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Fractional Calculus and Certain Integral Transform of Extended τ-Hypergeometric Function

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Abstract: In this paper, authors establish the some new integral and derivative formulas of generalized τ-hypergeometric functions ${}_3\Gamma_2^\tau(z)$ defined by (R. K. Gupta et al. [3, 4]). Furthermore by applying some integral transforms as Beta-transform, Verma transform, Whittaker transform, Laplace transform and P_α-transform on the resulting formulas.

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1. Introduction and Preliminaries

During the last some decades, number of authors have studied, in depth, the properties, applications and different extensions of various hypergeometric operators of fractional derivatives and integrals. A detailed account of such operators along with their properties and applications have been considered by several authors (see, [16–18]). The fractional derivative operators involving various special functions have significant importance and applications in various sub - field of applicable mathematical analysis. Extended τ hypergeometric function ${}_3\Gamma_2^\tau(z)$ was given by (R. K. Gupta et al. [3, 4]) and defined as follows:

$$ {}_3\Gamma_2^\tau(z) = {}_3\Gamma_2^\tau((\lambda,k),a,b;c,d;z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\lambda;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{z^n}{n!}, \tag{1}$$

$$(k\geq 0;\tau>0;|z|<1,\Re(d)>\Re(a)>0,\ \Re(c)>\Re(b)>0\,when\,k=0).$$

Notes:-

(1). If we put b = d, then (1) reduces to the extended τ-hypergeometric function ${}_2\Gamma_1^\tau(z)$ given by Parmer ([11] p.422, eq.(2.1)) as defined as

$$ {}_2\Gamma_1^\tau(z) = {}_2\Gamma_1^\tau((\lambda,k),b;c;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\lambda;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!} \tag{2}$$

$$(\lambda,b\in\mathbb{C},c\in\mathbb{C}\setminus\mathbb{Z}_0^-,k\geq 0;\tau>0;|z|<1,\Re(c)>\Re(b)>0\,when\,k=0).$$

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(2). If we put b = d and $\tau = 1$, then (1) reduces to the extended Gauss hypergeometric function ([12] p. 487, eq.17) given by

$$ {}_2F_1(z) = {}_2F_1((\lambda, k), b; c; z) = \sum_{n=0}^{\infty} \frac{[\lambda; k]_n\,(b)_n}{(c)n} \frac{z^n}{n!}. \tag{3}$$

(3). If we take $\tau = 1$ and $k = 0$ in (1), then it reduces to the classical Gauss's hypergeometric function as

$$ {}_3\Gamma_2(z) = {}_3\Gamma_2((\lambda, a, b; c, d; z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n\,(a)_n\,(b)_n}{(c)_n\,(d)_n} \frac{z^n}{n!} \tag{4}$$

(4). If we take $b = d$, $\tau = 1$ and $k = 0$ in (1), then it reduces to the classical Gauss's hypergeometric function as

$$ {}_2F_1(z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!}. \tag{5}$$

We need to recall the following pair of the Saigo hypergeometric fractional integral operators (see Saigo [7], Kiryakova [19]. For $x > 0$, $l, m, \xi \in \mathbb{C}$ and $\alpha > 0$, we have

$$(I_{0,x}^{l,m,\xi} f(t))(x) = \frac{x^{-l-m}}{\Gamma(l)} \int_0^x (x-t)^{l-1} {}_2F_1\left(l+m, -\xi; l; 1-\frac{t}{x}\right) f(t)\,dt, \tag{6}$$

$$(J_{x,\infty}^{l,m,\xi} f(t))(x) = \frac{1}{\Gamma(l)} \int_x^\infty (t-x)^{l-1} t^{-l-m} {}_2F_1\left(l+m, -\xi; l; 1-\frac{t}{x}\right) f(t)\,dt, \tag{7}$$

We also recall the Pochhammer symbol $(\lambda)_n$ defined (for $\lambda \in \mathbb{C}$) as (Rainville [8])

$$(\lambda)_n = \begin{cases} 1, & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & (n \in \mathbb{N}) \\ \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, & (\lambda \in \mathbb{C}/\mathbb{Z}_0^-) \end{cases} \tag{8}$$

where $\mathbb{Z}_0^-$ denotes the set of non positive integers.

The operator $I_{0,x}^{l,m,\xi}(\cdot)$ contains both the Riemann-Liouville $R_{0,x}^{l}(\cdot)$ and the Erdélyi-Kober $E_{0,x}^{l,\xi}(\cdot)$ fractional integral operators as particular cases, by means of the relationships:

$$(R_{0,x}^{l} f(t))(x) = (I_{0,x}^{l,-l,\xi} f(t))(x) = \frac{1}{\Gamma(l)} \int_0^x (x-t)^{l-1} f(t)\,dt,$$

$$(E_{0,x}^{l,\xi} f(t))(x) = (I_{0,x}^{l,0,\xi} f(t))(x) = \frac{x^{-l-\xi}}{\Gamma(l)} \int_0^x (x-t)^{l-1} t^{\eta} f(t)\,dt.$$

And also, note that the operator (7) incorporates the Weyl type and the Erdélyi-Kober fractional operators as follows:

$$(W_{x,\infty}^{l} f(t))(x) = (J_{x,\infty}^{l,-l,\xi} f(t))(x) = \frac{1}{\Gamma(l)} \int_x^\infty (t-x)^{l-1} f(t)\,dt,$$

$$(K_{x,\infty}^{l,\xi} f(t))(x) = (J_{x,\infty}^{l,0,\xi} f(t))(x) = \frac{x^{\xi}}{\Gamma(l)} \int_x^\infty (t-x)^{l-1} t^{-l-\xi} f(t)\,dt.$$

