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Basic Analogue of Fractional Derivative and Some Special Functions

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Abstract: The main objects of this paper is devoted to the basic analogues of fractional differentiation of classical functions and a theorem on term by term basic analogues of fractional differentiation has been derived, which is an extension of the result given by Yadav and Purohit (2004) [11]. Several special cases are also mentioned.

Keywords: The basic q-analogue, q-integral and differentia operator, Bessel-Maitland function, multiindex Mittag-Leffler function and F-function.

1. Introduction and Preliminaries

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades Fractional Calculus has been applied to almost every field of Mathematics like Special Functions etc., Science, Engineering and Technology. Many applications of Fractional Calculus can be found in Turbulence and Fluid Dynamics, Stochastic Dynamical System, Plasma Physics and Controlled Thermonuclear Fusion, Non-linear Control Theory, Image Processing, Non-linear Biological Systems and Astrophysics.

Definition 1.1. *The q-analogue of differential operator Al-Salam (1966) [1] is given by*

$$D_{x,q}f(x) = \frac{f(xq) - f(x)}{x(q-1)} \tag{1}$$

This is an inverse of the q-integral operator defined as

$$\int_x^\infty f(t)d(t;q) = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(xq^{-k}) \tag{2}$$

Where $0 < |q| < 1$.

Definition 1.2. *The fractional q-differential operator of order* α *is defined as follows*

$$D^{\alpha}_{x,q}f(x) = \frac{1}{\Gamma_q(-\alpha)}\int_0^x (x-yq)_{-\alpha-1} f(y)d(y;q) \tag{3}$$

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where $Re(\alpha) < 0$. As a particular case of (3), we have

$$D^{\alpha}_{x,q} x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1} \tag{4}$$

The Wright generalized hypergeometric function is given by Kilbas (2005) [4]

$$p\Psi q\,(z) = p\Psi q\left[{}^{(a1,A1),\dots,(ap,Ap)}_{(b1,B1),\dots,(bq,Bq)};z\right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(ai+rAj)\,z^r}{\prod_{j=1}^{q}\Gamma(bi+rBj)r!} \tag{5}$$

$$Aj > 0(j=1,2,...,p), Bj > 0(j=1,2,...,q); 1+\sum_{j=1}^{q} Bj - \sum_{j=1}^{p} Aj \geq 0$$

(Equality only for approximately bounded z)

Definition 1.3. *The Bessel-Maitland function is defined as*

$$J^{\upsilon}_{\delta}(x) = \Phi(\upsilon+1,\delta;-z) = \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(\delta k+\upsilon+1)k!}, (\delta \in R, \upsilon \in C)$$

The multiindex Mittag-Leffler function is defined by Kiryakova [5] by the power series

$$E(\tfrac{1}{\rho i}),(\mu i)(z) = \sum_{r=0}^{\infty} \varphi r\, z^r = \sum_{r=0}^{\infty} \frac{z^r}{\prod_{j=1}^{m}\Gamma(\mu j + \frac{r}{\rho j})} \tag{6}$$

Where $m > 1$ is an integer, ρj and μj are arbitrary real numbers.

The multiindex Mittag-Leffler function is an entire function and also gives its asypototic; estimate, order and type see Kiryakova [5]. The F-function of Robotnov and Hartley [3] is defined by the power series

$$F_{\omega}[a,x] = \sum_{n=0}^{\infty} \frac{a^n\, x^{(n+1)\omega-1}}{\Gamma((n+1)\omega)}, \omega > 0 \tag{7}$$

In 1993, Miller and Ross (1993) [6] introduced a function as the basis of the solution of fractional order initial value problem. It is defined as the vth integral of the exponential function, that is,

$$E_x[\nu,a] = \frac{d^{-\nu}}{dx^{-\nu}} e^{ax} = x^{\nu} e^{ax} \gamma^*(\nu,ax) = \sum_{n=0}^{\infty} \frac{a^n\, x^{n+\nu}}{\Gamma(n+\nu+1)}, \nu \in C \tag{8}$$

where $\gamma^*(\nu, ax)$ is the incomplete gamma function.

2. Main Results

In this section we shall prove a theorem on term by term fractional q-differentiation of a power series.

