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Some Fixed Point Theorems in Dislocated Quasi b-metric Spaces

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Abstract: In this paper we have given a fixed point theorem in dislocated quasi b-metric spaces which is generalization of dislocated quasi metric, partial b-metric spaces and the Banach contraction principle.

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1. Introduction

In 1922, Banach [3] proved a fixed point theorem for contraction mappings in metric spaces. Since then a number of fixed point theorems have been proved by different authors, and many more generalizations of this theorem have been established. The study of common fixed point of mappings satisfying certain contractive conditions have been at the center of vigorous research activity. Many generalizations of Banach's contraction theorem in different types of generalizations if metric spaces are made is clear from the literature. Some problems particularly the problems of convergence of measurable functions with respect to measure leads to the generalization of metric space. Czerwik [5] introduced the concept of b-metric space and proved Banach contraction theorem in so called b-metric spaces. Alaghamdi [2] introduced the notion of a b-metric like space which generalized the notion of a b-metric space, where they proved some exciting new fixed point results in b-metric like spaces. Recently Shukla [12] introduced the concept of partial b-metric space and gave some fixed point results and examples. The notion of dislocated metric spaces was introduced by Hitzler [6] as a part of the study of logic programming semantics. Zeyada [13] initiated the concept of dislocated quasi metric space and generalized the result of Hitzler [6]. Rehman and Sarwar [10] introduced the concept of dislocated quasi b-metric space and proved the Banach's contraction principle, Kannan [7] and Chatterjea [4] type fixed point results for self mappings in such spaces. Recently Srinivasa Rao [11], introduced the concept of dislocated quasi b-metric spaces with index k which in symmetric and establish some fixed point theorems which in the light of the new definitions. And generalized the Banach contraction Theorem in the context of d q b-metric space with index k. Banach's contraction principle has been generalized by various authors by putting different

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types of contractive conditions either on mappings or on the space. In this paper we generalize the results of Manish Sharma [9]. The following definitions and results are needed in the sequel.

Definition 1.1 ([1, 13]). *Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function, called a distance function, which satisfies one or more of the conditions*

(d_1). $d(x, x) = 0$.

(d_2). $d(x, y) = d(y, x) = 0$ *then* $x = y$.

(d_3). $d(x, y) = d(y, x)$.

(d_4). $d(x, y) \leq d(x, z) + d(z, y)$, *for all* $x, y, z \in X$.

If d satisfies the conditions d_1, d_2, d_3 and d_4 then d is called a metric on X. If d satisfies the conditions d_1, d_2 and d_4 then d is called a quasi metric on X. If d satisfies the conditions d_2, d_3 and d_4 then d is called a dislocated metric on X. If d satisfies the conditions d_2 and d_4 then d is called a dislocated quasi metric on X. A non empty set X with dislocated quasi metric d, i.e. (X, d) is called a dislocated quasi metric space.

Definition 1.2 ([5]). *Let X be a non empty set and $k \geq 1$ be real number. Then a mapping $d : X \times X \to [0, \infty)$ is called b-metric if*

(1). $d(x, x) = 0$.

(2). $d(x, y) = d(y, x) = 0$ *then* $x = y$.

(3). $d(x, y) = d(y, x)$

(4). $d(x, y) \leq k(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

Then the pair (X, d) is called b-metric space.

We observe that, b-metric is more general than usual metric.

Definition 1.3 ([5]). *Let X be a non empty set and $k \geq 1$ be real number. Then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi metric b-metric with index k if*

(1). $d(x, y) = d(y, x) = 0$ *then* $x = y$.

(2). $d(x, y) \leq k(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

Then the pair (X, d) is called a dislocated quasi b-metric space with index k (simply d q b-metric space).