We also use the following image formulas which are well known facts and easy consequences of the definitions of the operators (6) and (7) (see Saigo [7]):

$$(I_{0,x}^{l,m,\xi} t^{\lambda-1})(x) = \frac{\Gamma(\lambda)\Gamma(\lambda-m+\xi)}{\Gamma(\lambda-m)\Gamma(\lambda+l+\xi)} x^{\lambda-m-1}; \quad (\Re(\lambda) > 0, \Re(\lambda-m+\xi) > 0). \tag{9}$$

$$(J_{x,\infty}^{l,m,\xi} t^{\lambda-1})(x) = \frac{\Gamma(m-\lambda+1)\Gamma(\xi-\lambda+1)}{\Gamma(1-\lambda)\Gamma(m+l-\lambda+\xi+1)} x^{\lambda-m-1}; \ (\Re(m-\lambda+1)>0, \Re(\xi-\lambda+1)>0) \tag{10}$$

Then the generalized fractional derivative operators are defined as (Saigo [7])

$$\left(D_{0+}^{l,m,\xi} f\right)(x) = \left(I^{-l,-m,l+\xi} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-l+\xi,-m-\xi,l+\xi-n} f\right)(x); \ (\mathbb{R}(l)\geq 0, n=[\mathbb{R}(l)]+1). \tag{11}$$

$$\left(D_{0-}^{l,m,\xi} f\right)(x) = \left(I_{-}^{-l,-m,l+\xi} f\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-l+\xi,-m-\xi,l+\xi-n} f\right)(x); \ (\mathbb{R}(l)\geq 0, n=[\mathbb{R}(l)]+1). \tag{12}$$

The operator $\left(D_{0+}^{l,m,\xi}\right)(.)$ contains the Riemann-Liouville $D_{0+}^{l}(.)$ and Weyl fractional derivatives by means of the following relationships:

$$(D_{0+}^{l,-l,\xi} f)(x) = (D_{0+}^{l} f)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-l)} \int_0^x \frac{f(t)dt}{(x-t)^{l-n+1}}; \ (x>0, \ n=[\mathbb{R}(l)]+1, l\in\mathbb{C}, \mathbb{R}(l)\geq 0). \tag{13}$$

$$(D_{0-}^{l,-l,\xi} f)(x) = (D_{-}^{l} f)(x) = \left(-\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-l)} \int_x^\infty \frac{f(t)dt}{(t-x)^{l-n+1}}; \ (x>0, \ n=[\mathbb{R}(l)]+1, l\in\mathbb{C}, \mathbb{R}(l)\geq 0). \tag{14}$$

It is noted that the operators (11), (12) include also the Erdélyi-Kober fractional derivative operators (Kiryakova [9]) for $m=0$ and $l,\eta\in\mathbb{C}, \mathbb{R}(l)\geq 0$:

$$(D_{0+}^{l,0,\xi} f)(x) = (D_{\xi,l}^{+} f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+}^{-l+n,-l,-l+\xi-n} f)(x); \ (x>0, \ n=[\mathbb{R}(l)]+1, l\in\mathbb{C}).$$

$$(D_{0-}^{l,0,\xi} f)(x) = (D_{\xi,l}^{-} f)(x) = \left(-\frac{d}{dx}\right)^n (I_{-}^{-l+n,-l,-l+\xi-n} f)(x); \ (x>0, \ n=[\mathbb{R}(l)]+1, l\in\mathbb{C}).$$

We also use the following image formulae which are easy consequences of the operators definitions (Saigo [7]). Namely, for $l,m,\xi\in\mathbb{C}$ and $\Re(l)\geq 0$, $x>0$, $\lambda>-\min[0,\mathfrak{l}+\mathfrak{m}+\xi]$,

$$D_{0+}^{l,m,\xi}\left(x^{\lambda-1}\right) = \frac{\Gamma(\lambda)\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+m)\Gamma(\lambda+\xi)} x^{\lambda+m-1}, \tag{15}$$

and for $l,m,\xi\in\mathbb{C}$ and $x>0$, $\Re(l)\geq 0$, $\lambda<1+\min[(-m-n),(\mathfrak{l}+\xi)]$,

$$D_{0-}^{l,m,\eta}\left(x^{\lambda-1}\right) = \frac{\Gamma(1-\lambda-m)(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)} x^{\lambda+m-1}. \tag{16}$$

2. Definition of Certain Integral Transforms

We present some transforms, which exhibit the connection between the Euler, Varma, Pathway, Laplace and Whittaker integral transforms and the generalized incomplete τ- hypergeometric type function. We begin by recalling the following beta transform of a function (Sneddon [2]):

$$B\{f(t):l,m\} = \int_0^1 t^{l-1}(1-t)^{m-1} f(t)dt; \ l\ ,m>0. \tag{17}$$

The Verma transform of a function $f(t)$ is defined by the following integral equation (see Mathai [1], pp. 55):

$$V(f,k,m;s) = \int_0^\infty (st)^{m-\frac{1}{2}} \exp(-\frac{1}{2}st) \ W_{k,m}(st) \ f(t)dt, (\Re(s)>0), \tag{18}$$

where $W_{k,m}$ is the Whittaker function defined by (Mathai [1], pp. 55):

$$W_{k,m}(t) = \sum_{m,-m} \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)} M_{k,m}(t), \tag{19}$$

where the summation symbol indicates that the expression following it, a similar expression with m replaced by $-m$ is to be added and

$$M_{k,m}(t) = t^{m+\frac{1}{2}} \exp(-\frac{t}{2})\, {}_1F_1(\frac{1}{2} - k + m; 2m + 1; t).$$