Theorem 2.1. *If the series $\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(ai+nAj)a_n\, x^n}{\prod_{j=1}^{q}\Gamma(bi+nBj)n!}$ converges absolutely for $|x| < \rho$ then*

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(ai+nAj)a_n\, x^n}{\prod_{j=1}^{q}\Gamma(bi+nBj)n!}\right\} = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(ai+nAj)a_n}{\prod_{j=1}^{q}\Gamma(bi+nBj)n!} D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{9}$$

provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Proof. In view of (3) we have

$$= \frac{1}{\Gamma_q(-\mu)} \int\limits_0^x (x-yq)_{-\mu-1} y^{\lambda-1} \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(ai+nAj) a_n\, y^n}{\prod\limits_{j=1}^{q} \Gamma(bi+nBj) n!} d(y;q)$$

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int\limits_0^1 t^{\lambda-1}(1-tq)_{-\mu-1} \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(ai+nAj) a_n\, t^n\, x^n}{\prod\limits_{j=1}^{q} \Gamma(bi+nBj) n!} d(t;q). \tag{10}$$

Now the following observations are made

(1). $\sum\limits_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(ai+nAj) a_n\, t^n\, x^n}{\prod\limits_{j=1}^{q} \Gamma(bi+nBj) n!}$ converges absolutely and therefore uniformly in domain of x over the region of integration.

(2). $\int\limits_0^1 \left| t^{\lambda-1}(1-tq)_{-\mu-1} \right| d(t;q)$ is convergent, provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Therefore the order of integration and summation can be interchanged in (10) to obtain

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(ai+nAj) a_n\, x^n}{\prod\limits_{j=1}^{q} \Gamma(bi+nBj) n!} \int_0^1 t^{\lambda+n-1}(1-tq)_{-\mu-1} d(t;q)$$

$$= \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(ai+nAj) a_n}{\prod\limits_{j=1}^{q} \Gamma(bi+nBj) n!} D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{11}$$

This completes the prove of the Theorem (9). ☐

Theorem 2.2. *If the series* $\sum\limits_{n=0}^{\infty} \frac{a_n x^n}{\prod\limits_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})}$ *converges absolutely for* $|x| < \rho$ *then*

$$D^{\mu}_{x,q} \left\{ x^{\lambda-1} \sum_{n=0}^{\infty} \frac{a_n x^n}{\prod\limits_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})!} \right\} = \sum_{n=0}^{\infty} \frac{a_n D^{\mu}_{x,q}\{x^{\lambda+n-1}\}}{\prod_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})} \tag{12}$$

Provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Proof. In view of (3), we have

$$= \frac{1}{\Gamma_q(-\mu)} \int\limits_0^x (x-yq)_{-\mu-1} y^{\lambda-1} \sum_{n=0}^{\infty} \frac{a_n y^n}{\prod\limits_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})} d(y;q)$$

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int\limits_0^1 t^{\lambda-1}(1-tq)_{-\mu-1} \sum_{n=0}^{\infty} \frac{a_n t^n x^n}{\prod\limits_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})!} d(t;q). \tag{13}$$

Now the following observations are made

(1). $\sum\limits_{n=0}^{\infty} \frac{a_n t^n x^n}{\prod\limits_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})}$ converges absolutely and therefore uniformly in domain of x over the region of integration.

(2). $\int\limits_0^1 \left| t^{\lambda-1}(1-tq)_{-\mu-1} \right| d(t;q)$ is convergent, provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Therefore the order of integration and summation can be interchanged in (13) to obtain

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{n=0}^{\infty} \frac{a_n x^n}{\prod_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})} \int_0^1 t^{\lambda+n-1}(1-tq)_{-\mu-1} d(t;q)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{\prod_{j=1}^{m} \Gamma(\mu j + \frac{n}{\rho j})} D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{14}$$

This proves the Theorem (12) □

Theorem 2.3. *If the series* $\sum_{n=0}^{\infty} \frac{a_n(-x)^n}{\Gamma(\delta n+\upsilon+1)n!}$ *converges absolutely for* $|x| < \rho$ *then*

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{a_n(-x)^n}{\Gamma(\delta n+\upsilon+1)n!}\right\} = \sum_{n=0}^{\infty}\frac{a_n D^{\mu}_{x,q}\{(-x)^{\lambda+n-1}\}}{\Gamma(\delta n+\upsilon+1)n!} \tag{15}$$

Provided $Re(\lambda) > 0,\ Re(\mu) < 0,\ 0 < |q| < 1.$

Proof. In view of (3) we have

$$= \frac{1}{\Gamma_q(-\mu)} \int_0^x (x-yq)_{-\mu-1}(-y)^{\lambda-1} \sum_{n=0}^{\infty} \frac{a_n(-y)^n}{\Gamma(\delta n+\upsilon+1)n!} d(y;q)$$

$$= \frac{(-x)^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int_0^1 t^{\lambda-1}(1-tq)_{-\mu-1} \sum_{n=0}^{\infty} \frac{a_n(-t)^n x^n}{\Gamma(\delta n+\upsilon+1)n!} d(t;q). \tag{16}$$

Now the following observations are made

(1). $\sum_{n=0}^{\infty} \frac{a_n(-t)^n x^n}{\Gamma(\delta n+\upsilon+1)n!}$ converges absolutely and therefore uniformly in domain of x over the region of integration.

