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# Some New Modifications of Adomian Technique for Nonlinear Volterra Integral Equations

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**Abstract:** In this paper, the nonlinear Volterra integral equations are solved by using new modifications of the Adomian decomposition technique. The approximate solution of this equation is calculated in the form of a series with easily computable components. The accuracy of the proposed techniques are examined by comparison with exact results. The test problems are presented to illustrate the reliability and performance of the techniques.

**MSC:** 44A10, 49M27, 45D05.

**Keywords:** Adomian decomposition technique, Laplace transform, Nonlinear Volterra integral equation.



## 1. Introduction

In this paper, we consider the nonlinear Volterra integral equation of the form:

$$\Phi(s) = g(s) + \lambda \int_a^s \Psi(s,t)\Delta(\Phi(t))dt, \tag{1}$$

where $g(x)$ and the kernel $\Psi(s,t)$ are analytical functions on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, $\Delta(\Phi(t))$ is the nonlinear function of unknown function $\Phi(t)$ and $a, \lambda$ are positive integers. The Adomian decomposition method was introduced by George Adomian in 1980 [1]. Basically, the technique provides an infinite series solution of the equation and the nonlinear term is decomposed into an infinite series of Adomian polynomials [2–5, 11, 12]. Until recently, the application of the ADM and MADM [8–10, 13] in nonlinear problems has been developed by scientists and engineers, because this method deforms the difficult problem under study into a simple problem which is easy to solve. Moreover, convergence of the solution of nonlinear Volterra integral equations have been studied by several authors [6, 7].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated methods such as the Adomian decomposition method and the modified Laplace Adomian decomposition method for nonlinear Volterra integral equation. The rest of the paper is organized as follows: In Section 2, Adomian Decomposition Method is constructed for solving Volterra integral equations. In Section 2.1, modified Adomian Decomposition Method is constructed for solving Volterra integral equations. Section 2.2 a new modified on Adomian Decomposition Method is constructed for solving Volterra integral equations. Section 3 modified Laplace Adomian Decomposition Method is constructed for solving Volterra integral equations. Section 3.1 a new modified on Laplace Adomian Decomposition Method

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is also constructed for solving Volterra integral equations. Section 4 presents analytical examples to illustrate the accuracy of the methods used in this study. The final Section 5 gives a report on the paper along with a brief conclusion.

## 2. Adomian Decomposition Method

We consider the equation (1), the Adomian's method defines the solution $\Phi(s)$ by the series

$$\Phi(s) = \sum_{n=0}^{\infty} \Phi_n(s), \tag{2}$$

and the nonlinear function $F$ is decomposed as

$$\Delta(\Phi(t)) = \sum_{n=0}^{\infty} A_n(t), \tag{3}$$

where $A_n$ are the Adomian polynomials given by

$$\begin{aligned} A_0 &= \Delta(\Phi_0), \\ A_1 &= \Phi_1 \Delta^{'}(\Phi_0), \\ A_2 &= \Phi_2 \Delta^{'}(\Phi_0) + \frac{1}{2}\Phi_1^2 \Delta^{''}(\Phi_0), \\ A_3 &= \Phi_3 \Delta^{'}(\Phi_0) + \Phi_1 \Phi_2 \Delta^{''}(\Phi_0) + \frac{1}{3}\Phi_1^3 \Delta^{'''}(\Phi_0), \\ &\vdots \end{aligned} \tag{4}$$

The components $\Phi_0$, $\Phi_1$, $\Phi_2$,... are determined recursively by

$$\begin{aligned} \Phi_0 &= g(s), \\ \Phi_n &= \lambda \int_a^s \Psi(s,t) A_{n-1} dt, \quad n \geq 1. \end{aligned} \tag{5}$$

Having defined the components $\Phi_0$, $\Phi_1$, $\Phi_2$,..., the solution $\Phi$ in a series form defined by (2) follows immediately. It is important to note that the decomposition method suggests that the $0^{th}$ component $\Phi_0$ be defined by the initial conditions and the function $g(x)$ as described above. The other components namely $\Phi_1$, $\Phi_2$,..., are derived recurrently.

### 2.1. First Modified Adomian Decomposition Method

The modified Adomian decomposition method was introduced by Wazwaz [14, 15]. This method is based on the assumption that the function $g(s)$ can be divided into two parts, namely $g_1(s)$ and $g_2(s)$. Under this assumption we set

$$g(s) = g_1(s) + g_2(s). \tag{6}$$

The components $\Phi_0$, $\Phi_1$, $\Phi_2$,... are determined recursively by

$$\Phi_0 = g_1(s),$$

$$\Phi_1 = g_2(s) + \lambda \int_a^s \Psi(s,t) A_0 dt, \tag{7}$$

$$\Phi_n = \lambda \int_a^s \Psi(s,t) A_{n-1} dt, \quad n \geq 1. \tag{8}$$

### 2.2. Second Modified Adomian Decomposition Method

The main idea of the second modified technique is replacing the non-homogeneous function $g(s)$ by a series of infinite components. Ref. [9] expresses $g(s)$ in term of the Taylor series and introduces the recursive formula:

$$\begin{aligned}\Phi_0 &= g_1(s),\\ \Phi_k &= g_k(s) + \lambda\int_a^s \Psi(s,t)A_{k-1}dt, \quad k>0.\end{aligned} \tag{9}$$