It is interesting to observe that, for $k = -\nu + \frac{1}{2}$, the Verma transform defined by (18) reduces to the well-known Laplace transform of a function $f(t)$ (Sneddon [2]):

$$L\{f(t) : s\} = \int_0^\infty e^{-st} f(t)\, dt. \tag{20}$$

The pathway type transforms (P_ν-transforms) of a function $f(z)$ of a real variable z denoted by $P_\nu[f(z); s]$ is a function $F(s)$ of complex variable s, valid under certain conditions on $f(z)$ along with the condition $\nu > 1$, and is defined as (Kumar [10])

$$P_\phi[f(z); s] = F(s) = \int_0^\infty [1 + (\phi - 1)s]^{-\frac{z}{\phi-1}} f(z)\, dz. \tag{21}$$

For $\rho \in \mathbb{C}, \Re(\rho) > 0$ and $\phi > 1$, the P_ϕ-transform of power function is given by Kumar [10] as

$$P_\phi[z^{\rho-1}; s] = \left\{ \frac{\phi - 1}{\ln[1 + (\phi - 1)s]} \right\}^\rho \Gamma(\rho) = \frac{\Gamma(\rho)}{[\wedge(\phi; s)]^\rho} \tag{22}$$

where $\wedge(\phi; s) = \frac{\ln[1+(\phi-1)s]}{\phi-1}, min\{\Re(s), \Re(\rho)\} > 0; \phi > 1$. Furthermore, upon letting $\phi \to 1$ in (21), the P_ϕ-transform is reduces to classical Laplace transform of a function $f(z)$ ([2]) is given by (20). Agarwal [15] obtained solution of fractional volterra integral equation using integral transform of pathway type.

3. Integral Transform and Fractional Calculus

3.1. Beta transform of fractional derivative of incomplete τ hypergeometric function

Theorem 3.1. *Suppose that $t, p, q > 0$; $k \geq 0; \tau > 0$; $\Re(d) > \Re(a) > 0$; $\Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that $\min\{\Re(\lambda), \Re(m)\} > 0$ and $\Re(\lambda) > -\min\{0, \Re(l + m + \xi)\}$. Then the following beta-transform of fractional derivative formula holds:*

$$B\left[D_{0+}^{l,m,\xi} \left\{ t^{\lambda-1} {}_3\Gamma_2^\tau \left[\begin{matrix} (\sigma, k), a, b; \\ c, d; \end{matrix} \; zt \right] \right\} (x) : p,\, q \right] = x^{\lambda+m-1}\, B(p,\, q)$$
$$\frac{\Gamma(\lambda)\, \lambda + l + m + \xi}{\Gamma(\lambda + m)\, \Gamma(\lambda + \xi)}\, {}_3\Gamma_2^\tau\,(x) *\, {}_3F_3 \left[\begin{matrix} \lambda, \rho,\, \lambda + m + l + \xi; \\ p + q,\, \lambda + m,\, \lambda + \xi; \end{matrix} \; x \right] \tag{23}$$

and

$$B\left[D_{0+}^{l,m,\xi} \left\{ t^{\lambda-1} {}_3\gamma_2^\tau \left[\begin{matrix} (\sigma, k), a, b; \\ c, d; \end{matrix} \; zt \right] \right\} (x) : p,\, q \right] = x^{\lambda+m-1}\, B(p,\, q)$$
$$\frac{\Gamma(\lambda)\, \lambda + l + m + \xi}{\Gamma(\lambda + m)\, \Gamma(\lambda + \xi)}\, {}_3\gamma_2^\tau\,(x) *\, {}_3F_3 \left[\begin{matrix} \lambda, \rho,\, \lambda + m + l + \xi; \\ p + q,\, \lambda + m,\, \lambda + \xi; \end{matrix} \; x \right] \tag{24}$$

Proof. For convenience, we denote the left-hand side of the result (23) by ς and using (1), we find

$$\varsigma = B\left[D_{0+}^{l,m,\xi}\left\{t^{\lambda-1}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\sigma;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{(zt)^n}{n!}\right\}(x):p,\,q\right]$$

now using (15), (17) and changing order of integration and summation, we get-

$$\varsigma = x^{\lambda+m-1}\frac{\Gamma(\lambda)\,\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+m)\Gamma(\lambda+\xi)}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\sigma;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{x^n}{n!}$$
$$\frac{(\lambda)_n\,(\lambda+l+m+\xi)_n}{(\lambda+m)_n\,(\lambda+\xi)_n}\int_0^1 z^{p+n-1}\,(1-z)^{q-1}dz$$

using classical beta function and applying the Hadamard product series yields the desired result, equation (23). As in the proof of result (24), taking the operator (15) and the result (17) into account, one can easily prove result (24). ☐

