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Analytical and Numerical Analysis of Newell–Whitehead–Segel equations by HAM

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Abstract: In this paper, we employ Homotopy Analysis Method (HAM) to Newell–Whitehead–Segel equation which is used in different fields of sciences like bio-engineering, chemical engineering, biology, mechanical engineering and ecology etc. Approximate series solutions have been obtained and the results are compared with the closed form solutions of the equation which shows that this technique gives high accurate results. HAM is a reliable technique, easy to use and is widely applicable to a large class of non-linear differential equations.

Keywords: HAM, Newell–Whitehead–Segel equation.

1. Introduction

The Homotopy Analysis Method (HAM) was proposed by Liao [1] in 1992 which is an analytical method to solve the linear or non-linear differential problems. It provides an analytical approach to obtain a series solution. This method is widely applicable to different fields of sciences where linear or non-linear equations, including ordinary differential equations, algebraic equations, partial differential equations and coupled equations are formed. One of the biggest advantages of this method is that it gives us a convenient way to control and adjust the convergence region and the rate of approximate series according to the necessity. Another advantage of this method is that it is independent of any kind of small or large physical parameters present in the problem. Moreover, it ensures the convergence of the series solution and thus becomes valid even for strong non-linear problems. With this method, one has the liberty to choose the auxiliary things on his own, viz., non-zero auxiliary parameter and non-zero auxiliary function. Liao in his book [2] obtained analytical and numerical solutions to many non-linear differential equations. This book is very useful in better understanding of this method. His paper [3] may also be referred to clearly understand the difference between the functioning of the Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM). A number of researchers have done a substantial amount of work on this method to solve a large number of non-linear problems arising in different fields of sciences, e.g., Jafari [4] used HAM to solve a couple of evolution equations and made comparison with Adomian's decomposition method, Arora and Kumar [5] solved the Coupled Drinfeld's–Sokolov–Wilson System by HAM, Abbasbandy in 2006 [6] and 2007 [7], solved nonlinear equations arising in heat transfer, and a generalized Hirota-Satsuma coupled KdV equation by using HAM. Also, he in 2008 [8] found out soliton solutions for the 5th-order KdV equation by the HAM. Cheng [9] obtained series solutions of nano

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boundary layer flows by means of the HAM, Jafari [10] solved the fractional linear and non-linear diffusion wave equation, Liu and Liu [11] compared the HAM with the general series method, etc.

In this paper, the HAM method has been successfully employed on two types of the Newell–Whitehead–Segel equation. Their approximate analytical and numerical solutions have been found out. The results have been computed using the software package "Mathematica". The computed results are then compared with the known exact solutions of the corresponding differential equations, and the accuracy of the HAM has been shown with the help of absolute errors between the approximated solutions and the exact solutions. This method is very simple, effective and is being widely used to study the differential equations to a large scale.

2. Basic Concepts of the Homotopy Analysis Method

In order to describe the Homotopy Analysis Method, let us consider the following non-linear differential equation of the form:

$$N[u(x,t)] = 0, \tag{1}$$

where $u(x,t)$ is a dependent variable and an unknown function of x and t which denote the space and time variables, respectively, and N is a non-linear operator.

Definition 2.1. *Let ψ be a function of the homotopy parameter q, then*

$$D_n(\psi) = \frac{1}{n!}\frac{\partial^n \psi(x,t;q)}{\partial q^n}\bigg|_{q=0}, \quad where \quad n \geq 0, \tag{2}$$

is called the n^{th}-order homotopy derivative of ψ [1].

