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Bounds of Mean Code Word Lenth With Generalized Information Measure and its Application in Coding Theory

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Abstract: In this communication, we find a lower and upper bounds for mean code word length for complete and incomplete probability distribution with generalized information measure and we identify its application in coding theory.

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1. Introduction

Throughout the paper N denotes the set of the natural numbers and for $n \in N$ we set

$$\Delta_n = \left\{ R = (r_1, r_2, \dots, r_n); \quad 0 < r_k \le 1, \quad 0 < \sum_{k=1}^n r_k \le 1 \right\},$$
$$\Delta_n^* = \left\{ R = (r_1, r_2, \dots, r_n); \quad 0 < r_k \le 1, \quad \sum_{k=1}^n r_k = 1 \right\},$$

denote the sets of *n*-components, $n \ge 2$, generalized probability distributions and complete probability distributions respectively. For $(r_1, r_2, \ldots, r_n) = R \in \Delta_n, (s_1, s_2, \ldots, s_n) = S \in \Delta_n$, we define a non-additive measure of inaccuracy, denoted by $H(R, S; \omega)$ as

$$H(R, S; \omega) = \frac{1}{1 - \omega} \left[\left(\frac{\sum_{k=1}^{n} r_k^{\frac{\omega^2 - \omega + 1}{\omega}} s_k^{(\omega - 1)}}{\sum_{k=1}^{n} r_k} \right) - 1 \right]; \quad \omega > 1.$$

$$= -\frac{\sum_{k=1}^{n} r_k \log_2 s_k}{\sum_{k=1}^{n} r_k};; \quad \omega \to 1.$$
(1)

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If $s_k = r_k^{\frac{1}{\omega}}$, then $H(R, S; \omega)$ reduces to non-additive entropy. That is

$$H(R;\omega) = \frac{1}{1-\omega} \left[\left(\sum_{\substack{k=1\\ \sum \\ k=1}^{n} r_{k}} \sum_{k=1}^{n} r_{k} \right) - 1 \right]; \quad \omega > 0 (\neq 1).$$

$$= -\frac{\sum_{k=1}^{n} r_{k} \log_{2} r_{k}}{\sum_{k=1}^{n} r_{k}}; \quad \omega \to 1.$$
(2)

The entropy (2) was first of all characterized by Havrda and Charvat [7]. Later on, Daroczy [5] and M. Behara and P. Nath [2, 3] studied this entropy. Vajda [18] also characterized this entropy for finite discrete generalized probability distributions. Tsallis's [17] gave its applications in physics for $R \in \Delta_n^*$, and $\omega \to 1$, $H(R; \omega)$ reduces to Shannon [15] entropy. That is

$$H(R) = -\sum_{k=1}^{n} r_k \log_D r_k.$$
(3)

2. Formulation of the Problem

For $\omega \to 1$ and $R \in \Delta_n^*$, $S \in \Delta_n^*$, then an important property of Kerridge's inaccuracy [9] is that

$$H(R) \le H(R, S). \tag{4}$$

equality if and only if R = S. In other words, Shannon's entropy is the minimum value of Kerridge's inaccuracy. If $R \in \Delta_n$, $S \in \Delta_n$, then (4) is no longer necessarily true. Also, the corresponding inequality

$$H(R;\omega) \le H(R,S;\omega) \tag{5}$$

are not necessarily true even for generalized probability distributions. Hence, it is natural to ask the following question: "For generalized probability distributions, what are the quantity the minimum values of which are $H(R; \omega)$?" We give below an answer to the above question separately for $H(R; \omega)$ by dividing the discussion into two parts (i) $\omega \to 1$ and (ii) $1 \neq \omega$. Also we shall assume that $n \geq 2$, because the problem is trivial for n = 1.