## 3. Modified Laplace Adomian Decomposition Method

Consider the nonlinear Volterra integral equation (1) with difference kernel i.e. $k(x,t) = k(x-t)$ defined as:

$$\Phi(s) = g(x) + \int_a^s \Psi(s-t)\Delta(\Phi(t))dt. \tag{10}$$

We apply the Laplace transform to both sides of Eq.(10):

$$\mathcal{L}[\Phi(s)] = \mathcal{L}[g(x)] + \mathcal{L}[\int_a^s \Psi(s-t)\Delta(\Phi(t))dt]. \tag{11}$$

Using the differentiation property and convolution theorem of the Laplace transform [9] we get:

$$\mathcal{L}[\Phi(s)] = \mathcal{L}[g(x)] + \mathcal{L}[\Psi(s-t)]\mathcal{L}[\Delta(\Phi(t))dt]. \tag{12}$$

Substituting (3) and (4) into (2), we get

$$\mathcal{L}[\sum_{n=0}^{\infty}\Phi_n(s)] = \mathcal{L}[g(x)] + \mathcal{L}[\lambda\Psi(s-t)]\mathcal{L}[\sum_{n=0}^{\infty}A_n dt]. \tag{13}$$

Using the linearity property of Laplace transform, we get

$$\sum_{n=0}^{\infty}\mathcal{L}[\Phi_n(s)] = \mathcal{L}[g(x)] + \mathcal{L}[\lambda\Psi(s-t)]\sum_{n=0}^{\infty}\mathcal{L}[A_n dt]. \tag{14}$$

To determine the terms $\Phi_0, \Phi_1, \Phi_2, \cdots$ of infinite series, comparing both sides of (6), we have the following iterative scheme

$$\begin{aligned}\mathcal{L}[\Phi_0] &= \mathcal{L}[g_1(s)],\\ \mathcal{L}[\Phi_1] &= \mathcal{L}[g_2(s)] + \mathcal{L}[\lambda\int_a^s \Psi(s-t)A_0 dt], &(15)\\ \mathcal{L}[\Phi_n] &= \mathcal{L}[\lambda\int_a^s \Psi(s-t)A_{n-1}dt], \quad n\geq 1. &(16)\end{aligned}$$

Employing the inverse Laplace transform to (7) and (8), we get

$$\begin{aligned}\Phi_0 &= \mathcal{L}^{-1}[\mathcal{L}[g_1(s)]],\\ \Phi_1 &= \mathcal{L}^{-1}[\mathcal{L}[g_2(s)] + \mathcal{L}[\lambda\int_a^s \Psi(s-t)A_0 dt]], &(17)\\ \Phi_n &= \mathcal{L}^{-1}[\mathcal{L}[\lambda\int_a^s \Psi(s-t)A_{n-1}dt]], \quad n\geq 1. &(18)\end{aligned}$$

### 3.1. New Modified Laplace Decomposition Method

The main idea of the new modified technique is replacing the non-homogeneous function $g(s)$ by a series of infinite components. Ref. [14] expresses $g(s) = g_1(s) + g_2(s) + g_3(s) + g_4(s) + g_5(s) + \cdots$ in term of the Taylor series and introduces the recursive formula:

$$\begin{aligned}
\mathcal{L}[\Phi_0] &= \mathcal{L}[g_1(s)],\\
\mathcal{L}[\Phi_1] &= \mathcal{L}[g_2(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_0 dt],\\
\mathcal{L}[\Phi_2] &= \mathcal{L}[g_3(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_1 dt],\\
\mathcal{L}[\Phi_3] &= \mathcal{L}[g_4(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_2 dt],\\
\mathcal{L}[\Phi_4] &= \mathcal{L}[g_5(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_3 dt],\\
&\vdots
\end{aligned}$$

To determine the terms $\Phi_0, \Phi_1, \Phi_2, \cdots$ of infinite series, comparing both sides of (6), we have the following iterative scheme

$$\begin{aligned}
\Phi_0 &= \mathcal{L}^{-1}[\mathcal{L}[g_1(s)]],\\
\Phi_1 &= \mathcal{L}^{-1}[\mathcal{L}[g_2(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_0 dt]],\\
\Phi_2 &= \mathcal{L}^{-1}[\mathcal{L}[g_3(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_1 dt]],\\
\Phi_3 &= \mathcal{L}^{-1}[\mathcal{L}[g_4(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_2 dt]],\\
\Phi_5 &= \mathcal{L}^{-1}[\mathcal{L}[g_5(s)] + \mathcal{L}[\lambda \int_a^s \Psi(s-t)A_3 dt]],\\
&\vdots
\end{aligned}$$