Now, the following zero-order deformation equation is constructed:

$$(1-q)L[\psi(x,t;q) - u_0(x,t)] = q\hbar H(x,t)\, N[\psi(x,t;q)], \tag{3}$$

where L is an auxiliary linear operator having the property $Lf = 0$ when $f = 0$, $\psi(x,t;q)$ denotes an unknown function, $\hbar$ is a non-zero auxiliary parameter, q is an embedding parameter whose value lies in $[0,1]$, u_0 is an initial guess of u and H is a non-zero auxiliary function. One of the advantages of this method is that it contains some auxiliary things which facilitates us in choosing them with full liberty. Therefore, when the values of q are $q = 0$ and $q = 1$, we get

$$\begin{aligned} \psi(x,t;0) &= u_0(x,t), \\ \psi(x,t;1) &= u(x,t), \end{aligned} \tag{4}$$

i.e. the solution $\psi(x,t;q)$ ranges from the initial guess $u_0(x,t)$ to the solution $u(x,t)$ of the given problem, as the value of q increases from 0 to 1. Now, the function ψ is expanded in the Taylor's series about $q = 0$ in the following way:

$$\psi(x,t;q) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)q^n, \tag{5}$$

where

$$u_n(x,t) = D_n(\psi). \tag{6}$$

The series (5) converges at $q = 1$, if we choose the initial guess u_0, the auxiliary linear operator L, the non-zero auxiliary function H and parameter $\hbar$, appropriately. Then the series (5) will take the form

$$\psi(x,t;1) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t). \tag{7}$$

This equation should be one of the solutions of the original differential equation (1). Now, we differentiate n-times the zero-order deformation equation (3) w.r.t. q, put $q = 0$ and then divide the so-obtained equation throughout by $n!$, we obtain the n^{th}-order deformation equation, given by

$$L[u_n(x,t) - \sigma_n u_{n-1}(x,t)] = \hbar H\,(x,t)\, R_n(u_{n-1},x,t), \tag{8}$$

where

$$R_n(u_{n-1},x,t) = D_{n-1}\left(N[\psi(x,t;q)]\right) \quad and \quad \sigma_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1. \end{cases} \tag{9}$$

The n^{th}-order deformation equation (8) is linear and it can be solved easily by using computation software package like Mathematica, Matlab, Maple etc.

3. Application of the Homotopy Analysis Method (HAM)

In this section, we shall apply the Homotopy Analysis Method on non-linear partial differential equations with the given initial conditions.

3.1. The Newell–Whitehead–Segel Equation

The most general form of the Newell–Whitehead–Segel equation is given by

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \beta u - \gamma u^m,$$

where $(\alpha, \beta, \gamma, x) \in \Re$, $t \ge 0$, m is a positive integer and $\alpha > 0$. $u(x,t)$ is a function which represents the velocity of a fluid-flow in an infinitely long pipe with a small diameter or the nonlinear distribution of temperature in an infinitely long and thin rod. Such equations have huge applications in various fields of sciences like bio-engineering, chemical engineering, biology, mechanical engineering and ecology etc. Here, we shall solve two different forms of the Newell–Whitehead–Segel equation with different initial conditions. In Case 1, we take $\alpha = \beta = \gamma = 1$ and $m = 2$. And, in Case 2, we take $\alpha = 1$, $\beta = 3$, $\gamma = 4$ and $m = 3$.

Case 1: Let us consider the Newell–Whitehead–Segel equation [12]

$$u_t = u_{xx} + u - u^2, \tag{10}$$

subject to the initial condition

$$u(x,0) = \frac{1}{\left(1 + e^{\frac{x}{\sqrt{6}}}\right)^2}. \tag{11}$$

In order to apply the HAM, we first choose the linear operator L defined as

$$L[\psi(x,t;q)] = \frac{\partial \psi(x,t;q)}{\partial t}, \tag{12}$$

having the property $L(c) = 0$, where c is an arbitrary constant. Now, from the given equation (10), we define the nonlinear operator as

$$N[\psi(x,t;q)] = \frac{\partial \psi(x,t;q)}{\partial t} - \frac{\partial^2 \psi(x,t;q)}{\partial x^2} - \psi(x,t;q) + \psi^2(x,t;q). \tag{13}$$

