{2}, \frac{d(c_n, b_{n+1})}{2} \right\} + \mu \left\{ d\left(b_n, c_n\right) + d(b_{n+1}, c_{n-1})\right)\} \\
&= \gamma \left\{ d(c_{n-1}, c_n), \frac{d(c_{n-1}, c_n)}{2}, \frac{d(c_n, c_{n+1})}{2} \right\} + \mu \left\{ d\left(c_n, c_n\right) + d(c_{n+1}, c_{n-1})\right)\} \\
&= \gamma \left\{ d(c_{n-1}, c_n), \frac{d(c_{n-1}, c_n)}{2}, \frac{d(c_n, c_{n+1})}{2} \right\} + \mu \left\{ d\left(c_{n-1}, c_n\right) + d(c_n, c_{n+1})\right)\}
\end{aligned}$$

From above atleast one of the following cases holds:

(1). $d(c_n, c_{n+1}) \le \gamma d(c_{n-1}, c_n) + \mu \left\{ d\left(c_{n-1}, c_n\right) + d(c_n, c_{n+1})\right\},$

$$d(c_n, c_{n+1}) \le \frac{\gamma + \mu}{1 - \mu} d\left(c_{n-1}, c_n\right).$$

(2). $d(c_n, c_{n+1}) \le \gamma \frac{d(c_{n-1}, c_n)}{2} + \mu \left\{ d\left(c_{n-1}, c_n\right) + d(c_n, c_{n+1})\right\},$

$$d(c_n, c_{n+1}) \le \frac{(\frac{\gamma}{2} + \mu)}{(1 - \mu)} d\left(c_{n-1}, c_n\right).$$

(3). $d(c_n, c_{n+1}) \le \gamma \frac{d\left(c_n, c_{n+1}\right)}{2} + \mu \left\{ d\left(c_{n-1}, c_n\right) + d(c_n, c_{n+1})\right\},$

$$d(c_n, c_{n+1}) \le \frac{\mu}{(1 - \frac{\gamma}{2} - \mu)} d\left(c_{n-1}, c_n\right)..$$

From all the above cases it follows that,

$$d(c_n, c_{n+1}) \le kd\left(c_{n-1}, c_n\right)$$

where, $k = \max\{\frac{\gamma+\mu}{1-\mu}, \frac{(\frac{\gamma}{2}+\mu)}{(1-\mu)}, \frac{\mu}{(1-\frac{\gamma}{2}-\mu)}\}$.

Case 2: Let $c_n = b_n \in M$ but $c_{n+1} \ne b_{n+1}$. Then we have $c_{n+1} \in \partial M \cap seg[b_n, b_{n+1}]$. From equation (1) with $p = b$ we obtain

$$d(b, c) \le (1 - \gamma)\, d(a, b)$$

Also,

$$\begin{aligned}
d\left(a, b\right) &\le d\left(a, c\right) + d(c, b) \\
&\le \gamma d\left(a, b\right) + (1 - \gamma)\, d(a, b) \\
&= d(a, b)
\end{aligned}$$

Hence c $\in seg[a, b]$ implies $d(a, c) + d(c, b) = d(a, b)$. Since $c_{n+1} \in seg\left[b_n,\ b_{n+1}\right] = seg[c_n, b_{n+1}]$, we have

$$\begin{aligned}
d(c_n, c_{n+1}) &= d(b_n, c_{n+1}) \\
&= d(b_n, b_{n+1}) - d(c_{n+1}, b_{n+1}) \\
&\le d(b_n, b_{n+1})
\end{aligned}$$

From Case 1, we obtain, $d(b_n, b_{n+1}) \le kd\left(c_{n-1}, c_n\right)$ which implies that,

$$d(c_n, c_{n+1}) \le kd\left(c_{n-1}, c_n\right).$$

Case 3: Let $c_n \ne b_n$. Then $c_n \in \partial M \cap seg[fa_{n-2}, fa_{n-1}]$ that means $c_n \in \partial M \cap seg[b_{n-1}, b_n]$. From the above assumptions in possibilities, we have $c_{n+1} = b_{n+1}$ and $c_{n-1} = b_{n-1}$. Therefore we get,

$$d(c_n, c_{n+1}) = d(c_n, b_{n+1})$$

$$\begin{aligned} &\leq d(c_n, b_n) + d(b_n, b_{n+1}) \\ &= d(c_{n-1}, b_n) - d(c_n, c_{n-1}) + d(b_n, b_{n+1}) \\ &= d(b_{n-1}, b_n) - d(c_n, c_{n-1}) + d(b_n, b_{n+1}) \end{aligned} \tag{3}$$

