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Derivation of Black-Scholes-Merton Logistic Brownian Motion Differential Equation with Jump Diffusion Process

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Abstract: Black- Scholes formed the foundation of option pricing. However, some of the assumptions like constant volatility and interest among others are practically impossible to implement hence other option pricing models have been explored to help come up with a much reliable way of predicting the price trends of options. Black-scholes assumed that the daily logarithmic returns of individual stocks are normally distributed. This is not true in practical sense especially in short term intervals because stock prices are able to reproduce the leptokurtic feature and to some extent the "volatility smile". To address the above problem the Jump-Diffusion Model and the Kou Double-Exponential Jump-Diffusion Model were presented. But still they have not fully addressed the issue of reliable prediction because the observed implied volatility surface is skewed and tends to flatten out for longer maturities; The two models abilities to produce accurate results are reduced. This paper ventures into a research that will involve Black-Scholes-Merton logistic-type option pricing with jump diffusion. The knowledge of logistic Brownian motion will be used to develop a logistic Brownian motion with jump diffusion model for price process.

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1. Introduction

Financial Mathematics has been greatly applied in pricing of assets where options are valued in terms of derivatives and securities. Partial differential equations and probability and stochastic processes are the two main modelling approaches used in financial mathematics. Pricing of options and assets is partly derived from the interplay of demand and supply and partly from theoretical models. Standard Black-Scholes equation is derived under some strict assumptions that there are no transaction costs, the markets are liquid in nature, the rate of interest and volatility are known constants and the underlying asset follows a geometric Brownian motion. These assumptions are sometimes not applicable in real market world. Option pricing depends on the hypothesis that the dynamics of the underlying asset is sold if its price decreases and bought if its price increases in a perfectly liquid market. A standard model for price of stock as a function of time S(t) evolves according to geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{1}$$

where μ is the rate of growth of the asset, σ is the volatility and $dW(t)$ is the stochastic function Black [2]. This model is based on the idea that prices appear to be the previous price plus some random change and that these price changes are independent, prices being taken to follow some random walk-type behaviour. This is the basis for including the stochastic

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function. The demand and supply curves are also used to determine the quantity and price at which assets are bought and sold. The supply curve shows what the quantities the sellers are willing and able to sell at various prices whereas demand curve shows the quantities the consumers are willing and able to buy at different prices. The interplay between supply and demand brings in what is called market equilibrium. This is the situation where there is no tendency for change in security price and quantity. In other words there is no reason for the market price of products to rise or fall. In stock markets the price of an asset is assumed to respond to excess demand and is expressed as;

$$ED(S(t)) = Q_D(S(t)) - Q_S(S(t)), \tag{2}$$

where $ED(S(t)$ is excess demand, $Q_D(S(t))$ are quantities demanded and $Q_S(S(t))$ are quantities supplied at a given time t and price $S(t)$. The market structure with forces of demand and supply experience upward and downward shifts until a state of market equilibrium is achieved. A lot of literature dealing with pricing and hedging of contigent claims are based on a basic assumption that the asset's price follows a geometric Brownian motion. Emperical studies have shown that the models based on GBM may be limited in describing stock's price evolution hence inducing mispricing through overestimation or underestimation. To be able to produce more accurate option pricing, the jump diffusion models were introduced by Merton [5].The jump diffusion models unlike the famous Black- Scholes models do not make the same assumptions of normally distributed logarithmic returns. Stock prices may change due to the general economic factors such as demand and supply, changes in economic outlook and capitalization rates. These brings about small or marginal movements in stock's price hence modeled by a GBM. On the other hand the stock's price may fluctuates due to announcement of some important information causing over-reaction or under-reaction of the asset prices due to good and/or bad news. This information may emanate from the firm or industry. Such information that arrives at discrete points in time can only be modeled by a jump process. Stochastic process with jumps are better tools in modeling the price fluctuations due to the following reasons:

(i). Processes involving jumps are good tools to model calamitous events in the market.

(ii). The increased accessibility of high frequency data shows that the asset price path is not continuous in small time scales.

(iii). Jump processes as compared to diffusion models, produce rich structures on option implied surfaces and distribution of asset returns.

(iv). From Statistical evidence there is existence of "small" jumps along with the diffusion component in the asset price dynamics.