With the help of the above mentioned definitions, we form the zero-order deformation equation as follows:

$$(1-q)L[\psi(x,t;q) - u_0(x,t)] = q\hbar H N[\psi(x,t;q)], \tag{14}$$

and when $q = 0$ and $q = 1$, we have the following values

$$\psi(x,t;0) = u_0(x,t), \quad \psi(x,t;1) = u(x,t). \tag{15}$$

We, now, differentiate n-times the deformation equation (14) w.r.t. q, put $q = 0$ and then divide the so-obtained equation throughout by $n!$ to obtain the n^{th}-order deformation equation, given by

$$L[u_n(x,t) - \sigma_n u_{n-1}(x,t)] = \hbar H R_n(u_{n-1}, x, t), \tag{16}$$

subject to the initial condition

$$u_n(x,0) = 0, \tag{17}$$

where

$$R_n\left(u_{n-1}, x, t\right) = \frac{\partial u_{n-1}\left(x,t\right)}{\partial t} - \frac{\partial u_{n-1}^2\left(x,t\right)}{\partial x^2} - u_{n-1}(x,t) + u_{n-1}^2\left(x,t\right). \tag{18}$$

For convenience, we choose the value of the auxiliary function H as 1 and parameter $\hbar = h$, then the solution of the n^{th}-order deformation equation (16) takes the form, for $n \geq 1$, as

$$u_n(x,t) = \sigma_n u_{n-1}(x,t) + hL^{-1}\left[R_n(u_{n-1}, x, t)\right]. \tag{19}$$

The initial approximation to obtain the solution of the equation (10) is taken as $u_0(x,t) = u(x,0)$, given by the equation (11). Therefore, we obtain the following approximations:

$$u_0(x,t) + u_1(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3} + \frac{e^{\frac{x}{\sqrt{6}}}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3} - \frac{5hte^{\frac{x}{\sqrt{6}}}}{3\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}, \tag{20}$$

and so on. Due to the large expressions of the consecutive terms, we have not mentioned them here. On solving the equations (11) and (20), simultaneously, we may obtain the expression for $u_1(x,t)$. In the same way, by giving the values to n as $2, 3, ...$ in (19), we obtain $u_i(x,t)$ for $i \geq 2$. Here, we obtain the fifteen-terms approximation series solution for the equation (10), which is given by

$$u(x,t) = \sum_{i=0}^{14} u_i(x,t). \tag{21}$$

3.2. Evaluation of Convergence and Numerical Results of Case 1

Here, we observe that the series solution of the equation (10) contains the auxiliary parameter $\hbar = h$. This parameter is also known as the, "convergence control parameter" of the Homotopy Analysis Method, and, thus, governs the convergence of the series and rates of the approximation of this method. Now, in order to control the convergence of the approximation series, the value of the auxiliary parameter h is chosen appropriately with the help of the h-curve. In the h-curve, the valid region for the admissible values of h corresponds to the line segment which is either parallel or nearly parallel to the horizontal axis. In the Figure 1, the h-curve of $u_{tt}(1,0)$ of the equation (10) is drawn which is obtained by the fifteen-terms series approximation solution of the HAM, and a parallel line segment can easily be seen which yields the range for the admissible values of h. We can certainly determine the better approximations in the initial few terms only, if we choose a good enough auxiliary linear operator and initial guess. In case if these values are not good enough chosen but are moderate then also convergent results can be obtained just by choosing the value of the auxiliary parameter h appropriately.