Theorem 3.2. *Suppose that $t,\,p,\,q>0;\quad k\geq 0;\,\tau>0;\,\Re(d)>\Re(a)>0;\,\Re(c)>\Re(b)>0$ and $l,m,\xi,\lambda\in\mathbb{C}$ be parameters such that $\Re(l)\geq 0\ \Re(\lambda)>-\min\{0,\Re(l+m+\xi)\}$ Then the beta-transform of left-hand side fractional derivative formula holds:*

$$B\left[D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t}\\ c,d; & \end{matrix}\right]\right\}(x):p,\,q\right]=x^{\lambda+m-1}B(p,\,q)$$
$$\frac{\Gamma(\Gamma(1-\lambda-m)\,\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\,\Gamma(1-\lambda+\xi-m)}\,{}_3\Gamma_2^{\tau}\left(\frac{1}{x}\right)*{}_3F_3\left[\begin{matrix}p,\,1-\lambda-m,\,1-\lambda+l+\xi; & \frac{1}{x}\\ p+q,\,1-\lambda,\,1-\lambda+\xi-m; & \end{matrix}\right] \tag{25}$$

and

$$B\left[D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t}\\ c,d; & \end{matrix}\right]\right\}(x):p,\,q\right]=x^{\lambda+m-1}B(p,\,q)$$
$$\frac{\Gamma(\Gamma(1-\lambda-m)\,\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\,\Gamma(1-\lambda+\xi-m)}\,{}_3\gamma_2^{\tau}\left(\frac{1}{x}\right)*{}_3F_3\left[\begin{matrix}p,\,1-\lambda-m,\,1-\lambda+l+\xi; & \frac{1}{x}\\ p+q,\,1-\lambda,\,1-\lambda+\xi-m; & \end{matrix}\right] \tag{26}$$

Proof. For convenience, we denote the left-hand side of the result (25) by ς and using (1), we find

$$\varsigma = B\left[D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\sigma;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{(\frac{z}{t})^n}{n!}\right\}(x):p,\,q\right]$$

now using (16), (17) and changing order of integration and summation, we get

$$\varsigma = x^{\lambda+m-1}\frac{\Gamma(1-\lambda-m)\,\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\sigma;k]_n\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}$$
$$\frac{x^{-n}}{n!}\frac{(1-\lambda-m)_n\,(1-\lambda+l+\xi)_n}{(1-\lambda)_n\,(1-\lambda+\xi-m)_n}\int_0^1 z^{p+n-1}\,(1-z)^{q-1}dz$$

using classical beta function and applying the Hadamard product series yields the desired result, equation (25). As in the proof of result (26), taking the operator (16) and the result (17) into account, one can easily prove result (26). ☐

Further, if we replace m with 0 and m with $-l$ in Theorems 3.1 and 3.2 then we obtain the beta-transform of the Erdélyi-Kober and Riemann-Liouville fractional derivative of the generalized incomplete τ- hypergeometric type functions given by the following corollary:

Corollary 3.3. *Suppose that $t,\ p,\ q > 0;\ \ k \geq 0;\ \tau > 0;\ \Re(d) > \Re(a) > 0;\ \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that $\min\{\Re(\lambda), \Re(m)\} > 0$ and $\Re(\lambda) > -\min\{0, \Re(l+\xi)\}$. Then the following beta-transform of right-hand side Erdélyi-Kober fractional derivative formula holds:*

$$B\left[D_{0+}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \\ c,d; & \end{matrix} zt\right]\right\}(x):p,\,q\right]=x^{\lambda-1}B(p,\,q)\frac{\Gamma(\lambda+l+\xi}{\Gamma(\lambda+\xi)}\,{}_3\Gamma_2^{\tau}\,(x)*\,{}_2F_2\left[\begin{matrix}\rho,\,\lambda+l+\xi; \\ p+q,\,\lambda+\xi;\end{matrix}\ x\right] \tag{27}$$

and

$$B\left[D_{0+}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \\ c,d; & \end{matrix} zt\right]\right\}(x):p,\,q\right]=x^{\lambda-1}B(p,\,q)\frac{\Gamma(\lambda+l+\xi}{\Gamma(\lambda+\xi)}\,{}_3\gamma_2^{\tau}\,(x)*\,{}_2F_2\left[\begin{matrix}\rho,\,\lambda+l+\xi; \\ p+q,\,\lambda+\xi;\end{matrix}\ x\right] \tag{28}$$

Corollary 3.4. *Suppose that $t,\ p,\ q > 0;\ \ k \geq 0;\ \tau > 0;\ \Re(d) > \Re(a) > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that $\min\{\Re(\lambda)\} > 0$ and $\Re(\lambda) > -\min\{0, \Re(\xi)\}$. Then the following beta-transform of right-hand side Riemann-Liouville fractional derivative formula holds:*

$$B\left[D_{0+}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \\ c,d; & \end{matrix} zt\right]\right\}(x):p,\,q\right]=x^{\lambda-l-1}B(p,\,q)\frac{\Gamma(\lambda)}{\Gamma(\lambda-l)}\,{}_3\Gamma_2^{\tau}\,(x)*\,{}_2F_2\left[\begin{matrix}\lambda,\rho; \\ p+q,\,\lambda-l;\end{matrix}\ x\right] \tag{29}$$

and

$$B\left[D_{0+}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \\ c,d; & \end{matrix} zt\right]\right\}(x):p,\,q\right]=x^{\lambda-l-1}B(p,\,q)\frac{\Gamma(\lambda)}{\Gamma(\lambda-l)}\,{}_3\gamma_2^{\tau}\,(x)*\,{}_2F_2\left[\begin{matrix}\lambda,\rho; \\ p+q,\,\lambda-l;\end{matrix}\ x\right] \tag{30}$$

