Int. J. Math. And Appl., 7(2)(2019), 149–156
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Chebyshev Spectral Collocation Solution of Non-Linear Time Dependent Partial Differential Equation With Time Derivative Boundary Conditions

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Abstract: The present paper demonstrate the implementation of time derivative boundary conditions in the Chebyshev differentiation matrices. The non-linear time dependent partial differential equations with time derivative boundary conditions(mixed boundary conditions) is considered and the spectral collocation algorithm is developed and solutions are presented. Quasi-linearization technique is used to convert the non-linear partial differential equation into linear form by using Taylor series approximation about the initial guess. Lagrange interpolating polynomials are used as basis of the solution at the Gauss-Labatto grid points. Also the time derivative boundary conditions are incorporated within the Chebyshev differentiation matrices. MATLAB software is used to implement this algorithm and numerical results are depicted graphically. The case study problem is solved using this approach and the solution found by the this method is more accurate compared to the finite difference method with uniform grid points.

Keywords: Spectral-Collocation, Quasi-Linearization, Gauss-Labatto points, Lagrange polynomials, Cheb function, Cauchy-Boundaries.

1. Introduction

The partial differential equations plays a very important role in the mathematical modelling of industrial and engineering problems. There are several phenomenons are non-linear in nature and hence the mathematical modelling of those phenomenons leads to non-linear partial differential equations. These differential equations to be solved with respect different types of boundary conditions coincides with real world situations. The time derivative boundary conditions or mixed Cauchy type of conditions exits in many industrial and chemical engineering problems in particular the drainage and holdup of liquid foams [1, 2]. In the present study it is demonstrated the implementation of the time derivative boundary condition such as Cauchy's or mixed boundary conditions in a Chebyshev differential matrix along with Gauss-Labatto collocation points. A case study problem is taken from chemical engineering application which leads to a non-linear partial differential equation with time derivative boundary condition [1, 2].

A general non-linear partial differential equation of the form given below is considered in this study with suitable initial and boundary conditions:

$$U_t = F(z, t, U, U_z, U_z z) \tag{1}$$

subject to the initial and mixed boundary conditions respectively;

$$U(z, 0) = f(z) \tag{2}$$

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$$U_t(z_0, t) + U(z_0, t) = g_1(t) \tag{3}$$

and

$$U_t(z_1, t) + U(z_1, t) = g_2(t) \tag{4}$$

Many researchers are solved this partial differential equations of these categories using numerical methods such as finite difference and finite elements. In present work the above said non-linear time dependent PDE is solved using Chebyshev spectral collocation method. To obtain the solution of non-linear time dependent partial differential equations, the algorithm developed is as follows. First the non-linear PDE is decomposed in to sum of linear and no-linear terms. Then by using Taylor's series expansion about the guessed solution the non-linear terms are taken to right hand side and they are not unknowns any more. Further, the derivatives in the resultant equations are written in terms of Chebyshev differential matrix by assuming the basis of the solution by Lagrange polynomials about the Gauss-Labatto collocation points.Finally, the boundary conditions are implemented inside these matrices and 1 and $N+1$ rows and the system of equations are solved for the unknown. The implementation of time derivative boundary conditions in the Chebyshev differentiation matrices is considered to be the uniqueness of the present study compared to that of other works by several authors. The method discussed in this paper is found to be accurate and fast solver of non-linear partial differential equations.

2. Case Study Problem

To demonstrate the method and theory developed in this paper it is applied to the following case study problem in the form of non-linear partial differential equation which arises in the chemical engineering while modelling the drainage of foams and it is given [1, 2] in terms of partial differential equation:

$$\frac{\partial W}{\partial t} = \frac{k_0 W_0^3}{2} + k_1 W_1^2 - c_1 \left[2k_2 W \frac{\partial W}{\partial z} - c_2 \left(\frac{\sqrt{W}}{2} \frac{\partial^2 W}{\partial z^2} + W^{-0.5} \frac{\partial W}{\partial z} \right) \right] \tag{5}$$

in the domain $z \epsilon [0, 1]$. Where W_0 and W_1 are some known functions of z and t representing, mass balance for film and Mass balance for horizontal Plateau borders, respectively. Subject to the time derivative mixed (Cauchy) boundary conditions at $z = 0$ and $z = 1$ in non-dimensional form;

$$\frac{\partial W}{\partial t} = \frac{k_0 W_0^3}{2} + k_1 W_1^2 - c_1 \left[2k_2 W^2 (0, t) - c_2 \frac{\sqrt{W (0, t)}}{2} \frac{\partial W (0, t)}{\partial z} \right] \qquad at \qquad z = 0 \tag{6}$$