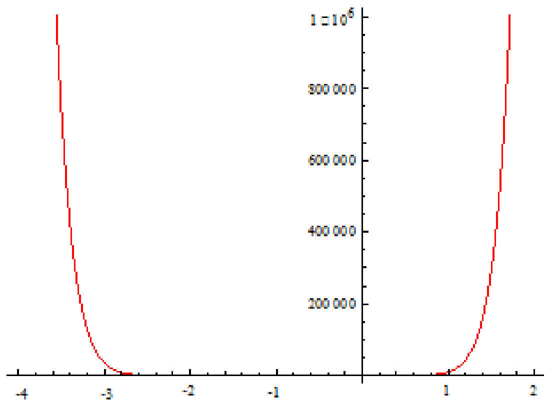


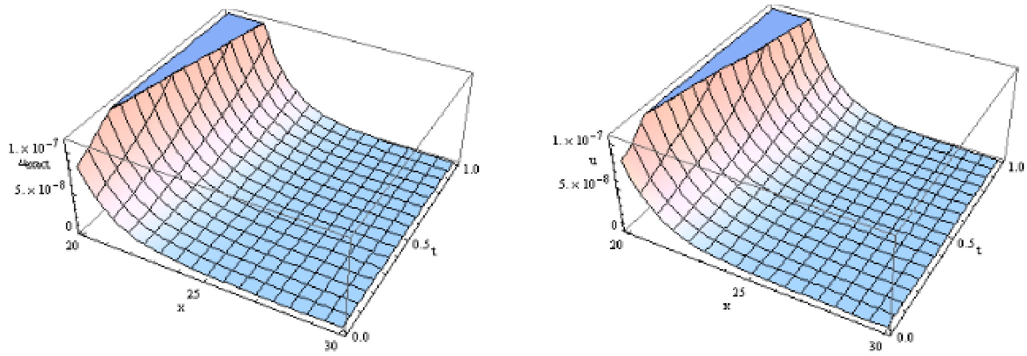
Figure 1: The h-curve of $u_{tt}(1,0)$ of the equation (10) obtained by the fifteen-terms approximation solution of the Homotopy Analysis Method

From the Figure 1, we choose the most appropriate value of h for our problem to be $h = -2.0$. Now, we will show the efficiency of the HAM by comparing the so-obtained approximate solution (21) with the exact solution of the equation (10), given by

$$u(x,t) = \frac{1}{\left(1 + e^{\frac{x}{\sqrt{6}} - \frac{5t}{6}}\right)^2} \tag{22}$$

	t					
x	0.0	0.2	0.4	0.6	0.8	1.0
10	5.963E-19	1.019E-05	2.329E-05	1.779E-05	4.895E-05	3.752E-05
20	3.044E-22	2.812E-09	6.461E-09	5.237E-09	1.277E-08	1.010E-08
30	4.621E-25	7.994E-13	1.837E-12	1.490E-12	3.628E-12	2.868E-12
40	4.181E-29	2.274E-16	5.225E-16	4.239E-16	1.032E-15	8.156E-16
50	3.852E-34	6.468E-20	1.486E-19	1.206E-19	2.935E-19	2.320E-19

Table 1: Absolute errors for u(x, t) obtained by the fifteen-terms approximate solution of the HAM for $h = -2.0$.



(a) (b)

Figure 2: (a). $u(x,t)$ exact and (b). $u(x,t)$ computed of the Newell–Whitehead–Segel equation (10)

Case 2: Let us consider another form of Newell–Whitehead–Segel equation [12]

$$u_t = u_{xx} + 3u - 4u^3, \tag{23}$$

subject to the initial condition

$$u(x,0) = \frac{\sqrt{3}e^{\sqrt{6}x}}{2\left(e^{\sqrt{6}x} + e^{\sqrt{6}x/2}\right)}. \tag{24}$$

In order to apply the HAM, we first choose the linear operator L defined as

$$L[\psi(x,t;q)] = \frac{\partial \psi(x,t;q)}{\partial t}, \tag{25}$$

having the property $L(c) = 0$, where c is an arbitrary constant. Now, from the given equation (23), we define the nonlinear operator as

$$N[\psi(x,t;q)] = \frac{\partial \psi(x,t;q)}{\partial t} - \frac{\partial^2 \psi(x,t;q)}{\partial x^2} - 3\psi(x,t;q) + 4\psi^3(x,t;q). \tag{26}$$