Volatility is a measure of how uncertain we are about the future of stock price, hence the estimation of volatility is crucial for implementation and valuation of asset and derivative pricing. Volatility forecast affects investment choice and is the key input to valuation of corporate and public liabilities. It gives the idea about the stability of stock prices. Volatility forecast is also the most important parameter affecting prices of market-listed options of which trading volume has increased in the last decade. Volatility of an asset used by Black-Scholes model Black [2] and Merton [5] is assumed to be constant throughout the duration of derivative. The market equilibrium has made it possible to apply the idea of logistic equation in finance. It is with this reason that we derive a new Black-Scholes-Merton logistic-type option pricing with jump diffusion.

2. Preliminaries

In this section, we look at some fundamental concepts that will be of importance to our study:

2.1. Stochastic process

Any variable whose value changes over time in uncertain way is said to follow a stochastic process. Hence it obeys laws of probability. Mathematically, a stochastic process $X = [X(t); t\epsilon(0,\alpha)]$ is a collection of random variables such that for each t in the index set $(0,\alpha)$, $X(t)$ is a random variable where $X(t)$ is the state of the process at time t. A discrete time stochastic is the one where the value of the variable can only change at a certain fixed points in time. On the other hand continuous time stochastic, change can take any value within a certain range.

2.2. Markov Process

This is a particular type of Stochastic process where only the present value of the variable is relevant for predicting the future. It is believed that the current price already contain what is relevant from the past. It implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past Hull [3]. Stock prices are assumed to follow Markov process.

2.3. Wiener process or Brownian motion

Is a particular type of Markov Stochastic process with a mean change of zero and a variance of 1.0 per year. It follows a stochastic process where μ is the mean of the probability distribution and σ is the standard deviation. That is $W(t) \sim N(\mu,\sigma)$ then for Wiener process $W(t) \sim N(0,1)$ which means $W(t)$ is a normal distribution with $\mu = 0$ and $\sigma = 1$. If a variable W follows a Wiener process then it has the following properties;

(i). The change ΔW for any two different short time intervals of time $\Delta W = \epsilon\sqrt{\Delta t}$, where ϵ has a standardized normal distribution; $\phi(0,1)$.

(ii). The values of ΔW for any two different short time intervals of time , Δt, are independent that is $Var(\Delta W_i, \Delta W_j) = 0$, $i \neq j$ it follows that from the first property that itself has a normal distribution with mean of $\Delta W = 0$ and standard deviation of $\Delta W = \sqrt{\Delta t}$ and variance $\Delta W = \Delta t$ that is $\Delta W \longrightarrow N(0,\sqrt{\Delta t}))$ the second property implies that W follows a Markov process.

Consider the change in the value of W during a relatively long period of time T. This can be denoted by $W(T) - W(0)$. It can be regarded as the sum of the changes in W in N small time intervals of length Δt, where $N = \frac{T}{t}$ thus $W(T) - W(0) = \sum iN\epsilon_i\sqrt{\Delta t}$ where $\epsilon_i (i = 1,2,3,\ldots,N)$ are distributed $\phi(0,1)$. From the second property of Wiener process, ϵ_i are independent of each other. It follows that $W(T) - W(0)$ is normally distributed with
Mean of $W(T) - W(0) = 0$; Variance of $W(T) - W(0) = n\Delta t = T$ thus Standard deviation of $W(T) - W(0)$ is $\sqrt{T}$. Hence $W(T) - W(0) \longrightarrow N(0,\sqrt{T})$.

2.4. Generalised Wiener process

The basic Wiener process, dW that has been developed so far has a drift rate of zero and a variance rate of 1.0. the drift rate of zero means that the expected value of W at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in W in time interval of length T equals T. A generalised Wiener process for a variable X can be defined in terms of dW as

$$dX = adt + bdW \tag{3}$$

Where mean rate a and variance rate b are constants, adt is the expectation of dX and bdW is the addition of noise or variability to the path followed by X, while b is the diffusivity. In a small interval Δt, the change in the value of $X, \Delta X$ is of the form

$$\Delta X = a\Delta t + b\epsilon\Delta t \quad (4)$$

where as already defined ϵ is a random variable drawing from standardised normal distribution thus the distribution of ΔX is $Mean = E(\Delta X) = a\Delta t$ $Variance(\Delta X) = b^2\Delta t$ thus, Standard deviation of $\Delta X = b\sqrt{\Delta t}$. Hence $\Delta X \sim N(a\Delta t, b\sqrt{\Delta t})$. Similar argument to those given for a Wiener process show that the change in the value of X in any time interval T is normally distributed with mean of change in $X = aT$ Standard deviation of change in $X = bT$, Variance of change in $X = b^2T$ Hence dX $N(aT, b\sqrt{T})$