$$\frac{\partial W}{\partial t} = \frac{k_0 W_0^3}{2} + k_1 W_1^2 - c_1 \left[2k_2 W^2 (1, t) - c_2 \frac{\sqrt{W (1, t)}}{2} \frac{\partial W (1, t)}{\partial z} - \left(c_3 - c_5 W^{-0.5} + c_6 \right) \right] \qquad at \qquad z = 1 \tag{7}$$

where k_0, k_1, c_1 and c_2 are some non-dimensional parameters. This equation is of practical importance in many chemical engineering experiments representing the shape of the polyhedral bubbles in foam is assumed to be pentagonal dodecahedral. Hence, a polyhedral foam consists of three essential parts, viz. films, horizontal and vertical Plateau borders. The mass balance equations for films and horizontal Plateau borders remain same as that of [1]. However, the equation for vertical Plateau border changes from that of [1] due to the inclusion of Plateau border suction in overall foam drainage.

3. Numerical Solution

3.1. Quasi-linearization

The partial differential equation considered in above section is non-linear in nature, hence the closed form solution of it does not exists. In this section we present the detailed quasi-linearization process to reduce this differential equation in to linear form [3]. Let

$$
\begin{aligned}
F &= \frac{\partial W}{\partial t} - \frac{k_1 W_0^3}{2} - k_2 W_1^2 \\
G &= -2c_1 k_3 W \frac{\partial W}{\partial z} + \frac{c_1 c_2 \sqrt{W}}{2} \frac{\partial^2 W}{\partial z^2} + \frac{c_1 c_2 W^{-0.5}}{2} \frac{\partial W}{\partial z} \\
H &= \frac{\partial W}{\partial t} - \frac{k_0 W_0^3}{2} + k_1 W_1^2, \\
J &= -c_1 \left[2k_2 W^2 (0,t) - c_2 \frac{\sqrt{W(0,t)}}{2} \frac{\partial W(0,t)}{\partial z} \right] \\
L &= \frac{\partial W}{\partial t} - \frac{k_0 W_0^3}{2} - k_1 W_1^2 \\
\textit{and } N &= -c_1 \left[2k_2 W^2 (1,t) - c_2 \frac{\sqrt{W(1,t)}}{2} \frac{\partial W(1,t)}{\partial z} - \left(c_3 - c_5 W^{-\frac{1}{2}} + c_6 \right) \right]
\end{aligned}
$$

here F, H, and L are linear parts of equations (5)-(7) and G, J and N are non-linear parts of equations (5)-(7), respectively. Hence equations (5)-(7) can be written as

$$F + G = 0 \tag{8}$$

$$H + J = 0 \tag{9}$$

$$L + N = 0 \tag{10}$$

To reduce the non-linear parts G, J, and N of equations (8)-(10) to linear form we make use of quasi-linearization technique. In this technique we assume W_r be the approximate solution of the non-linear partial differential in r^{th} iteration and $W_{r+1} - W_r \ll 1$ then G, J and N can be expanded as multivariate Taylor series about W_r, respectively given by

$$G(W, W', W'') \approx G(W_r, W_r', W_r'') + (W_{r+1} - W_r).\nabla G[W_r, W_r', W_r''] = 0 \tag{11}$$

$$J(W(0,t), W'(0,t), W''(0,t)) \approx J(W_r, W_r', W_r'') + (W_{r+1} - W_r).\nabla J[W_r, W_r', W_r''] = 0 \tag{12}$$

$$N(W(1,t), W'(1,t), W''(1,t)) \approx N(W_r, W_r', W_r'') + (W_{r+1} - W_r).\nabla N[W_r, W_r', W_r''] = 0 \tag{13}$$

The above equations are obtained after ignoring second and higher order terms. Further simplifying these equations by taking the known terms purely in W_r to right hand side, we get

$$a_{2r} W_{r+1}'' + a_{1r} W_{r+1}' + a_{0r} W_{r+1} - \frac{\partial W_r}{\partial t} = R_1 \tag{14}$$

$$b_{1r}(0,t) W_{r+1}'(0,t) + b_{0r}(0,t) W_{r+1}(0,t) - \frac{\partial W_r(0,t)}{\partial t} = R_2(0,t) \tag{15}$$

$$c_{1r}(1,t) W_{r+1}'(1,t) + c_{0r}(1,t) W_{r+1}(1,t) - \frac{\partial W_r(1,t)}{\partial t} = R_3(1,t) \tag{16}$$