Corollary 3.5. *Suppose that $t,\ p,\ q > 0;\ \ k \geq 0;\ \tau > 0;\ \Re(d) > \Re(a) > 0;\ \Re(c) > \Re(b) > 0$ and $l, m, \xi, \lambda \in \mathbb{C}$ be parameters such that $\min\{\Re(\lambda), \Re(l)\} > 0$ and $\Re(\lambda) > -\min\{0, \Re(\xi)\}$. Then the beta-transform of left-hand side Erdélyi-Kober fractional derivative formula holds:*

$$B\left[D_{0-}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t} \\ c,d; & \end{matrix}\right]\right\}(x):p,\,q\right]=x^{\lambda-1}B(p,\,q)\frac{\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi)}\,{}_3\Gamma_2^{\tau}\left(\frac{1}{x}\right)*\,{}_2F_2\left[\begin{matrix}p,\,1-\lambda+l+\xi; & \frac{1}{x} \\ p+q,\,1-\lambda+\xi; & \end{matrix}\right] \tag{31}$$

and

$$B\left[D_{0-}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t} \\ c,d; & \end{matrix}\right]\right\}(x):p,\,q\right]=x^{\lambda-1}B(p,\,q)\frac{\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi)}\,{}_3\Gamma_2^{\tau}\left(\frac{1}{x}\right)*\,{}_2F_2\left[\begin{matrix}p,\,1-\lambda+l+\xi; & \frac{1}{x} \\ p+q,\,1-\lambda+\xi; & \end{matrix}\right] \tag{32}$$

Corollary 3.6. *Suppose that $t,\ p,\ q > 0;\ k \geq 0;\ \tau > 0;\ \Re(d) > \Re(a) > 0;\ \Re(c) > \Re(b) > 0$ and $l, m, \xi, \lambda \in \mathbb{C}$ be parameters such that $\min\{\Re(\lambda), \Re(l)\} > 0$. Then the beta-transform of left-hand side Riemann-Liouville fractional derivative formula holds:*

$$B\left[D_{0-}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t} \\ c,d; & \end{matrix}\right]\right\}(x):p,\,q\right]=x^{\lambda-l-1}B(p,\,q)\frac{\Gamma(1-\lambda+l)}{\Gamma(1-\lambda)}\,{}_3\Gamma_2^{\tau}\left(\frac{1}{x}\right)*\,{}_2F_2\left[\begin{matrix}p,\,1-\lambda+l; & \frac{1}{x} \\ p+q,\,1-\lambda; & \end{matrix}\right] \tag{33}$$

and

$$B\left[D_{0-}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t} \\ c,d; & \end{matrix}\right]\right\}(x):p,\,q\right]=x^{\lambda-l-1}B(p,\,q)\frac{\Gamma(1-\lambda+l)}{\Gamma(1-\lambda)}\,{}_3\gamma_2^{\tau}\left(\frac{1}{x}\right)*\,{}_2F_2\left[\begin{matrix}p,\,1-\lambda+l; & \frac{1}{x} \\ p+q,\,1-\lambda; & \end{matrix}\right] \tag{34}$$

3.2. Pathway transform of fractional calculus of incomplete τ hypergeometric function

Theorem 3.7. *Suppose that $t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ and $l,m,\xi,\lambda\in\mathbb{C}$ be parameters such that $\Re(\lambda)>0,\Re(\lambda-m+\xi)>0$ and $\phi>1$, and $\Re(\lambda)>-\min\{0,\Re(l+m+\xi)\}$. Then the following image formula holds:*

$$P_\phi\left[z^{\rho-1}\,I_{0,x}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right]=\frac{x^{\lambda-m-1}}{[\wedge(\phi;s)]^{\rho}}\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda-m)}$$
$$\frac{\Gamma(\lambda-m+\xi)}{\Gamma(\lambda+l+\xi)}\,{}_3\Gamma_2^{\tau}\left(\frac{x}{[\wedge(\phi;s)]}\right)*{}_3F_2\left[\begin{matrix}\lambda,\rho,\ \lambda-m+\xi;\\ \lambda-m,\ \lambda+l+\xi;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{35}$$

and

$$P_\phi\left[z^{\rho-1}\,I_{0,x}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right]=\frac{x^{\lambda-m-1}}{[\wedge(\phi;s)]^{\rho}}\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda-m)}$$
$$\frac{\Gamma(\lambda-m+\xi)}{\Gamma(\lambda+l+\xi)}\,{}_3\gamma_2^{\tau}\left(\frac{x}{[\wedge(\phi;s)]}\right)*{}_3F_2\left[\begin{matrix}\lambda,\rho,\ \lambda-m+\xi;\\ \lambda-m,\ \lambda+l+\xi;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{36}$$

Proof. For convenience, we denote the left-hand side of the result (35) by ς and using (1), we find

$$\varsigma=P_\phi\left[z^{\rho-1}\,I_{0,x}^{l,m,\xi}\left\{t^{\lambda-1}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\sigma;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{(zt)^n}{n!}\right\}(x):s\right]$$

Now using (9), (22) and changing the order of integration and summation, we obtain

$$\varsigma=x^{\lambda-m-1}\frac{\Gamma(\lambda)\,\Gamma(\lambda-m+\xi)}{\Gamma(\lambda-m)\,\Gamma(\lambda+l+\xi)}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{[\sigma;k]_n\,\Gamma(a+\tau n)\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)\,\Gamma(d+\tau n)}\frac{x^n}{n!}$$
$$\frac{(\lambda)_n\,(\lambda-m+\xi)_n}{(\lambda-m)_n\,(\lambda+l+\xi)_n}\frac{\Gamma(\rho+n)}{[\wedge\phi;s]^{\rho+n}}$$

Using (8) and applying the Hadamard product series yields the desired result, equation (35). As in the proof of result (36), taking the operator (9) and the result (22) into account, one can easily prove result (36). Therefore, we omit the details of the proof. □