(2). $\int_0^1 \left|t^{\lambda-1}(1-tq)_{-\mu-1}\right| d(t;q)$ is convergent, provided $Re(\lambda) > 0,\ Re(\mu) < 0,\ 0 < |q| < 1.$

Therefore the order of integration and summation can be interchanged in (16) to obtain

$$= \frac{(-x)^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{n=0}^{\infty} \frac{a_n(-x)^n}{\Gamma(\delta n+\upsilon+1)n!} \int_0^1 t^{\lambda+n-1}(1-tq)_{-\mu-1} d(t;q)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\delta n+\upsilon+1)n!} D^{\mu}_{x,q}\{(-x)^{\lambda+n-1}\} \tag{17}$$

This completes the proof of the Theorem (15). □

Proof. If the series $\sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^{(n+1)\omega-1}}{\Gamma((n+1)\omega)}$ converges absolutely for $|x| < \rho$ then

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{b_n\, a^n\, x^{(n+1)\omega-1}}{\Gamma((n+1)\omega)}\right\} = \sum_{n=0}^{\infty}\frac{b_n\, a^n}{\Gamma((n+1)\omega)} D^{\mu}_{x,q}\{x^{\lambda+(n+1)\omega-2}\} \tag{18}$$

Provided $Re(\lambda) > 0,\ Re(\mu) < 0,\ 0 < |q| < 1.$ □

Proof. In view of (3), we have

$$= \frac{1}{\Gamma_q(-\mu)} \int_0^x (x-yq)_{-\mu-1} y^{\lambda-1} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^{(n+1)\omega-1}\, y^n}{\Gamma((n+1)\omega)} d(y;q)$$

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int_0^1 t^{\lambda-1}(1-tq)_{-\mu-1} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^{(n+1)\omega-1}\, t^{(n+1)\omega-1}}{\Gamma((n+1)\omega)} d(t;q). \tag{19}$$

Now the following observations are made

(1). $\sum\limits_{n=0}^{\infty} \frac{b_n\, a^n\, x^{(n+1)\omega-1}\, t^{(n+1)\omega-2}}{\Gamma((n+1)\omega)}$ converges absolutely and therefore uniformly in domain of x over the region of integration.

(2). $\int\limits_0^1 \left|t^{\lambda-1}(1-tq)_{-\mu-1}\right| d(t;q)$ is convergent, provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Therefore the order of integration and summation can be interchanged in (19) to obtain

$$= \frac{x^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^{(n+1)\omega-1}}{\Gamma((n+1)\omega)} \int_0^1 t^{\lambda+(n+1)\omega-1}(1-tq)_{-\mu-1} d(t;q)$$

$$= \sum_{n=0}^{\infty} \frac{b_n\, a^n}{\Gamma((n+1)\omega)} D_{x,q}^{\mu}\{x^{\lambda+(n+1)\omega-2}\} \tag{20}$$

This proves the Theorem (18). □

Theorem 2.4. *If the series* $\sum\limits_{n=0}^{\infty} \frac{b_n\, a^n\, x^{n+\nu}}{\Gamma(n+\nu+1)}$ *converges absolutely for* $|x| < \rho$ *then*

$$D_{x,q}^{\mu}\left\{x^{\lambda-1} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^{n+\nu}}{\Gamma(n+\nu+1)}\right\} = \sum_{n=0}^{\infty} \frac{b_n\, a^n}{\Gamma(n+\nu+1)} D_{x,q}^{\mu}\{x^{\lambda+n+\nu-1}\} \tag{21}$$

Provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Proof. In view of (3), we have

$$= \frac{1}{\Gamma_q(-\mu)} \int\limits_0^x (x-yq)_{-\mu-1} y^{\lambda-1} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, y^{n+\nu}}{\Gamma(n+\nu+1)} d(y;q)$$

$$= \frac{x^{\nu+\lambda-\mu-1}}{\Gamma_q(-\mu)} \int\limits_0^1 t^{\lambda-1}(1-tq)_{-\mu-1} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^n\, t^n}{\Gamma(n+\nu+1)} d(t;q). \tag{22}$$

Now the following observations are made

(1). $\sum\limits_{n=0}^{\infty} \frac{b_n\, a^n\, x^n\, t^n}{\Gamma(n+\nu+1)}$ converges absolutely and therefore uniformly in domain of x over the region of integration.