Where

$$
\begin{aligned}
a_{0r} &= -2c_1 k_3 \frac{\partial W_r}{\partial z} + \frac{c_1 c_2 W_r^{-0.5}}{4} \frac{\partial^2 W_r}{\partial z^2} + \frac{c_1 c_2 W_r^{-0.5}}{4} \frac{\partial W_r}{\partial z}, \\
a_{1r} &= -2c_1 k_3 W_r + \frac{c_1 c_2 W_r^{-0.5}}{2}, \\
a_{2r} &= \frac{c_1 c_2 W_r^{0.5}}{2}, \\
b_{0r} &= -2c_1 k_3 W_r(0,t) + \frac{c_1 c_2 W_r^{-0.5}(0,t) W_r'(0,t)}{4}, \\
b_{1r} &= \frac{c_1 c_2 W_r^{0.5}(0,t)}{2},
\end{aligned}
$$

$$
\begin{aligned}
c_{0r} &= -2c_1k_3W_r(1,t) + \frac{c_1c_2W_r^{-0.5}(1,t)W_r'(1,t)}{4} - \frac{c_1c_5W_r^{-1.5}(1,t)}{2},\\
c_{1r} &= \frac{c_1c_2W_r^{0.5}(1,t)}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
R1 &= -2c_1k_3W_rW_r' + \frac{c_1c_2W_r'W_r^{0.5}}{2} + \frac{c_1c_2W_r''W_r^{0.5}}{4} - \frac{k_1W_0^3}{2} - k_2W_1^2\\
R2 &= -\frac{k_1W_0^3}{2} + k_2W_1^2 - c_1k_3W_r^2(0,t) + c_1c_2\frac{W_r^{0.5}(0,t)}{2}\frac{\partial W_r(0,t)}{\partial z}\\
R3 &= -\frac{k_1W_0^3}{2} + k_2W_1^2 - c_1k_3W_r^2(1,t) + c_1c_2\frac{W_r^{0.5}(1,t)}{2}\frac{\partial W_r(1,t)}{\partial z} + c1\left(c_4 - c_5W_r^{-0.5}(1,t) + c_6\right)
\end{aligned}
$$

The equations (11)-(13) are in linear form, hence they can be solved by any method. In the present work these equations are solved by using spectral Chebyshev collocation technique discussed in the next section below.

3.2. Bivariate Spectral Collocation

To obtain the numerical solution of equations (11)-(13) by spectral method the domain $[0, 1]$ is transformed to $[-1, 1]$ by using $N + 1$, $M + 1$ Gauss-Labatto grid points in space as well as time respectively and are given by (see [3**?**])

$$
Z_i = \cos\left(\frac{\pi i}{N}\right), \qquad i = 0, 1, 2, \ldots .N \tag{17}
$$

and

$$
T_j = \cos\left(\frac{\pi j}{N}\right), \qquad j = 0, 1, 2, \ldots .M \tag{18}
$$

these grid points are in computational domain for spectral collocation method. They can be transformed to physical domain by using

$$
z = \frac{(Z+1)}{2} \tag{19}
$$

$$
t = \frac{Lt(T+1)}{2} \tag{20}
$$

for some time interval $[0, Lt]$. The solution W_r is assumed to be of the sum of product of Lagrangian polynomials in z and t as given below

$$
W_r(z,t) = \sum_{k=0}^{M}\sum_{j=0}^{N} c_{jk}\phi_j(z)\phi_k(t) \tag{21}
$$

Where ϕ as the Kronecker delta property. Further these solution at collocation points becomes

$$
W_r(z_i,t_p) = \sum_{k=0}^{M}\sum_{j=0}^{N} c_{jk}\phi_j(z_i)\phi_k(t_p) \tag{22}
$$

Similarly the partial derivatives at these collocation points transform to

$$
\frac{\partial W_r}{\partial t}\Big|_{(z_i,t_j)} = 2\sum_{k=0}^{M}\overline{d}_{jk}W(Z_i,T_k) = 2\sum_{k=0}^{M}d_{jk}W_k \tag{23}
$$

$$
\frac{\partial W_r}{\partial z}\Big|_{(z_i,t_j)} = 2\sum_{p=0}^{N}\overline{W}(Z_p,T_j)D_{ip} = DW_j \tag{24}
$$