Proposition 1.4 ([10]). *Suppose X is a non empty set and d^* is d q-metric and d^{**} is a d q b-metric with $k \geq 1$ on X. Then the function $d : X \times X \to [0, \infty)$ defined by $d(x, y) = d^*(x, y) + d^{**}(x, y)$ for all $x, y \in X$ is d q- metric on X.*

Definition 1.5 ([11]). *Suppose (X, d) is a d q b-metric space with index k. A sequence $\{x_n\}$ in X is called d q b-convergent to x in X if for $n \geq N$ we have $d(x_n, x) + d(x, x_n) \to 0$ as $n \to \infty$. Also x is called d q b-limit of the sequence $\{x_n\}$.*

Note: In this case we write $x_n \to x$ as $n \to \infty$.

Definition 1.6 ([11]). *A sequence $\{x_n\}$ in a dislocated quasi b-metric is called a Cauchy sequence if $d(x_m, x_n) \to 0$ as $m.n \to \infty$. That is, given $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_m, x_n) < \varepsilon$ for $m, n \geq n_0$.*

Definition 1.7 ([11]). *A d q b-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.*

Definition 1.8 ([11]). *A self map $T : X \to X$ is said to be continuous if $x_n \to x \Rightarrow Tx_n \to Tx$.*

Lemma 1.9 ([11]). *Limit of convergent sequence in a d q b-metric space with index k is unique.*

Lemma 1.10 ([11]). *Let (X, d) be a d q b-metric space with index k, $0 \leq \alpha k < 1$ and $\{x_n\}$ be a sequence in X, such that $d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)$ and $d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1})$ for $n = 1, 2, 3, \ldots$. Then $\{x_n\}$ is a Cauchy sequence.*

Definition 1.11 ([11]). *Let $T : X \to X$ be such that $d(Tx, Ty) \leq \alpha d(x, y) \quad \forall \ x, y \in X$, where, $0 \leq \alpha k < 1$. Then T is called a contraction.*

Khan [7] proved the following fixed point theorem using rational type contraction in complete metric spaces.

Theorem 1.12 ([7]). *Let T be a self map defined on a complete metric space (X, d). Further, let T satisfies the following contractive condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(x.Tx).d(x, Ty) + (dy, Ty).d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

For all $x, y \in X$ and $\alpha \in [0, 1)$ then T has a unique fixed point.

Recently, Manish Sharma [9] modified Theorem 1.11 in the context of d q b-metric space as follows.

Theorem 1.13 ([9]). *Let (X, d) be a complete d q b-metric space with $k \geq 1$ and let $T : X \to X$ be a continuous self mapping satisfying the condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(x.Tx).d(x, Ty) + d(y, Ty).d(y, Tx)}{d(x, Ty) + d(y, Tx)} + \beta d(x, y)$$

$\forall \ x, y \in X$ and $\alpha, \beta \geq 0$, $d(x, Ty) + d(y, Tx) \neq 0$ with $k(2\alpha + \beta) < 1$. Then T has a unique fixed point in X.

2. Main Result

In this section we will prove our main result. In proving this Theorem in [9], continuity of T is assumed.

Theorem 2.1. *Let (X, d) be a complete d q b- metric space with index k and let $T : X \to X$ be a contraction with $\alpha, \beta \geq 0$ and $k(2\alpha + \beta) < 1$. Suppose*

$$d(Tx, Ty) \leq \alpha (d(x, Tx) + d(y, Ty)) + \beta d(x, y) \tag{1}$$

holds $\forall \ x, y \in X$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in X such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots$ and in general $x_n = T^n x_0$ for $n = 1, 2, 3, \ldots$. Now,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d\left(T^n x_0, T^{n+1} x_0\right) \\
&= d\left(T\left(T^{n-1} x_0\right), T\left(T^n x_0\right)\right) \\
&= \alpha\left(d\left(T^{n-1} x_0, T^n x_0\right) + d\left(T^n x_0, T^{n+1} x_0\right)\right) + \beta d\left(T^{n-1} x_0, T^n x_0\right) \\
&= \alpha\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) + \beta d(x_{n-1}, x_n)
\end{aligned}$$

$$\Rightarrow (1 - \alpha) d(x_n, x_{n+1}) = (\alpha + \beta) d(x_{n-1}, x_n)$$

$$\therefore \quad d(x_n, x_{n+1}) = \gamma d(x_{n-1}, x_n), \quad \text{where } \gamma = \frac{(\alpha+\beta)}{(1-\alpha)} < 1.$$