(2). $\int\limits_0^1 \left|t^{\lambda-1}(1-tq)_{-\mu-1}\right| d(t;q)$ is convergent, provided $Re(\lambda) > 0$, $Re(\mu) < 0$, $0 < |q| < 1$.

Therefore the order of integration and summation can be interchanged in (22) to obtain

$$= \frac{x^{\nu+\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{n=0}^{\infty} \frac{b_n\, a^n\, x^n}{\Gamma(n+\nu+1)} \int_0^1 t^{\lambda+n-1}(1-tq)_{-\mu-1} d(t;q)$$

$$= \sum_{n=0}^{\infty} \frac{b_n\, a^n}{\Gamma(n+\nu+1)} D_{x,q}^{\mu}\{x^{\lambda+n+\nu-1}\} \tag{23}$$

This proves the Theorem (21). □

3. Special Cases

Let us consider some special cases of above result:

(1). If we take $a_n = 1$, $\delta = \upsilon = 0$, $x \to -x$ in equation (15) it reduces to fractional q-derivative of exponential function [10]

$$D_{x,q}^{\mu}\left\{x^{\lambda-1} \sum_{n=0}^{\infty} \frac{x^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{24}$$

Or equivalently,

$$D_{x,q}^{\mu}\left\{x^{\lambda-1} e^x\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{25}$$

(2). If we take $a_n = 1$ in equation (15) it reduces to fractional q-derivative of Bessel-Maitland function denoted by $J_\delta^\upsilon(x)$

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(\delta n+\upsilon+1)n!}\right\}=\sum_{n=0}^{\infty}\frac{a_n D_{x,q}^{\mu}\{(-x)^{\lambda+n-1}\}}{\Gamma(\delta n+\upsilon+1)n!} \tag{26}$$

If we put $a_n = n!$, $\upsilon = 0$, $x \to -x$ in equation (15) then it reduces to fractional q-derivative of Mittag-Leffler function [7]

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\delta n+1)}\right\}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\delta n+1)}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{27}$$

Or equivalently,

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}E_\delta(x)\right\}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\delta n+1)}D_{x,q}^{\mu}\left\{x^{\lambda+n-1}\right\} \tag{28}$$

(3). If we put $a_n = n!$, $\upsilon \to \upsilon - 1$, $x \to -x$ in equation(15) then it reduces to fractional q-derivative of generalized Mittag-Leffler function [7]

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\delta n+\upsilon)}\right\}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\delta n+\upsilon)}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{29}$$

(4). If we take $a_n = 1$ in equation (12) it reduces to fractional q-derivative of multiindex Mittag-Leffler function

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{\prod_{j=1}^{m}\Gamma(\mu j+\frac{n}{\rho j})}\right\}=\sum_{n=0}^{\infty}\frac{1}{\prod_{j=1}^{m}\Gamma(\mu j+\frac{n}{\rho j})}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{30}$$

Or equivalently,

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}E(\tfrac{1}{\rho i}),(\mu i)(z)\right\}=\sum_{\mathrm{n}=0}^{\infty}\frac{1}{\prod_{j=1}^{m}\Gamma(\mu j+\frac{n}{\rho j})}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{31}$$

(5). If we take $m = 2$, $a_n = 1$ in (12) it reduces to

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu 1+\frac{n}{\rho 1})\Gamma(\mu 2+\frac{n}{\rho 2})}\right\}=\sum_{\mathrm{n}=0}^{\infty}\frac{1}{\Gamma(\mu 1+\frac{n}{\rho 1})\Gamma(\mu 2+\frac{n}{\rho 2})}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{32}$$

Which is the fractional q-derivative of the function shown by Dzrbashjan [2] that it is an entire function of order?

$$\rho=\frac{\rho 1\rho 2}{\rho 1+\rho 2}\quad \mathit{and\ \ type}\quad \sigma=(\tfrac{\rho 1}{\rho})^{\frac{\rho}{\rho 1}}(\tfrac{\rho 2}{\rho})^{\frac{\rho}{\rho 2}}$$

(6). If we put $m = 1$, $\rho j = \frac{1}{\alpha}$, $\mu j = \beta$ in (12), we get

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\right\}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{33}$$

Which is the fractional q- derivative of generalized Mittag-Leffler function denoted by $E\alpha,\beta(z)$.