Where d_{jk} and D are called Chebyshev differentiation matrices of order $M+1 \times M+1$ and $N+1 \times N+1$ respectively. Replacing the continuous derivatives by derivatives at collocation points [see [3–5]] equations (11)-(13) becomes

$$a_{2r}D^2W_{r+1} + a_{1r}DW_{r+1} + a_{0r}W_{r+1} - 2\sum_{k=0}^{M} d_{jk}W_k = R_1 \tag{25}$$

$$b_{1r}(N,:)DW_{r+1}(N,:) + b_{0r}(N,:)W_{r+1}(N,:) - 2\sum_{k=0}^{M} d_{jk}W_k(N,:) = R_2(N,:) \tag{26}$$

$$c_{1r}(1,:)DW_{r+1}(1,:) + c_{0r}(1,:)W_{r+1}(1,:) - 2\sum_{k=0}^{M} d_{jk}W_k(1,:) = R_3(1,:) \tag{27}$$

Further by using initial condition at $j = M+1$ the solution is known for spacial grids the above equations reduces to

$$a_{2r}D^2W_{r+1} + a_{1r}DW_{r+1} + a_{0r}W_{r+1} - 2\sum_{k=0}^{M-1} d_{jk}W_k = K_1 + 2d(:,M)W(:,M) \tag{28}$$

$$[b_{1r}(N,:)D + b_{0r}(N,:)]\,W_{r+1}(N,:) - 2\sum_{k=0}^{M-1} d_{jk}W_k(N,:) = K_2(N,:) + 2d(N,M)W(N,M) \tag{29}$$

$$c_{1r}(1,:)DW_{r+1}(1,:) + c_{0r}(1,:)W_{r+1}(1,:) - 2\sum_{k=0}^{M-} d_{jk}W_k(1,:) = K_3(1,:) + 2d(1,M)W(1,M) \tag{30}$$

$$\begin{aligned}
K1 &= -2c_1k_3W_rDW_r + \frac{c_1c_2DW_rW_r^{0.5}}{2} + \frac{c_1c_2D^2W_rW_r^{0.5}}{4} - \frac{k_1W_0^3}{2} - k_2W_1^2 + +2d(:,M)W(:,M)\\
K2 &= -\frac{k_1W_0^3}{2} + k_2W_1^2 - c_1k_3W_r^2(N,:) + c_1c_2\frac{W_r^{0.5}(N,:)}{2}DW_r(N,:) + 2d(N,M)W(N,M)\\
K3 &= -\frac{k_1W_0^3}{2} + k_2W_1^2 - c_1k_3W_r^2(1,:) + c_1c_2\frac{W_r^{0.5}(1,:)}{2}DW_r(1,:) + c1\left(c_4 - c_5W_r^{-0.5}(1,:) + c_6\right) + 2d(:,M)W(:,M)
\end{aligned}$$

Expanding equation(22) for all $j = 0, 1, 2...M$, we get

$$(A_0 - d_{0,0}I)W_0 - (d_{0,1}W_1 + d_{0,2}W_2 + ... + d_{0,M-1}W_{M-1}) = K1_0$$

$$-d_{1,0}W_0 + (A_2 - d_{1,1}I)W_1 - (d_{1,2}W_2 + ... + d_{1,M-1}W_{M-1}) = K1_1$$

$$-d_{2,0}W_0 - d_{2,1}W_1 + (A_3 - d_{2,2}I)W_2 - (d_{2,3}W_3 + ... + d_{2,M-1}W_{M-1}) = K1_2$$

and so on

$$-d_{M-1,0}W_0 - d_{M-1,1}W_1 - .... - d_{M-1,M-2}W_{M-2} + (A_{M-1} - d_{M-1,M-1}I)W_{M-1} = K1_{M-1}$$

These equations can be put in the following matrix form $AW = K$

$$\begin{pmatrix} A_{0,0} & A_{0,1} & . \; . \; . & A_{0,M-1}\\ A_{1,0} & A_{1,1} & . \; . \; . & A_{1,M-1}\\ A_{2,0} & A_{2,1} & . \; . \; . & A_{2,M-1}\\ . & . & . \; . \; . & .\\ . & . & . \; . \; . & .\\ A_{M-1,0} & A_{M-1,1} & . \; . \; . & A_{M-1,M-1} \end{pmatrix} \begin{pmatrix} W_0\\ W_1\\ W_2\\ .\\ .\\ W_{M-1} \end{pmatrix} = \begin{pmatrix} K1_0\\ K1_1\\ K1_2\\ .\\ .\\ K1_{M-1} \end{pmatrix} \tag{31}$$

Hence by using the boundary conditions given equation (29) and (30) in this matrix system and one can easily obtain W by using $W = A\backslash K$. This entire algorithm is implemented in MATLAB with suitable initial guess and results are presented in next section.