Theorem 3.8. *Suppose that $t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ and $l,m,\xi,\lambda\in\mathbb{C}$ be parameters such that $\Re(m-\lambda+1)>0,\Re(\xi-\lambda+1)>0$ and $\phi>1$, and $\min\{\Re(\lambda),\Re(m)\}>0$ and $\Re(\lambda)>-\min\{0,\Re(l+m+\xi)\}$. Then the following pathway transform formula holds:*

$$P_\phi\left[z^{\rho-1}\,J_{x,\infty}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ \frac{z}{t}\right]\right\}(x):s\right]=\frac{x^{\lambda-m-1}}{[\wedge(\phi;s)]^{\rho}}\frac{\Gamma(\rho)\,\Gamma(m-\lambda+1)}{\Gamma(1-\lambda)}$$
$$\frac{\Gamma(\xi-\lambda+1)}{\Gamma(m+l-\lambda+\xi+1)}\,{}_3\Gamma_2^{\tau}\left(\frac{1}{x\,[\wedge(\phi;s)]}\right)*{}_3F_2\left[\begin{matrix}\rho,m-\lambda+1,\ \xi-\lambda+1;\\ 1-\lambda,\ m+l-\lambda+\xi+1;\end{matrix}\ \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{37}$$

and

$$P_\phi\left[z^{\rho-1}\,J_{x,\infty}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^{\tau}\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ \frac{z}{t}\right]\right\}(x):s\right]=\frac{x^{\lambda-m-1}}{[\wedge(\phi;s)]^{\rho}}\frac{\Gamma(\rho)\,\Gamma(m-\lambda+1)}{\Gamma(1-\lambda)}$$
$$\frac{\Gamma(\xi-\lambda+1)}{\Gamma(m+l-\lambda+\xi+1)}\,{}_3\gamma_2^{\tau}\left(\frac{1}{x\,[\wedge(\phi;s)]}\right)*{}_3F_2\left[\begin{matrix}\rho,m-\lambda+1,\ \xi-\lambda+1;\\ 1-\lambda,\ m+l-\lambda+\xi+1;\end{matrix}\ \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{38}$$

Proof. As in the proof of Theorem 3.7, taking the operator (10) and the result (22) into account, one can easily prove result (37) and (38). Therefore, we omit the details of the proof. □

Theorem 3.9. *Suppose that $t > 0,\ k \geq 0; \tau > 0; \Re(d) > \Re(a) > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi, \lambda \in \mathbb{C}$ be parameters such that $\min\{\Re(\lambda), \Re(m)\} > 0$ and $\Re(\lambda) > -\min\{0, \Re(l+m+\xi)\}$. Then the following image formula holds:*

$$P_\phi\left[z^{\rho-1}\, D_{0+}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; \\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right] = \frac{x^{\lambda+m-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda+m)}$$
$$\frac{\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+\xi)}\,{}_3\Gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right) * {}_3F_2\left[\begin{matrix}\lambda,\rho,\ \lambda+m+l+\xi; \\ \lambda+m,\ \lambda+\xi;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{39}$$

and

$$P_\phi\left[z^{\rho-1}\, D_{0+}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; \\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right] = \frac{x^{\lambda+m-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda+m)}$$
$$\frac{\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+\xi)}\,{}_3\gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right) * {}_3F_2\left[\begin{matrix}\lambda,\rho,\ \lambda+m+l+\xi; \\ \lambda+m,\ \lambda+\xi;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{40}$$

Proof. As in the proof of Theorem 3.7, taking the operator (15) and the result (22) into account, one can easily prove result (39) and (40). Therefore, we omit the details of the proof. □

Theorem 3.10. *Suppose that $\phi > 1\ \ t > 0,\ k \geq 0; \tau > 0; \Re(d) > \Re(a) > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi, \lambda \in \mathbb{C}$ be parameters such that $\Re(l) \geq 0\ \Re(\lambda) > -\min\{0, \Re(l+m+\xi)\}$ Then the P_ϕ-transform of left-hand side fractional derivative formula holds:*

$$P_\phi\left[z^{\rho-1}\, D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; \\ c,d;\end{matrix}\ \frac{z}{t}\right]\right\}(x):s\right] = \frac{x^{\lambda+m-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(1-\lambda-m)}{\Gamma(1-\lambda)}$$
$$\frac{\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi-m)}\,{}_3\Gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right) * {}_3F_2\left[\begin{matrix}\rho,\ 1-\lambda-m, 1-\lambda+l+\xi; \\ 1-\lambda,\ 1-\lambda+\xi-m;\end{matrix}\ \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{41}$$

and

$$P_\phi\left[z^{\rho-1}\, D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; \\ c,d;\end{matrix}\ \frac{z}{t}\right]\right\}(x):s\right] = \frac{x^{\lambda+m-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(1-\lambda-m)}{\Gamma(1-\lambda)}$$
$$\frac{\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi-m)}\,{}_3\gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right) * {}_3F_2\left[\begin{matrix}\rho,\ 1-\lambda-m, 1-\lambda+l+\xi; \\ 1-\lambda,\ 1-\lambda+\xi-m;\end{matrix}\ \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{42}$$

Proof. As in the proof of Theorem 3.7, taking the operator (16) and the result (22) into account, one can easily prove result (41) and (42). Therefore, we omit the details of the proof. □

Further, if we replace m with 0 and m with $-l$ in Theorems 3.7, 3.8, 3.9 and 3.10 then we obtain the P_ϕ-transform of the Erdélyi-Kober, Weyl and Riemann-Liouville fractional integral and derivative of the generalized incomplete τ- hypergeometric type functions given by the following corollary:

Corollary 3.11. *Suppose that $t > 0,\ k \geq 0; \tau > 0; \Re(d) > \Re(a) > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi, \lambda \in \mathbb{C}$ be parameters such that $\Re(\lambda) > 0, \Re(\lambda - m + \xi) > 0$ and $\phi > 1,$ and $\Re(\lambda) > -\min\{0, \Re(l+m+\xi)\}$. Then the P_ϕ-transform of right hand side*

Erdélyi-Kober fractional integral formulae holds:

$$P_\phi\left[z^{\rho-1}\, I_{0,x}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\; zt\right]\right\}(x):s\right]= \frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(\lambda+\xi)}{\Gamma(\lambda+l+\xi)}\,{}_3\Gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,\ \lambda+\xi;\\ \lambda+l+\xi;\end{matrix}\; \frac{x}{[\wedge(\phi;s)]}\right] \tag{43}$$

and

$$P_\phi\left[z^{\rho-1}\, I_{0,x}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\; zt\right]\right\}(x):s\right]= \frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(\lambda+\xi)}{\Gamma(\lambda+l+\xi)}\,{}_3\gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,\ \lambda+\xi;\\ \lambda+l+\xi;\end{matrix}\; \frac{x}{[\wedge(\phi;s)]}\right] \tag{44}$$

Corollary 3.12. *Suppose that* $t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi,\lambda\in\mathbb{C}$ *be parameters such that* $\Re(\lambda)>0,\Re(\lambda-m+\xi)>0$ *and* $\phi>1,$ *and* $\min\{\Re(\lambda),\Re(m)\}>0$ *and* $\Re(\lambda)>-\min\{0,\Re(\xi)\}$*. Then the* P_ϕ*-transform of right hand side Riemann-liouville fractional integral formulae holds:*

$$P_\phi\left[z^{\rho-1}\, I_{0,x}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\; zt\right]\right\}(x):s\right]= \frac{x^{\lambda+l-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda+l)}\,{}_3\Gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\lambda,\rho;\\ \lambda+l;\end{matrix}\; \frac{x}{[\wedge(\phi;s)]}\right] \tag{45}$$

and

$$P_\phi\left[z^{\rho-1}\, I_{0,x}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\; zt\right]\right\}(x):s\right]= \frac{x^{\lambda+l-1}}{[\wedge(\phi;s)]^\rho}\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda+l)}\,{}_3\gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\lambda,\rho;\\ \lambda+l;\end{matrix}\; \frac{x}{[\wedge(\phi;s)]}\right] \tag{46}$$

Corollary 3.13. *Suppose that* $t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi,\lambda\in\mathbb{C}$ *be parameters such that* $\Re(m-\lambda+1)>0,\Re(\xi-\lambda+1)>0$ *and* $\phi>1,$ *and* $\min\{\Re(\lambda),\Re(m)\}>0$ *and* $\Re(\lambda)>-\min\{0,\Re(l+m+\xi)\}$*. Then the* P_ϕ*-transform of right hand side Erdélyi-Kober fractional integral formula holds :*

$$P_\phi\left[z^{\rho-1}\, K_{x,\infty}^{l,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\; \frac{z}{t}\right]\right\}(x):s\right]=\frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho} \frac{\Gamma(\rho)\,\Gamma(\xi-\lambda+1)}{\Gamma(l-\lambda+\xi+1)}\,{}_3\Gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,\xi-\lambda+1;\\ l-\lambda+\xi+1;\end{matrix}\; \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{47}$$

and

$$P_\phi\left[z^{\rho-1}\, K_{x,\infty}^{l,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\; \frac{z}{t}\right]\right\}(x):s\right]=\frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho} \frac{\Gamma(\rho)\,\Gamma(\xi-\lambda+1)}{\Gamma(l-\lambda+\xi+1)}\,{}_3\gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,\xi-\lambda+1;\\ l-\lambda+\xi+1;\end{matrix}\; \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{48}$$

Corollary 3.14. *Suppose that* $t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi,\lambda\in\mathbb{C}$ *be parameters such that* $\Re(m-\lambda+1)>0,\Re(\xi-\lambda+1)>0$ *and* $\phi>1,$ *and* $\min\{\Re(\lambda),\Re(m)\}>0$ *and* $\Re(\lambda)>-\min\{0,\Re(l+m+\xi)\}$. *Then the* P_ϕ*-transform of right hand side Weyl fractional integral*

$$P_\phi\left[z^{\rho-1}\,W^{l,-l,\xi}_{x,\infty}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ \frac{z}{t}\right]\right\}(x):s\right]=\frac{x^{\lambda+l-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(1-\lambda-l)}{\Gamma(1-\lambda)}\,{}_3\Gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,1-\lambda-l;\\ 1-\lambda;\end{matrix}\ \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{49}$$

and

$$P_\phi\left[z^{\rho-1}\,W^{l,-l,\xi}_{x,\infty}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ \frac{z}{t}\right]\right\}(x):s\right]=\frac{x^{\lambda+l-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(1-\lambda-l)}{\Gamma(1-\lambda)}\,{}_3\gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,1-\lambda-l;\\ 1-\lambda;\end{matrix}\ \frac{1}{x\,[\wedge(\phi;s)]}\right] \tag{50}$$