(7). If we take $m = 1$, $\rho j = \frac{1}{\alpha}$, $\mu j = 1$ in (12), we get

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\right\}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+1)}D_{x,q}^{\mu}\{x^{\lambda+n-1}\} \tag{34}$$

Which is the fractional q- derivative of Mittag-Leffler function denoted by $E\alpha(z)$

(8). If we take $\alpha = 1$ in (34), we get the fractional q-derivative of the Exponential function [9] e^x

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{z^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{35}$$

(9). If we take $a_n = 1$ in equation (9) it reduces to fractional q-derivative of The Wright generalized hypergeometric function

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(ai+nAj)\,x^n}{\prod_{j=1}^{q}\Gamma(bi+nBj)n!}\right\} = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(ai+nAj)}{\prod_{j=1}^{q}\Gamma(bi+nBj)n!}D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{36}$$

Or equivalently,

$$D^{\mu}_{x,q}\{x^{\lambda-1}p\Psi q\,(z)\} = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(ai+nAj)}{\prod_{j=1}^{q}\Gamma(bi+nBj)n!}D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{37}$$

(10). If we take $a_n = 1$, $p = 0$, $q = 1$, $bi = \alpha$ and $Bj = \beta$ in (36) it reduces to

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\beta n+\alpha)n!}\right\} = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\beta n+\alpha)n!}D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{38}$$

Which is the fractional q-derivative of the Wright function denoted by $\Phi(\alpha,\beta;x)$ and introduced by Wright

(11). If we put $\beta = \delta$, $\alpha = \upsilon + 1$ and $x \to -x$ in (38), we get

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(\delta n+\upsilon+1)n!}\right\} = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\delta n+\upsilon+1)n!}D^{\mu}_{x,q}\{(-x)^{\lambda+n-1}\} \tag{39}$$

Which is the fractional q- derivative of the Wright generalized Bessel function denoted by $J^{\delta}_{\upsilon}(x)$

(12). If we take $\alpha = 1$, $\beta = 0$ in (38), we get the fractional q-derivative of the exponential function [9]

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}D^{\mu}_{x,q}\{x^{\lambda+n-1}\} \tag{40}$$

(13). If we take $b_n = 1$ in equation (18) it reduces to fractional q-derivative of the F-function of Robotnov and Hartley [3] denoted by $F_{\omega}[a,x]$

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{a^n\,x^{(n+1)\omega-1}}{\Gamma((n+1)\omega)}\right\} = \sum_{n=0}^{\infty}\frac{a^n}{\Gamma((n+1)\omega)}D^{\mu}_{x,q}\{x^{\lambda+(n+1)\omega-2}\} \tag{41}$$

(14). If we take $a = \omega = 1$ in (41) it reduces to

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)}\right\} = \sum_{n=0}^{\infty}\frac{D^{\mu}_{x,q}\{x^{\lambda+n-1}\}}{\Gamma(n+1)} \tag{42}$$

Which is the fractional q-derivative of the Mittag-Leffler function [7] $E_1(x)$ or generalized Mittag-Leffler function [8] $E_{1,1}(x)$ or Exponential function [10] e^x.

(15). If we take $b_n = 1$ in equation (21) it reduces to fractional q-derivative of the function $E_x[\nu,a]$ defined by Miller and Ross [6]

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{a^n\,x^{n+\nu}}{\Gamma(n+\nu+1)}\right\} = \sum_{n=0}^{\infty}\frac{a^n}{\Gamma(n+\nu+1)}\,D^{\mu}_{x,q}\{x^{\lambda+n+\nu-1}\} \tag{43}$$

(16). If we take $a = 1$, $\nu = 0$ in (43) it reduces to

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)}\right\} = \sum_{n=0}^{\infty}\frac{D^{\mu}_{x,q}\{x^{\lambda+n-1}\}}{\Gamma(n+1)} \tag{44}$$

Which is the fractional q-derivative of the Mittag-Leffler function [7] $E_1(x)$ or generalized Mittag-Leffler function [8] $E_{1,1}(x)$ or Exponential function [10] e^x.

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