With the help of the above mentioned definitions, we form the zero-order deformation equation as follows:

$$(1-q)L[\psi(x,t;q) - u_0(x,t)] = q\hbar H N[\psi(x,t;q)], \tag{27}$$

and when $q = 0$ and $q = 1$, we have the following values

$$\psi(x,t;0) = u_0(x,t), \qquad \psi(x,t;1) = u(x,t). \tag{28}$$

We, now, differentiate n-times the deformation equation (27) w.r.t. q, put $q = 0$ and then divide the so-obtained equation throughout by $n!$, we obtain the n^{th}-order deformation equation, given by

$$L[u_n(x,t) - \sigma_n u_{n-1}(x,t)] = \hbar H R_n(u_{n-1}, x, t), \tag{29}$$

subject to the initial condition

$$u_n(x,0) = 0, \tag{30}$$

where

$$R_n\left(u_{n-1}, x, t\right) = \frac{\partial u_{n-1}\left(x,t\right)}{\partial t} - \frac{\partial u_{n-1}^2\left(x,t\right)}{\partial x^2} - 3u_{n-1}(x,t) + 4u_{n-1}^3\left(x,t\right). \tag{31}$$

For convenience, we choose the value of the auxiliary function H as 1 and parameter $\hbar = h$, then the solution of the n^{th}-order deformation equation (29) takes the form, for $n \geq 1$, as

$$u_n(x,t) = \sigma_n u_{n-1}(x,t) + \hbar L^{-1}\left[R_n(u_{n-1},x,t)\right]. \tag{32}$$

The initial approximation to obtain the solution of the equation (23) is taken as $u_0(x,t) = u(x,0)$, given by the equation (24). Therefore, we obtain the following approximations:

$$u_0(x,t) + u_1(x,t) = \frac{\sqrt{3}e^{\sqrt{\frac{3}{2}}x}}{2\left(1+e^{\sqrt{\frac{3}{2}}x}\right)^2} + \frac{\sqrt{3}e^{\sqrt{6}x}}{2\left(1+e^{\sqrt{\frac{3}{2}}x}\right)^2} - \frac{9\sqrt{3}hte^{\sqrt{\frac{3}{2}}x}}{4\left(1+e^{\sqrt{\frac{3}{2}}x}\right)^2}, \tag{33}$$

and so on. Due to the large expressions of the consecutive terms, we have not mentioned them here. On solving the equations (24) and (33), simultaneously, we may obtain the expression for $u_1(x,t)$. In the same way, by giving the values to n as $2, 3, \ldots$ in (32), we obtain $u_i(x,t)$ for $i \geq 2$. Here, we obtain the eleven-terms approximation series solution for the equation (23), which is given by

$$u(x,t) = \sum_{i=0}^{10} u_i(x,t). \tag{34}$$

3.3. Evaluation of Convergence and Numerical Results of Case 2

Here, we again observe that the series solution of the equation (23) contains the auxiliary parameter $\hbar = h$. This parameter is also known as the, "convergence control parameter" of the Homotopy Analysis Method, and, thus, governs the convergence of the series and rates of the approximation of this method. Now, in order to control the convergence of the approximation series, the value of the auxiliary parameter h is chosen appropriately with the help of the h-curve. In the h-curve, the valid region for the admissible values of h corresponds to the line segment which is either parallel or nearly parallel to the horizontal axis. In the Figure 3, the h-curve of $u_{tt}(1,0)$ of the equation (23) is drawn which is obtained by the eleven-terms approximation solution of the HAM, and a parallel line segment can easily be seen which yields the range for the admissible values of h. We can certainly determine the better approximations in the initial few terms only, if we choose a good enough auxiliary linear operator and initial guess. In case if these values are not good enough chosen but are moderate then also convergent results can be obtained just by choosing the value of the auxiliary parameter h appropriately.