We need to find $d(b_{n-1}, b_n)$ and $d(b_n, b_{n+1})$. Since $c_{n-1} = b_{n-1}$ we have that, $d(b_{n-1}, b_n) \leq kd(c_{n-2}, c_{n-1})$ from Case 2. Also,

$$\begin{aligned} d(b_n, b_{n+1}) &= d(fa_{n-1}, fa_n) \\ &\leq \gamma \left\{ d(Ta_{n-1}, Ta_n), \frac{d(Ta_{n-1}, fa_{n-1})}{2}, \frac{d(Ta_n, fa_n)}{2} \right\} + \mu \{ d(fa_{n-1}, Ta_n) + d(fa_n, Ta_{n-1})) \} \\ &= \gamma \left\{ d(c_{n-1}, c_n), \frac{d(c_{n-1}, b_n)}{2}, \frac{d(c_n, b_{n+1})}{2} \right\} + \mu \{ d(b_n, c_n) + d(b_{n+1}, c_{n-1})) \} \\ &= \gamma \left\{ d(c_{n-1}, c_n), \frac{d(c_{n-1}, b_n)}{2}, \frac{d(c_n, c_{n+1})}{2} \right\} + \mu \{ d(b_n, c_n) + d(c_{n+1}, c_{n-1})) \} \\ &= \gamma \left\{ d(c_{n-1}, c_n), \frac{d(b_{n-1}, b_n)}{2}, \frac{d(c_n, c_{n+1})}{2} \right\} + \mu \{ d(c_{n-1}, c_n) + d(c_n, c_{n+1})) \} \end{aligned}$$

Again the following cases hold:

(1). $d(b_n, b_{n+1}) \leq (\gamma + \mu) d((c_{n-1}, c_n) + \mu d(c_n, c_{n+1})$

(2). $$\begin{aligned} d((b_n, b_{n+1}) &\leq \gamma \left(\frac{d(b_{n-1}, b_n)}{2} \right) + \mu \{ d(c_{n-1}, c_n) + d(c_n, c_{n+1})) \} \\ &\leq \frac{\gamma k}{2} d(c_{n-2}, c_{n-1}) + \mu \{ d(c_{n-1}, c_n) + d(c_n, c_{n+1})) \} \end{aligned}$$

(3). $d((b_n, b_{n+1}) \leq \gamma \left\{ \frac{d(c_n, c_{n+1})}{2} \right\} + \mu \{ d(c_{n-1}, c_n) + d(c_n, c_{n+1})) \}$

Substituting all the above in (3) we get three cases.

(4). $$\begin{aligned} d(c_n, c_{n+1}) &\leq kd(c_{n-2}, c_{n-1}) - d(c_{n-1}, c_n) + (\gamma + \mu) d((c_{n-1}, c_n) + \mu d(c_n, c_{n+1}) \\ d(c_n, c_{n+1})(1 - \mu) &\leq (\gamma + \mu - 1) d((c_{n-1}, c_n) + kd(c_{n-2}, c_{n-1}) \\ d(c_n, c_{n+1})(1 - \mu) &\leq kd(c_{n-2}, c_{n-1}) \\ d(c_n, c_{n+1}) &\leq \frac{k}{1 - \mu} d(c_{n-2}, c_{n-1}) \\ d(c_n, c_{n+1}) &\leq kd(c_{n-2}, c_{n-1}) \end{aligned}$$

(5). $$\begin{aligned} d(c_n, c_{n+1}) &\leq kd(c_{n-2}, c_{n-1}) - d(c_{n-1}, c_n) + \frac{\gamma k}{2} d(c_{n-2}, c_{n-1}) + \mu \{ d(c_{n-1}, c_n) + d(c_n, c_{n+1})) \} \\ d(c_n, c_{n+1})(1 - \mu) &\leq d(c_{n-2}, c_{n-1}) \left(k + \frac{\gamma k}{2} \right) + d(c_{n-1}, c_n)(\mu - 1) \\ &\leq d(c_{n-2}, c_{n-1}) \frac{(k + \frac{\gamma k}{2})}{1 - \mu} \\ d(c_n, c_{n+1}) &\leq kd(c_{n-2}, c_{n-1}) \end{aligned}$$

(6). $$\begin{aligned} d(c_n, c_{n+1}) &\leq kd(c_{n-2}, c_{n-1}) - d(c_{n-1}, c_n) + \gamma \left\{ \frac{d(c_n, c_{n+1})}{2} \right\} + \mu \{ d(c_{n-1}, c_n) + d(c_n, c_{n+1})) \} \\ d(c_n, c_{n+1})(1 - \mu - \frac{\gamma}{2}) &\leq kd(c_{n-2}, c_{n-1}) + d(c_{n-1}, c_n)(\mu - 1) \\ d(c_n, c_{n+1}) &\leq \frac{k}{1 - \mu - \frac{\gamma}{2}} d(c_{n-2}, c_{n-1}) \\ d(c_n, c_{n+1}) &\leq kd(c_{n-2}, c_{n-1}) \end{aligned}$$