4. Results and Discussion

The non-linear time dependent partial differential equation with respect to time derivative boundary conditions is solved using spectral collocation method with Lagrange interpolating polynomials as basis of the solution with an initial guess $W(z, 0) = \phi(z)$. The numerical results are presented graphically in figures (1)-(5) are well coincides with the experimental results of [1, 2]. Also these results are more accurate than the results produced by [1] using finite difference method.

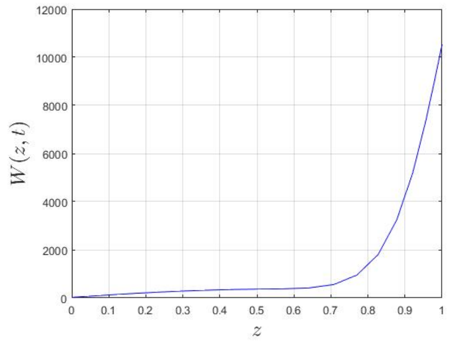

Figure 1. **Graph of $W(z, t)$ vs. z for various values of t**

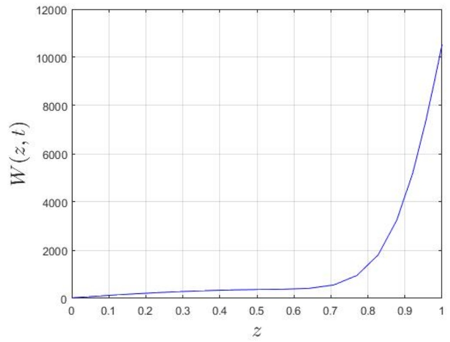

Figure 2. **Graph of$W(z, t)$ vs. z for various values of t**

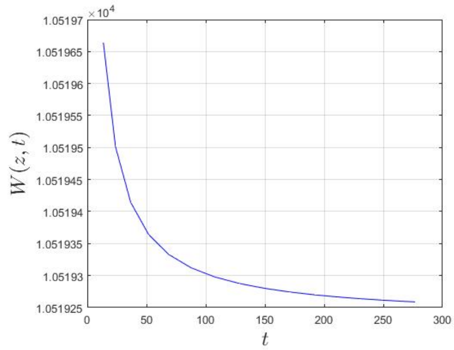

Figure 3. **Graph of $W(z, t)$ vs. t for various values of z**

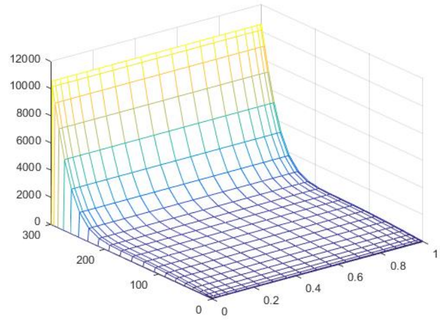

Figure 4. **Graph of $W(z,t)$ vs. z for various values of t**

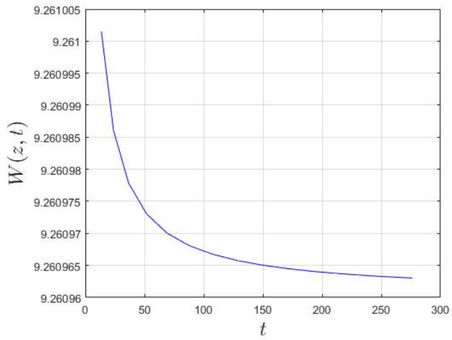

Figure 5. **Graph of $W(z,t)$ vs. t for various values of z**

5. Conclusion

The non-linear partial differential equation is solved using Chebyshev spectral collocation method by implementing the time derivative boundary conditions with in the Chebyshev differential matrices. The results presented in above section are accurate and coincide with the results obtained by previous authors. Also the number of collocation points required to get accuracy is less when compared with finite difference method. Further the cost and computational time of the method presented in this study is better than the finite difference and finite element methods. Hence it is concluded that the spectral collocation method described in this paper is faster and accurate compared to other numerical techniques.

Acknowledgments

The author Chandra Shekara G. wishes to thank the BMS College of Engineering, Bengaluru-19, India for giving their support to carry out this research work under Technical Education Quality Improvement Programme(TEQIP-II) of the MHRD, Government of India

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