Similarly, we can prove that $d(x_{n+1}, x_n) = \gamma d(x_n, x_{n-1})$. Let $A_n = d(x_n, x_{n+1})$ and $A'_n = d(x_{n+1}, x_n)$, $A_{n-1} = d(x_{n-1}, x_n)$ and $A'_{n-1} = d(x_n, x_{n-1})$ write $A''_n = \max\{A_n, A'_n\}$. Then $A''_n \leq \gamma A''_{n-1} \leq \gamma^2 A''_{n-2} \leq \cdots \leq \gamma^n A_0 \to 0$ as $n \to \infty$. Therefore $A''_n \to 0$, $A_n \to 0$ and $A'_n \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned} A''_n + A''_{n+1} + \cdots + A''_{n+1} &\leq A''_n + \gamma A''_n + \gamma^2 A''_n + \cdots + \gamma^1 A''_n \\ &= \left(1 + \gamma + \gamma^2 + \cdots + \gamma^1\right) A''_n \\ &\leq \frac{1}{1-\gamma} A''_n \to 0 \quad as \quad n \to \infty \end{aligned}$$

Therefore $\{A''_n\}$ is a Cauchy sequence.

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_n, x_{n+1}) + kd(x_{n+1}, x_{n+1}) \\ &\leq kd(x_n, x_{n+1}) + k^2 d(x_{n+1}, x_{n+2}) + k^2 d(x_{n+2}, x_{n+1}) \\ &= kd(x_n, x_{n+1}) + k^2 d(x_{n+1}, x_{n+2}) + k^3 d(x_{n+2}, x_{n+3}) + \cdots + k^1 d(x_{n+2}, x_{n+1}) \\ &\qquad \begin{pmatrix} \because \; d(x_{n+1}, x_{n+2}) \leq \gamma k d(x_n, x_{n+1}) \\ d(x_{n+2}, x_{n+3}) \leq \gamma k d(x_{n+1}, x_{n+2}) \leq \gamma^2 d(x_n, x_{n+1}) \end{pmatrix} \end{aligned}$$

$$\therefore d(x_{n+1}, x_{n+1+1}) \leq \gamma^1 d(x_n, x_{n+1})$$

Therefore by mathematical induction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_n, x_{n+1}) + k^2\gamma d(x_n, x_{n+1}) + k^3\gamma^2 d(x_n, x_{n+1}) + \cdots + k^1\gamma^{l-1} d(x_n, x_{n+1}) \\ &= k\left(1 + k\gamma + k^2\gamma^2 + \cdots + k^{l-1}\gamma^{l-1}\right) d(x_n, x_{n+1}) \end{aligned}$$
$$\frac{k}{1-k\gamma} d(x_n, x_{n+1}) \to 0 \quad as \quad n \to \infty.$$

Similarly, we can prove that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in X. So there exist $x \in X$ such that $d(x_n, x) \to 0$ and $d(x, x_n) \to 0$. Now

$$\begin{aligned} d(x_{n+1}, Tx) &= d(Tx_n, Tx) \\ &\leq \alpha\left(d(x_n, Tx_n) + d(x, Tx)\right) + \beta d(x_n, x) \\ &= \alpha\left(d(x_n, x_{n+1}) + d(x, Tx)\right) + \beta d(x_n, x) \end{aligned}$$

But

$$\begin{aligned} d(x, Tx) &\leq kd(x, x_{n+1}) + kd(x_{n+1}, Tx) \\ &= kd(x, x_{n+1}) + k\alpha d(x_n, x_{n+1}) + k\alpha d(x, Tx) + k\beta d(x_n, x) \end{aligned}$$
$$\therefore \quad (1-k\alpha) d(x, Tx) \leq kd(x, x_{n+1}) + k\alpha d(x_n, x_{n+1}) + k\beta d(x_n, x) \to 0 \quad as \quad n \to \infty$$

Therefore $(1-k\alpha) d(x, Tx) = 0$ $(\because \;\; (1-k\alpha) \to 0)$. Hence $d(x, Tx) = 0$. Similarly, we can prove that $d(Tx, x) = 0$. Therefore $x = Tx \Rightarrow Tx = x$. Therefore x is a fixed point of T.