Corollary 3.15. *Suppose that* $t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi,\lambda\in\mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda),\Re(l)\}>0$ *and* $\Re(\lambda)>-\min\{0,\Re(\xi)\}$. *Then the* P_ϕ*-transform of right-hand side Erdélyi-Kober fractional derivative formulae holds:*

$$P_\phi\left[z^{\rho-1}\,D^{l,0,\xi}_{0+}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right]=\frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(\lambda+l+\xi)}{\Gamma(\lambda+\xi)}\,{}_3\Gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,\ \lambda+l+\xi;\\ \lambda+\xi;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{51}$$

and

$$P_\phi\left[z^{\rho-1}\,D^{l,0,\xi}_{0+}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right]=\frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(\lambda+l+\xi)}{\Gamma(\lambda+\xi)}\,{}_3\gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\rho,\ \lambda+l+\xi;\\ \lambda+\xi;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{52}$$

Corollary 3.16. *Suppose that* $\phi>1\ \ t>0,\ k\geq 0;\tau>0;\Re(d)>\Re(a)>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi,\lambda\in\mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda),\Re(l)\}>0$. *Then the* P_ϕ*-transform of the right-hand side Riemann-Liouville fractional derivative formulae holds*

$$P_\phi\left[z^{\rho-1}\,D^{l,-l,\xi}_{0+}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right]=\frac{x^{\lambda-l-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda-l)}\,{}_3\Gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\lambda,\rho;\\ \lambda-l;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{53}$$

and

$$P_\phi\left[z^{\rho-1}\,D^{l,-l,\xi}_{0+}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b;\\ c,d;\end{matrix}\ zt\right]\right\}(x):s\right]=\frac{x^{\lambda-l-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(\lambda)}{\Gamma(\lambda-l)}\,{}_3\gamma_2^\tau\left(\frac{x}{[\wedge(\phi;s)]}\right)*\,{}_2F_1\left[\begin{matrix}\lambda,\rho;\\ \lambda-l;\end{matrix}\ \frac{x}{[\wedge(\phi;s)]}\right] \tag{54}$$

Corollary 3.17. *Suppose that* $\phi > 1$ $t > 0,\ k \geq 0; \tau > 0; \Re(d) > \Re(a) > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi, \lambda \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > -\min\{0, \Re(\xi)\}$. *Then the* P_ϕ*-transform of left-hand side Erdélyi-Kober fractional derivative formulae holds:*

$$P_\phi\left[z^{\rho-1}\, D_{0-}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t}\\ c,d; & \end{matrix}\right]\right\}(x):s\right] = \frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi)}\,{}_3\Gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right) *\,{}_2F_1\left[\begin{matrix}\rho,\ 1-\lambda+l+\xi; & \frac{1}{x\,[\wedge(\phi;s)]}\\ 1-\lambda+\xi; & \end{matrix}\right] \tag{55}$$

and

$$P_\phi\left[z^{\rho-1}\, D_{0-}^{l,0,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t}\\ c,d; & \end{matrix}\right]\right\}(x):s\right] = \frac{x^{\lambda-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi)}\,{}_3\gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right) *\,{}_2F_1\left[\begin{matrix}\rho,\ 1-\lambda+l+\xi; & \frac{1}{x\,[\wedge(\phi;s)]}\\ 1-\lambda+\xi; & \end{matrix}\right] \tag{56}$$

Corollary 3.18. *Suppose that* $\phi > 1$ $t > 0,\ k \geq 0; \tau > 0; \Re(d) > \Re(a) > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi, \lambda \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$. *Then the* P_ϕ*-transform of the left-hand side Riemann-Liouville fractional derivative formulae holds:*

$$P_\phi\left[z^{\rho-1}\, D_{0-}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\Gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t}\\ c,d; & \end{matrix}\right]\right\}(x):s\right] = \frac{x^{\lambda-l-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(1-\lambda+l)}{\Gamma(1-\lambda)}\,{}_3\Gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right) *\,{}_2F_1\left[\begin{matrix}\rho,\ 1-\lambda+l; & \frac{1}{x\,[\wedge(\phi;s)]}\\ 1-\lambda; & \end{matrix}\right] \tag{57}$$

and

$$P_\phi\left[z^{\rho-1}\, D_{0-}^{l,-l,\xi}\left\{t^{\lambda-1}\,{}_3\gamma_2^\tau\left[\begin{matrix}(\sigma,k),a,b; & \frac{z}{t}\\ c,d; & \end{matrix}\right]\right\}(x):s\right] = \frac{x^{\lambda-l-1}}{[\wedge(\phi;s)]^\rho}$$
$$\frac{\Gamma(\rho)\,\Gamma(1-\lambda+l)}{\Gamma(1-\lambda)}\,{}_3\gamma_2^\tau\left(\frac{1}{x\,[\wedge(\phi;s)]}\right) *\,{}_2F_1\left[\begin{matrix}\rho,\ 1-\lambda+l; & \frac{1}{x\,[\wedge(\phi;s)]}\\ 1-\lambda; & \end{matrix}\right] \tag{58}$$

Remark 3.19. *It is interesting to observe that for* $\phi \to 1$ *in pathway integral transform defined by* (21) *reduces to Laplace transform, we obtain above results for Laplace transform.*

Remark 3.20. *In the above similar manner we will find Verma transform and Whittekar trasform of fractional derivative of incomplete* τ*-hypergeometric function* ${}_3\Gamma_2^\tau(z)$.

4. Conclusion

This article provides certain image formulas pertaining incomplete generalized τ-hypergeometric function and Pathway integral transform. Furthermore, the implementation of the Beta, Laplace and pathway integral transform on fractional calculus operators of incomplete generalized τ-hypergeometric function have been discussed. The importance of using P_Φ-transform method is that we get a wider class of integrals varying from binomial to exponential function(See [5, 6]). The method could lead to a promising approach for many applications in applied sciences.

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