Case 1: Let $\omega \to 1$. If $R \in \Delta_n^*$, $S \in \Delta_n^*$, then as remarked earlier (4) is true. For $R \in \Delta_n$, $S \in \Delta_n$, it can be easily seen by using Jenson's inequality that (4) is true if $\sum_{k=1}^n r_k \ge \sum_{k=1}^n s_k$, equality in (4) holding if and only if

$$\frac{r_1}{s_1} = \frac{r_2}{s_2} = \dots = \frac{r_n}{s_n} = \frac{\sum_{k=1}^n r_k}{\sum_{k=1}^n s_k}.$$

Case 2: Let $1 \neq \omega$. Since (5) is not necessarily true, we need an inequality

$$\sum_{k=1}^{n} r_k^{\omega - 1} s_k^{\omega} \le \sum_{k=1}^{n} r_k^{\omega}; \ \omega > 1$$
(6)

such that $H(R;\omega) \leq H(R,S;\omega)$ and equality holds if and only if $s_k = r_k^{\frac{1}{\omega}}$.

Remark 2.1. If $\omega = 1$, the inequality (6) reduces to case 1.

Since $\omega > 1$, by reverse Hölder inequality, that is, if $n = 2, 3, ..., \gamma > 1$ and $x_1, ..., x_n, y_1, ..., y_n$ are positive real numbers then

$$\left(\sum_{k=1}^{n} x_k^{\frac{1}{\gamma}}\right)^{\gamma} \cdot \left(\sum_{k=1}^{n} y_k^{-\frac{1}{(\gamma-1)}}\right)^{-(\gamma-1)} \le \sum_{k=1}^{n} x_k y_k.$$

$$\tag{7}$$

Let $\gamma = \frac{\omega}{\omega-1}$, $x_k = r_k^{\frac{\omega^2 - \omega + 1}{\omega-1}} s_k$, $y_k = r_k^{\frac{\omega}{1-\omega}}$ (k = 1, 2, 3, ...n). Putting these values into (7), we get

$$\left(\sum_{k=1}^n r_k^{\frac{\omega^2 - \omega + 1}{\omega}} s_k^{\omega - 1}\right)^{\frac{\omega}{\omega - 1}} \left(\sum_{k=1}^n r_k^{\omega}\right)^{\frac{1}{1 - \omega}} \le \sum_{k=1}^n r_k^{\omega - 1} s_k^{\omega} \le \sum_{k=1}^n r_k^{\omega},$$

where we used (6), too. This implies however that

$$\left(\sum_{k=1}^{n} r_k^{\frac{\omega^2 - \omega + 1}{\omega}} s_k^{\omega - 1}\right) \le \left(\sum_{k=1}^{n} r_k^{\omega}\right).$$
(8)

Or

$$\begin{pmatrix}
\sum_{k=1}^{n} r_k^{\frac{\omega^2 - \omega + 1}{\omega}} s_k^{\omega - 1} \\
\sum_{k=1}^{n} r_k
\end{pmatrix} \leq \begin{pmatrix}
\sum_{k=1}^{n} r_k^{\omega} \\
\sum_{k=1}^{n} r_k
\end{pmatrix}$$
(9)

using (9) and the fact that $\omega > 1$, we get (5).

Particular's Case: If $\omega = 1$, then (5) becomes

$$H\left(R\right) \le H\left(R,S\right)$$

which is Kerridge's inaccuracy [9].

3. Mean Codeword Length and Their Bounds

We will now give an application of inequality (5) in coding theory for

$$\Delta_n^* = \left\{ R = (r_1, r_2, ..., r_n); \ 0 < r_k \le 1, \ \sum_{k=1}^n r_k = 1 \right\}$$

Let a finite set of n input symbols

$$X = \{x_1, x_2, \dots, x_n\}$$

be encoded using alphabet of D symbols, then it has been shown by Feinstien [6] that there is a uniquely decipherable code with lengths N_1, N_2, \ldots, N_n if and only if the Kraft inequality holds that is,

$$\sum_{k=1}^{n} D^{-N_k} \le 1.$$
 (10)

Where D is the size of code alphabet. Furthermore, if

$$L = \sum_{k=1}^{n} N_k r_k \tag{11}$$

is the average codeword length, then for a code satisfying (10), the inequality

$$L \ge H\left(R\right) \tag{12}$$

113

is also fulfilled and equality holds if and only if

$$N_k = -\log_D(r_k) \ (k = 1, ..., n), \tag{13}$$

and that by suitable encoded into words of long sequences, the average length can be made arbitrarily close to H(R), (see Feinstein [6]). This is Shannon's noiseless coding theorem. By considering Renyi's entropy (see e.g. [14]), a coding theorem and analogous to the above noiseless coding theorem has been established by Campbell [4] and the authors obtained bounds for it in terms of

$$H_{\omega}(R) = \frac{1}{1-\omega} \log_D \sum r_k^{\omega}, \omega > 0 (\neq 1).$$