2.5. It$\widehat{o}$ s Process

This is the generalised Wiener process in which the parameters a and b are functions of the value of the underlying variable X and time t. An It$\widehat{o}$ process can be written algebraically as

$$dX = a(X,t)dt + b(X,t)dW \quad (5)$$

Both the expected drift rate and variance rate of an It$\widehat{o}$ process are liable to change over time. In a small time interval between t and $t + \Delta t$, the changes from X to $X + \Delta X$, is expressed as

$$\Delta X = a(X,t)\Delta t + b(X,t)\epsilon\sqrt{\Delta t} \quad (6)$$

This relationship involves a small approximation. It assumes that the drift and variance rate of X remain constant, equal to $a(X,t)\Delta t$ and $b^2(X,t)\Delta t$ respectively during the interval between t and $t + \Delta t$ hence $\Delta X \sim N(a(X,t)\Delta t, b(X,t)\sqrt{\Delta t})$

2.6. It$\widehat{os}$ Lemma

This is the formula used for solving stochastic differential equations. Suppose that the value of a variable X follows It$\widehat{o}$ s Process

$$dX = a(X,t)dt + b(X,t)dW, \quad (7)$$

where dW is a wiener process and a and b are functions of X and t. The variable X has a drift rate of a and a variance of b^2. It$\widehat{o}$ s Lemma shows that a function G(X,t) twice differentiable in X and once in t, is also an It$\widehat{o}$ process given by

$$dG = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}bdW \quad (8)$$

Where the dZ is the same Wiener process, thus G also follows an It$\widehat{o}$ Process with a drift rate of $(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2)$ and a variance rate of $(\frac{\partial G}{\partial X})^2 b^2$

2.7. Geometric Brownian Motion

A specific It$\hat{o}$ Process is the geometric Brownian motion of the form

$$dX = aXdt + bXdW \quad (9)$$

Where $a(X,t) = aX$ and $b(X,t) = bX$. In the above equation geometric Brownian motion has been applied in stock pricing and is given as

$$dS = \mu S dt + \sigma S dW, \tag{10}$$

where S is the stock price μ is the expected rate of return per unit time and σ is the volatility of the stock price. The equation can be written as;

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{11}$$

This model is the most widely used model of stock price behaviour. A review of this model gives a discrete time model ,

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}, \tag{12}$$

where ΔS is the change in stock price S within a small interval of time Δt and ϵ is a random variable drawn from standardised normal distribution with mean zero and standard deviation 1. Hence in a short time Δt, the expected value of return is $\mu \Delta t$ and the stochastic component of the return is $\sigma \epsilon \sqrt{\Delta t}$. The variance of the fractional rate of return is $\sigma^2 \Delta t$ and $\sigma\sqrt{\Delta t}$ is the standard deviation. Therefore $\frac{\Delta S}{S}$ is normally distributed with mean $\mu \Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$ or $\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma\sqrt{\Delta t})$

3. Main results

We begin with deriving Black-Scholes option pricing model.

3.1. The Black-Scholes Option Pricing model

Black-Scholes and Merton Black [2] used Samuelson's model Samuelson [11] to derive an explicit solution to the problem of pricing and hedging a European call and put options on a non-dividend paying asset. In their model, the option price is only determined by observable variables of the asset: the current price, $S(t)$ of the underlying asset, the strike price K,the expiration date, T of the contract, the interest rate r, and the volatility σ of the underlying asset. The model presented do not require knowledge of either the traders' taste or beliefs about expected returns on the underlying common asset.