Thus in all the cases, we get $d(c_n, c_{n+1}) \leq kd(c_{n-2}, c_{n-1})$. That is, $d(c_n, c_{n+1}) \leq k\theta_n$, where $\theta_n \in \{d((c_{n-2}, c_{n-1}), d(c_{n-1}, c_n)\}$. By Assad and Kirk [3] procedure we get, by induction for $n > 1$,

$$d(c_n, c_{n+1}) \leq k^{\frac{n-1}{2}}\theta_2 \tag{4}$$

where $\theta_2 \in \{d((c_0, c_1), d(c_1, c_2)\}$. For $n > m$ and using (4) and the triangle inequality we have

$$\begin{aligned} d(c_n, c_m) &\leq d(c_n, c_{n-1}) + d(c_{n-1}, c_{n-2}) + \ldots + d(c_{m+1}, c_m) \\ &\leq \left(k^{\frac{n-1}{2}} + k^{\frac{n-2}{2}} + \cdots + k^{\frac{m-1}{2}}\right)\theta_2 \\ &\leq \frac{\sqrt{k^{m-1}}}{1-\sqrt{k}}.\theta_2 \to 0 \ as \ m \to \infty. \end{aligned}$$

This shows that it is Cauchy sequence. Since $c_n = fa_{n-1} \in fM \cap M$ is complete, there is some $c \in fM \cap M$ such that $c_n \to c$. Let h in M be such that $Th = c$. From the construction of $\{c_n\}$, there is a subsequence $\{c_{nk}\}$ such that $c_{nk} = b_{nk} = fa_{nk-1}$ and $fa_{nk-1} \to c$. We will prove that $fh = c$.

$$\begin{aligned} d(fh, c) &\leq d(fh, fa_{nk-1}) + d(fa_{nk-1}, c) \\ &\leq \gamma\left\{d(Th, Ta_{nk-1}), \frac{d(Th, fh)}{2}, \frac{d(Ta_{nk-1}, fa_{nk-1})}{2}\right\} + \mu\left\{d(fh, Ta_{nk-1}) + d(fa_{nk-1}, Th))\right\} \\ &\leq \gamma\left\{d(c, c), \frac{d(c, fh)}{2}, \frac{d(c, fh)}{2}\right\} + \mu\left\{d(c, fh) + d(c, fh))\right\} \\ &\leq \gamma\left\{0, \frac{d(c, fh)}{2}, \frac{d(c, fh)}{2}\right\} + \mu\left\{2d(c, fh)\right\} \end{aligned}$$

From which we get in all cases

$$d(fh, c) \leq (\gamma + \mu)d(c, fh).$$

Since $\gamma + \mu < 1$ we get $d(fh, c) = 0$. Hence $c = fh$. If T and f are weakly compatible, then we have $c = fh = Th$ which implies $fc = fTh = Tfh = Tc$. We next prove that $c = fc = Tc$. Suppose $c \neq fc$ then using (2) we obtain

$$\begin{aligned} d(fc, c) &= d(fc, fh) \\ &\leq \gamma\left\{d(Tc, Th), \frac{d(Tc, fc)}{2}, \frac{d(Th, fh)}{2}\right\} + \mu\left\{d(fc, Th) + d(fh, Tc))\right\} \\ &\leq \gamma\left\{d(c, c), \frac{d(c, fc)}{2}, \frac{d(c, c)}{2}\right\} + \mu\left\{d(c, fc) + d(c, c))\right\} \\ d(fc, c) &\leq \left(\frac{\gamma}{2} + \mu\right)d(c, fc) \end{aligned}$$

which is a contradiction. This implies that $c = fc$. Therefore we get $c = fc = Tc$. Thus T and f have a common fixed point and it is also unique. □

Corollary 2.2. *Let (X, d) be metric space of hyperbolic type, M a non-empty closed subset of X and ∂M the boundary of M. Let ∂M be nonempty such that f: $M \to M$ satisfies the condition*

$$d(fa, fb) \leq \gamma\left\{d(a, b), \frac{d(a, fa)}{2}, \frac{d(b, fb)}{2}\right\} + \mu\left\{d(fa, b) + d(fb, a))\right\} \tag{5}$$

for every a,b in M and γ, μ are positive real numbers such that $(\gamma + 2\mu) < 1$ and f has the additional property that for each $x \in \partial M$ and $fx \in M$. Then f has a unique fixed point.

Corollary 2.3. *If (X, d) is a metric space of hyperbolic type and M a non-empty closed subset of X such that for each ∂M be nonempty and let the two non-self mappings be $T : M \to X$ and $f : M \cap T(M) \to X$ satisfying the condition*

$$d(fa, fb) \leq \gamma \left\{ d(Ta, Tb), \frac{d(Tb, fb)}{2} \right\}$$

for every a, b in M and γ, μ are positive real numbers such that $0 < \gamma < \frac{1}{2}$. If

(1). $\partial M \subseteq TM$, $fM \cap M \subseteq TM$,

(2). $Ta \in \partial M$ implies $fa \in M$,

(3). $fM \cap M$ is complete.

Then there exist a coincidence point z of f and T. Moreover if f and T are weakly compatible, then z is the unique common fixed point in M.

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