Kieffer [11] defined a class rules and showed $H_{\omega}(R)$ is the best decision rule for deciding which of the two sources can be coded with expected cost of sequences of length N when $N \to \infty$, where the cost of encoding a sequence is assumed to be a function of length only. Further, in Jelinek [8] it is shown that coding with respect to Campbell's mean length is useful in minimizing the problem of buffer overflow which occurs when the source symbol is produced at a fixed rate and the code words are stored temporarily in a finite buffer. Concerning Campbell's mean length the reader can consult [4]. It may be seen that the mean codeword length (11) had been generalized parametrically by Campbell [4] and their bounds had been studied in terms of generalized measures of entropies. Here we give another generalization of (11) and study its bounds in terms of generalized entropy of order ω . Generalized coding theorems by considering different information measure under the condition of unique decipherability were investigated by several authors, see for instance the papers [6, 10, 12, 13]. An investigation is carried out concerning discrete memoryless sources possessing an additional parameter ω which seems to be significant in problem of storage and transmission (see [8], [11] and [12]).

In this section we study a coding theorem by considering a new information measure depending on two parameters. Our motivation is among others that this quantity generalizes some information measures already existing in the literature such as the Arndt [1] entropy, which is used in physics.

Definition 3.1. Let $n \in N$, $\omega > 0$ ($\neq 1$) be arbitrarily fixed, then the mean length $L(\omega)$ corresponding to the generalized information measure $H(R;\omega)$ is given by the formula

$$L(\omega) = \frac{1}{1-\omega} \left[\left(\sum_{k=1}^{n} r_k^{\frac{\omega^2 - \omega + 1}{\omega}} D^{N_k(1-\omega)} \right) - 1 \right], \tag{14}$$

where $R = (r_1, \ldots, r_n) \in \Delta_n^*$ and D, N_1, N_2, \ldots, N_n are positive integers so that

$$\sum_{k=1}^{n} r_k^{\omega - 1} D^{-N_k \omega} \le \sum_{k=1}^{n} r_k^{\omega}.$$
(15)

Since (15) reduces to Kraft inequality when $\omega = 1$, therefore it is called generalized Kraft inequality and codes obtained under this generalized inequality are called personal codes.

Theorem 3.2. Let $n \in N$, $\omega > 1$ be arbitrarily fixed. Then there exist code length N_1, \ldots, N_n so that

$$H(R;\omega) \le L(\omega) < D^{1-\omega}H(R;\omega) + \frac{-1+D^{1-\omega}}{1-\omega}$$
(16)

holds under the condition (15) and equality holds if and only if

$$N_k = -\log_D\left(r_k^{\frac{1}{\omega}}\right); \quad k = 1, 2, ..., n.$$

$$(17)$$

Where $H(R;\omega)$ and $L(\omega)$ are given by (2) and (14) respectively.

Proof. First of all we shall prove the lower bound of $L(\omega)$. By reverse Hölder inequality, that is, if $n = 2, 3, ..., \gamma > 1$ and $x_1, ..., x_n, y_1, ..., y_n$ are positive real numbers then

$$\left(\sum_{k=1}^{n} x_k^{\frac{1}{\gamma}}\right)^{\gamma} \cdot \left(\sum_{k=1}^{n} y_k^{-\frac{1}{(\gamma-1)}}\right)^{-(\gamma-1)} \le \sum_{k=1}^{n} x_k y_k.$$
(18)