In general, Black-Scholes-Merton model assumes that the market consists of one risky asset whose price evolves according to Brownian motion:

$$\frac{dS(t)}{S} = \mu dt + \sigma dW(t), t\varepsilon[0,T], \tag{13}$$

Where μ is the expected growth rate in the underlying asset price, $\sigma > 0$ is the constant volatility, and $\{W(t), t \geq 0\}$ is the standard Brownian motion. In this equation (13), the part $dS(t) = \mu S(t)dt$ is the deterministic, predictable or anticipated return of the stock during a period of dt. The additional term $\sigma S(t)dW(t)$ is the random part which makes the equation stochastic, thus it reflects the response of stock prices to external effects such as unexpected news. The random part has the term $W(t)$ which is the standard Brownian motion that is normally distributed with mean zero and standard deviation $\sqrt{T}$. It is assumed that the prices of the underlying asset are lognormally distributed meaning that the returns of the underlying asset are normally distributed. The asset price at a time $t \geq 0\}$ is given by

$$S(t) = S_0 exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma W(t)\right], W(t_0) = 0, \tag{14}$$

3.2. Black-Scholes-Merton Differential equation

We consider stock price process that follow a geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW \tag{15}$$

Let us denote $f(S,t))$ to be the value of a European call or put option which depends only on the asset price S at time t. Consider π to denote the value of the portfolio containing one long option position and short position of Δ units of the underlying asset such that the value of the portfolio by defination will be given by;

$$\Pi = f(S,t) - \Delta S \tag{16}$$

The value of the change of the portfolio by a very short period of time dt is;

$$d\Pi = df(S,t) - \Delta dS, \tag{17}$$

as Δ remains constant during the time step dt. From the It$\hat{o}'s$ Lemma we have,

$$df = \left(\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dW \tag{18}$$

We do delta hedging by setting the $\Delta = \frac{\partial f}{\partial S}$. In this case we shall have eliminated the risk hence the randomness reduced to zero. The value of the portfolio therefore simplifies to;

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)dt \tag{19}$$

Equation (19) shows that the portfolio is completely riskless during the time dt by not involving dW. This implies that the portfolio must instantaneously earn the same rate of return as other short term riskless assets. According to the assumption of Black-Scholes-Merton differential equation, if it earned more than this return,traders could take advantage by making a riskless profit by borrowing money to buy the portfolio. On the other hand if it earned less, they could make profit with no risk by taking the short position of the option and buy a riskless assets. This is called arbitraging. Therefore using the non-arbitrage principle, the change in value of the portfolio must be the same as the growth one could get if he/she puts an equivalent amount of cash in a riskless interest bearing account. Equivalently, under risk-neutral probability measure, the future expected value of the financial derivative is discounted at the risk-free rate. Hence;

$$d\Pi = r\Pi dt, \tag{20}$$

Where r is the riskless interest rate. Substituting for Π and $d\Pi$ in the above equation we get,

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)dt = r\left(f(S,t) - \frac{\partial f}{\partial S}S\right)dt \tag{21}$$

On simplifying we obtain;

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0, S > 0, 0 \le t \le T \tag{22}$$

This is the Black-Scholes-Merton differential equation.

3.3. Derivation of Black-Scholes-Merton logistic Brownian motion differential equation with jump diffusion

We consider a logistic jump diffusion model given by

$$dS(t) = (\mu - \lambda k)S(t)(S^* - S(t))dt + \sigma S(t)S^* - S(t))dW + S(t)S^* - S(t))(q-1)dN \tag{23}$$

Where μ is the growth rate, σ is the volatility, λ is the rate at which the jumps happen, k is the average jump size measured as a proportional increase in asset price, S^* is the equilibrium price and N is the Poisson process generating jumps Oduor [9]. dW and dN are assumed to be independently and identically distributed Nyakinda [7].

Suppose in the small time interval dt the asset price jumps from S to qS (we call q as absolute price jump size). So the relative price jump size (i.e percentage change in the asset price caused by the jump) is

$$\frac{dS}{S} = \frac{qS - S}{S} = q - 1 \tag{24}$$

If f is the price of a call option or other derivative contingent twice differentiable in S and once in t, the value of f must be function of S and t. Hence from $It\hat{o}s$ process on logistic Brownian motion we have;

$$df = \left[\frac{\partial f}{\partial t} + (\mu - \lambda k)S(t)\varphi\frac{\partial f}{\partial S} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right]dt + S(t)\varphi\sigma\frac{\partial f}{\partial S}dW + [f(qS,t) - f(S,t)]dN, \tag{25}$$