Uniqueness: Let x, y be two fixed points of T. Then

$$
\begin{aligned}
d(x,y) = d(Tx,Ty) &\le \alpha\left(d(x,Tx) + d(y,Ty)\right) + \beta d(x,y) \\
&= \alpha\left(d(x,x) + d(y,y)\right) + \beta d(x,y) \\
&\qquad \left(\begin{array}{c} \because \text{ if x is a fixed point, then} \\ d(x,x) \le \alpha\left(d(x,x) + d(x,x)\right) + \beta d(x,x) = (2\alpha+\beta)\, d(x,x) \\ \text{Hence } d(x,x) = 0 \text{ similarly } d(y,y) = 0 \end{array}\right) \\
&= \beta d(x,y) \\
\therefore\ d(x,y) = \beta d(x,y) &\quad (\because \quad \beta < 1) \\
\therefore\ d(x,y) = 0.
\end{aligned}
$$

Similarly, we can prove that $d(y,x) = 0$. Therefore $x = y$. Therefore T has a unique common fixed point. □

Note: If we put $\alpha = 0$ in 2.1.1 of our main result, we get Banach contraction principle. Thus Theorem 2.1 is the generalization of Banach contraction principle. The above Theorem improves the result of [9] in complete d q b-metric space, since continuity of T is not assumed here.

Corollary 2.2. *Let (X,d) be a complete d q b- metric space with $k \ge 1$ and let $T : X \to X$ be a continuous self mapping satisfying the condition:*

$$d(Tx,Ty) \le \alpha \frac{d(x.Tx).d(x,Ty) + d(y,Ty).d(y,Tx)}{d(x,Ty) + d(y,Tx)} + \beta d(x,y)$$

$\forall\ x, y \in X$ and $\alpha, \beta \ge 0$, $d(x,Ty) + d(y,Tx) \ne 0$ with $k(2\alpha+\beta) < 1$. Then T has a unique fixed point in X.

Proof. Since

$$d(Tx,Ty) \le \alpha \frac{d(x.Tx).d(x,Ty) + d(y,Ty).d(y,Tx)}{d(x,Ty) + d(y,Tx)} + \beta d(x,y)$$

Write $A = d(x,Ty)$ and $B = d(y,Tx)$. Then $d(Tx,Ty) \le \alpha \frac{d(x.Tx).A + d(y,Ty).B}{d(x,Ty)+d(y,Tx)} + \beta d(x,y)$. Write

$$
\begin{aligned}
c &= \max\{A, B\} \\
&= \alpha \frac{d(x.Tx).c + d(y,Ty).c}{c + c} + \beta d(x,y) \\
&\le \alpha\left(d(x,Tx) + d(y,Ty)\right) + \beta d(x,y)
\end{aligned}
$$

Therefore the proof of the Theorem 2.1 follows and T has unique fixed point. □

Note: Our main theorem improves the result of Manish Sharma in [9].

Corollary 2.3. *Let (X,d) be a complete d q b- metric space with $k \ge 1$ and let $T : X \to X$ be a continuous self mapping satisfying the condition:*

$$d(Tx,Ty) \le \alpha \frac{d(x.Tx).d(x,Ty) + d(y,Ty).d(y,Tx)}{1 + d(x,Ty) + d(y,Tx)} + \beta d(x,y)$$

$\forall\ x, y \in X$ and $\alpha, \beta \ge 0$ with $k(2\alpha+\beta) < 1$. Then T has a unique fixed point in X.

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