## 4. Illustrative Examples

**Example 4.1.** *Consider the non-linear Volterra integral equation:*

$$\Phi(s) = \sec s + \tan s - \int_0^s \Phi^2(t)dt, \tag{19}$$

To investigate the first modified technique, we split $g(s)$ into two parts, say

$$g(s) = \sec s + \tan s = g_1(s) + g_2(s)$$

The modified recursive Formula (3.2) reads

$$\begin{aligned}
\Phi_0(s) &= \sec s,\\
\Phi_1(s) &= \tan s - \int_0^s A_0 dt = \tan s - \int_0^s \sec^2 t dt = 0,\\
\Phi_k(s) &= 0, \quad k \geq 1.
\end{aligned}$$

This leads to the exact solution $\Phi(s) = \sec s$. To investigate the second modified technique, let us first expand the function $g(s)$ in term of Taylor series expansion. This reads

$$g(s) = 1 + s + \frac{1}{2}s^2 + \frac{1}{3}s^3 + \frac{5}{24}s^4 + \frac{2}{15}s^5 + \frac{61}{720}s^6 + \cdots$$

Next, the second modified recursive formula (12) gives:

$$\begin{aligned}
\Phi_0(s) &= 1, \\
\Phi_1(s) &= s - \int_0^s A_0 dt = 0, \\
\Phi_2(s) &= = \frac{1}{2}s^2 - \int_0^s A_1 dt = \frac{1}{2}s^2, \\
\Phi_3(s) &= \frac{1}{3}s^3 - \int_0^s A_2 dt = 0, \\
\Phi_4(s) &= \frac{5}{24}s^4 - \int_0^s A_3 dt = \frac{5}{24}s^4,
\end{aligned}$$

This leads to

$$Phi(s) = 1 + \frac{1}{2}s^2 + \frac{5}{24}s^4 + \frac{61}{720}s^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} E_{2n} s^{2n}.$$

This is the Taylor series expansion of $\sec s$, where $E_n$ represents the $n^{th}$ Euler number.

**First Laplace Modification:** Using the recursion relation (3), we get

$$\begin{aligned}
\mathcal{L}\{\Phi_0(s)\} &= \mathcal{L}\{\sec s\}, \\
\mathcal{L}\{\Phi_1(s)\} &= \mathcal{L}\{\tan s\} - \mathcal{L}\{\int_0^s A_0 dt\} = \mathcal{L}\{\tan s\} - \mathcal{L}\{\int_0^s \sec^2 t dt\} = 0, \\
\mathcal{L}\{\Phi_k(s)\} &= -\mathcal{L}\{\int_0^s A_{k-1} dt\} = 0, \quad k \geq 1. \\
\mathcal{L}\{\Phi(s)\} &= \sum_{k=0}^{\infty} \mathcal{L}\{\Phi_k(s)\} = \mathcal{L}\{\sec s\} + 0 + 0 + \cdots .
\end{aligned}$$

Taking Laplace inverse transform on both sides of the above equation, this leads to the exact solution $\Phi(s) = \sec s$.

**Second Laplace Modification:** To apply the second modified technique, let us first expand the function $g(s)$ in terms of Taylor series expansion. The recursive formula (3.3) gives:

$$\begin{aligned}
\mathcal{L}\{\Phi_0(s)\} &= \mathcal{L}\{1\}, \\
\mathcal{L}\{\Phi_1(s)\} &= \mathcal{L}\{s\} - \mathcal{L}\{\int_0^s A_0 dt\} = 0, \\
\mathcal{L}\{\Phi_2(s) &= \mathcal{L}\{\frac{1}{2}s^2\} - \mathcal{L}\{\int_0^s A_1 dt\} = \mathcal{L}\{\frac{1}{2}s^2\}, \\
\mathcal{L}\{\Phi_3(s)\} &= \mathcal{L}\{\frac{1}{3}s^3\} - \mathcal{L}\{\int_0^s A_2 dt\} = 0, \\
\mathcal{L}\{\Phi_4(s)\} &= \mathcal{L}\{\frac{5}{24}s^4\} - \mathcal{L}\{\int_0^s A_3 dt\} = \mathcal{L}\{\frac{5}{24}s^4\},
\end{aligned}$$

This leads to

$$\mathcal{L}\{\Phi(s)\} = \mathcal{L}\{1\} + \mathcal{L}\{\frac{1}{2}s^2\} + \mathcal{L}\{\frac{5}{24}s^4\} + \cdots .$$

Taking Laplace inverse transform on both sides of the above equation, this is the Taylor series expansion of $\sec s$.

# 5. Conclusion

This paper successfully applied the application of four modifications of decomposition method for solving nonlinear Volterra integral equations. Exact solutions of the tested problems, arising in many physical and biological models are calculated by using modifications. We note that second modification minimizes the size of the calculations which produced in the first modification.

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