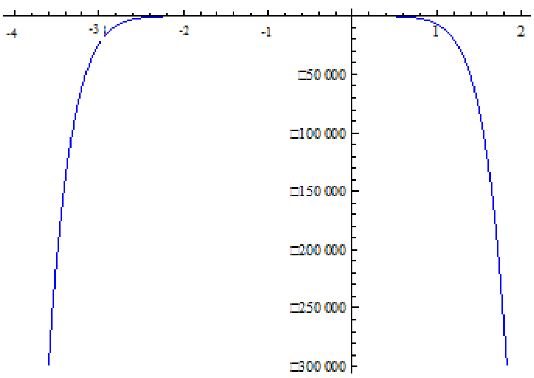


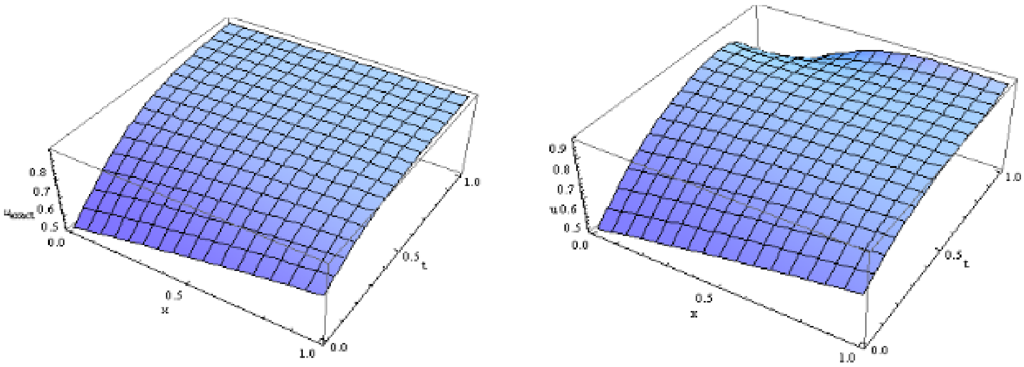
Figure 3: The h-curve of $u_{tt}(1,0)$ of the equation (23) obtained by the eleven-terms approximation solution of the Homotopy Analysis Method

From the Figure 3, we choose the most appropriate value of h for our problem to be $h = -0.6$. Now, we will show the efficiency of the HAM by comparing the so-obtained approximate solution (34) with the exact solution of the equation (23), given by

$$u(x,t) = \frac{\sqrt{3}e^{\sqrt{6}x}}{2\left(e^{\sqrt{6}x} + e^{\frac{\sqrt{6}x}{2} - \frac{9t}{2}}\right)}. \tag{35}$$

	t					
x	0	0.05	0.1	0.2	0.6	1.0
-30	9.861E-32	9.190E-21	5.045E-20	4.706E-19	7.441E-17	1.865E-15
-20	3.231E-27	1.916E-15	1.052E-14	9.810E-14	1.551E-11	3.887E-10
-10	0	3.993E-10	2.192E-09	2.045E-08	3.232E-06	8.090E-05
10	3.442E-15	2.686E-12	3.228E-11	3.946E-11	2.931E-11	6.321E-10
20	6.661E-15	6.661E-15	6.772E-15	6.439E-15	6.439E-15	9.659E-15
30	6.106E-15	6.106E-15	6.106E-15	6.106E-15	6.106E-15	6.106E-15

Table 2: Absolute errors for $u(x,t)$ obtained by the eleven-terms approximate solution of the HAM for $h = -0.6$.



(a) (b)

Figure 4: (a). $u(x,t)$ exact and (b). $u(x,t)$ computed of the Newell–Whitehead–Segel equation (23)

4. Conclusion

In the present paper, the HAM technique is employed on two types of Newell-Whitehead-Segel equation to obtain approximate analytical and numerical solutions. Unlike other analytical methods, this method ensures the convergence of series solutions. The convergence can be controlled by choosing a proper value of the convergence-control parameter h. The software package “MATHEMATICA” is used to compute the numerical results. These numerical results are compared with the known exact solutions of the given problems. These results show the accuracy of this method with the help of absolute errors. The minimum the absolute error, the better is the accuracy of the method. In the present paper, very less absolute errors are obtained by the HAM. This method provides highly accurate numerical solutions and can be applied to a wide class of non-linear problems.

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