Let $\gamma = \frac{\omega}{\omega-1}$, $x_k = r_k^{\frac{\omega^2 - \omega + 1}{\omega-1}} D^{-N_k}$, $y_k = r_k^{\frac{\omega}{1-\omega}}$ $(k = 1, 2, 3, \dots, n)$. Putting these values into (18), we get

$$\left(\sum_{k=1}^{n} r_{k}^{\frac{\omega^{2}-\omega+1}{\omega}} D^{-N_{k}(\omega-1)}\right)^{\frac{\omega}{\omega-1}} \left(\sum_{k=1}^{n} r_{k}^{\omega}\right)^{\frac{1}{1-\omega}} \leq \sum_{k=1}^{n} r_{k}^{\omega-1} D^{-N_{k}\omega} \leq \sum_{k=1}^{n} r_{k}^{\omega} ; \ \omega > 1.$$

where we used (15), too. This implies however that

$$\left(\sum_{k=1}^{n} r_{k}^{\frac{\omega^{2}-\omega+1}{\omega}} D^{-N_{k}(\omega-1)}\right)^{\frac{\omega}{\omega-1}} \leq \left(\sum_{k=1}^{n} r_{k}^{\omega}\right)^{\frac{\omega}{\omega-1}}$$
(19)

For $\omega > 1$, (19) becomes

$$\left(\sum_{k=1}^{n} r_{k}^{\frac{\omega^{2}-\omega+1}{\omega}} D^{-N_{k}(\omega-1)}\right) \leq \left(\sum_{k=1}^{n} r_{k}^{\omega}\right)$$

$$(20)$$

using (20) and the fact that $\omega > 1$, we get

$$H(R;\omega) \le L(\omega) \tag{21}$$

From (17) and after simplification, we get

$$r_k^{\frac{\omega^2-\omega+1}{\omega}}D^{-N_k(\omega-1)} = r_k^{\omega}$$

This implies

$$\left(\sum_{k=1}^{n} r_k^{\frac{\omega^2 - \omega + 1}{\omega}} D^{-N_k(\omega - 1)}\right) = \left(\sum_{k=1}^{n} r_k^{\omega}\right)$$
(22)

which gives $L(\omega) = H(R; \omega)$. Then equality sign holds in (21). Now we will prove the inequality (16) for upper bound of $L(\omega)$. We choose the codeword lengths N_k , k = 1, ..., n in such a way that

$$-\log_D\left(r_k^{\frac{1}{\omega}}\right) \le N_k < -\log_D\left(r_k^{\frac{1}{\omega}}\right) + 1.$$
(23)

is fulfilled for all k = 1, ..., n. From the left inequality of (23), we have

$$D^{-N_k} \le r_k^{\frac{1}{\omega}},\tag{24}$$

raising power both side by $\omega - 1$ and multiplying both sides by $r_k^{\frac{\omega^2 - \omega + 1}{\omega}}$ and then taking sum over k, we get the generalized inequality (15). So there exists a generalized code with code lengths N_k , k = 1, ..., n. Since $\omega > 1$, then (23) can be written as

$$r_{k}^{\left(\frac{\omega-1}{\omega}\right)} \ge D^{-N_{k}(\omega-1)} > r_{k}^{\left(\frac{\omega-1}{\omega}\right)} D^{-\left(\frac{\omega-1}{\omega}\right)}$$

$$\tag{25}$$

Multiplying (25) throughout by $r_k^{\frac{\omega^2-\omega+1}{\omega}}$ and then summing up from k=1 to n, we obtain inequality

$$\left(\sum_{k=1}^{n} r_{k}^{\omega}\right) \geq \left(\sum_{k=1}^{n} r_{k}^{\frac{\omega^{2}-\omega+1}{\omega}} D^{N_{k}\left(\frac{1-\omega}{\omega}\right)}\right)^{\omega} > \left(\sum_{k=1}^{n} r_{k}^{\omega}\right) D^{(1-\omega)}.$$
(26)

Since $1 - \omega < 0$ for $\omega > 1$, we get from (26) the inequality (16).

Particular Case: For $\omega \to 1$, then (16) becomes

$$\frac{H(P)}{\log D} \le L < \frac{H(P)}{\log D} + 1.$$

Which is the Shannon [15] classical noiseless coding theorem.

$$N_k = -\log_D(r_k)$$
; $k = 1, 2, \dots, n.$ (27)

Where H(R) and L are given by (3) and (11) respectively.

4. Conclusion

In this paper we prove a generalization of Shannon's inequality for the case of Tsallis entropy with the help of reverse Hölder inequality and define a new mean codeword length. Considering Theorem 3.2 we remark that the optimal code length depends on ω in contrast with the optimal code lengths of Shannon which do not depend on a parameter. However, it is possible to prove coding theorem with respect to (1) such that the optimal code lengths are identical to those of Shannon.

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