Where $S(t) = S$, $\varphi = S^* - S(t)$ and $[f(qS,t) - f(S,t)]dN$ describes the difference in the option value when a jump occurs. The discrete versions of equation (23) and (25) are

$$\Delta S(t) = (\mu - \lambda k)S(t)(S^* - S(t))\Delta t + \sigma S(t)S^* - S(t))\Delta W + S(t)S^* - S(t))(q-1)\Delta N \tag{26}$$

and

$$\Delta f = \left[\frac{\partial f}{\partial t} + (\mu - \lambda k)S(t)\varphi\frac{\partial f}{\partial S} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right]\Delta t + S(t)\varphi\sigma\frac{\partial f}{\partial S}\Delta W + [f(qS,t) - f(S,t)]\Delta N, \tag{27}$$

Where ΔS and Δf are changes in S and f in small interval Δt.

We seek to eliminate the Wiener process by choosing the portfolio and the derivative. Suppose that an investment buys one call option $f(t)$ and simultaneously sells Δ shares of the underlying asset at time t. We consider investing portfolio $f(t) - S(t)\Delta$ which is composed of a single option of an asset worth $S(t)$ and simultaneously selling a number Δ of this asset. Then the holder of this portfolio is short and long an amount of $\frac{\partial f}{\partial S}$ of shares. In this case we say that the portfolio is self-financing. We define $\Pi(t)$ as the value of the portfolio as

$$\Pi(t) = f - \frac{\partial f}{\partial S}\varphi S \tag{28}$$

The change $\Delta\Pi(t)$ in the value of the option in the time interval Δt is given by;

$$\Delta\Pi(t) = \Delta f - \frac{\partial f}{\partial S}\Delta\varphi S \tag{29}$$

Substituting (26) and (27) in (29) we obtain

$$d\Pi = \left[\frac{\partial f}{\partial t} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right]dt + \left[f(qS,t) - f(S,t) - \frac{\partial f}{\partial S}S\varphi(q-1)\right]dN \tag{30}$$

Equation (30) shows that the average rate of growth of asset μ has been eliminated from the equation meaning that average rate of growth in the asset price will not influence the valuation of the option. If the source of jumps is such information, then the jump component of the stock's return will represent non-systematic risk. This means that the market and the jump component will be uncorrelated. According to Merton he assumed that the risk associated with the jumps in the asset price are uncorrelated with the market as a whole. The Capital Asset Pricing Model (CAPM) says that the jumps terms offer no risk premium and the asset still grows at the risk free rate r. The no-arbitrage argument implies that the percentage return of the portfolio over time interval dt should be equal to r that is;

$$Ed\Pi(t) = r\Pi(t)dt \tag{31}$$

$$E\left\{\left[\frac{\partial f}{\partial t} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right]dt + \left[f(qS,t) - f(S,t) - \frac{\partial f}{\partial S}S\varphi(q-1)\right]dN\right\} = r\left(f - \frac{\partial f}{\partial S}S\varphi\right)dt \tag{32}$$

$$\left[\frac{\partial f}{\partial t} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\right]dt + E\left[f(qS,t) - f(S,t) - \frac{\partial f}{\partial S}S\varphi(q-1)\right]\lambda dt = r\left(f - \frac{\partial f}{\partial S}S\varphi\right)dt \tag{33}$$

On simplifying we get;

$$\frac{\partial f}{\partial t} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2} + \lambda E[f(qS,t) - f(S,t)] - \lambda\frac{\partial f}{\partial S}S\varphi E(q-1) + r\frac{\partial f}{\partial S}S\varphi - rf = 0 \tag{34}$$

This is the developed logistic Brownian motion differential equation with jump diffusion when jump is expected. If no jump is expected the equation reduces to Black-Scholes PDE.

$$\frac{\partial f}{\partial t} + \frac{\varphi^2 S^2\sigma^2}{2}\frac{\partial^2 f}{\partial S^2} + r\varphi S\frac{\partial f}{\partial S} - rf = 0 \tag{35}$$

4. Conclusion

In this paper we have developed logistic Brownian motion differential equation with jump diffusion when jump is expected. The results obtained are useful to long term investors to know the impact of jump diffusion behaviour of stocks on assets before making decisions on trading strategies.

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