Int. J. Math. And Appl., 7(1)(2018), 23–30
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

On $\mathbb{R}$-Complex Finsler Space with Matsumoto Metric

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Abstract: In this paper, we determined the fundemental tensor fields $(\tilde{g}_{ij}, \tilde{g}_{i\bar{j}})$ and inverse of these tensor fields, their determinant. Further, we studied some properties of non-Hermitian $\mathbb{R}$-complex Finsler space with Matsumoto metric.

MSC: 53C25, 53C40, 53D15.

Keywords: $\mathbb{R}$-Complex Finsler space, $\mathbb{R}$-complex Finsler space with (α, β) metics, $\mathbb{R}$-complex Finsler space with Matsumoto metric.

1. Introduction

The studies of $\mathbb{R}$-Complex Finsler spaces are new concept in Finsler geometry. In [11], Munteanu and Purcuru have extended the notion of a Complex Finsler spaces to new class of Finsler space called $\mathbb{R}$-Complex Finsler spaces by reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called $\mathbb{R}$-Complex Finsler spaces. In [14], the authors Nicolta Alda and Gheorghe Munteanu were studied the (α, β)-Complex Finsler metrics and also determined the fundamental metric tensor and some properties of Hermitian of the Complex Randers metrics. Then, some important results on $\mathbb{R}$-Complex Finsler spaces have been obtained in ([10, 16]). In the present paper, following the ideas from real Finsler spaces with class of Matsumoto metrics, we introduce the notions on $\mathbb{R}$-Complex Finsler space with Matsumoto metric.

2. Preliminaries

Let M be a complex Finsler manifold, $dim_c M = n$. The complexified of the real tangent bundle $T_c M$ splits into the sum of holomorphic tangent bundle $T^{'}M$ and its conjugates $T''M$.The bundle $T^{'}M$ is in its turn a complex manifold, the local coordinate in a chart will be denoted by (z^k, η^k) and these are changed by the rules,

$$z^{'k} = z^{'k}(z), \quad \eta^{'k} = \frac{\partial z^{'k}}{\partial z^j}\eta^j. \tag{1}$$

The complexified tangent bundle of $T^{'}M$ is decompsed as $T_c(T^{'}M) = T^{'}(T^{'}M) \oplus T^{''}M$. A natural local frame for $T^{'}(T^{'}M)$ is $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}$ which is changes by the rules obtained with Jacobi matrix of the above transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives. A complex nonlinear connection breifly (c.n.c) is a supplementory distribution $H(T^{'}M)$ to a verticle distribution $V(T^{'}M)$ in $T^{'}(T^{'}M)$. The vertical distribution is spanned by $\frac{\partial}{\partial \eta^k}$ and an

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adapted frame in $H(T^{'}M)$ is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}$, where N^j_k are the coefficient of the c.n.c and they have a certain rule of changes at (1), so that $\frac{\delta}{\delta z^k}$ transform like vectors on the base manifold M. Next, we use the abbreviations $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $\partial_{\bar{k}}$, $\dot{\partial}_k$, $\delta_{\bar{k}}$ for their conjugates. The dual adapted basis of $\delta_k, \dot{\partial}_k$ are $\left\{dz^k, \delta\eta^k = d\eta^k + N^k_j dz^j\right\}$ and $\left\{d\bar{z}^k, \delta\bar{\eta}^k\right\}$ their conjugates. We recall, that the homogenity of the metric function of a complex Finsler space (see more [2, 7, 8, 12, 15]) is with respect to complex scalars and the metric tensor of the space is a Hermitian one. In [11] slightly changed the definition of complex Finsler spaces as:

Definition 2.1. *An ℝ-complex Finlser metric on M is continuous function $F : T^{'}M \longrightarrow \mathbb{R}$ satisfying:*

(1). $L = F^2$ is a smooth on $\widetilde{T^{'}M}/0$;

(2). $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

(3). $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \mid \lambda \mid F(z, \eta, \bar{z}, \bar{\eta})$, for all $\lambda \in \mathbb{R}$.

It follows that L is (2,0) homogeneous with respect to the real scalar λ and is proved that the following identities are fulfilled [10];

$$\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i}\bar{\eta}^i = 2L; \qquad g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}, \tag{2}$$

$$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ij}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0; \qquad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0, \tag{3}$$

$$2L = g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{4}$$

where,

$$g_{ij} = \frac{\partial^2 L}{\eta^i \eta^j}; \quad g_{i\bar{j}} = \frac{\partial^2 L}{\eta^i \bar{\eta}^j}; \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial\bar{\eta}^i \partial\bar{\eta}^j}.$$

Definition 2.2. *An ℝ-complex Finsler space (M, F) is called (α, β)-metric if the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is ℝ-homogeneous by means of functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$-depends on $z^i, \eta^i, \bar{z}^i$ and $\bar{\eta}^i$, (i=1,2,...,n) by means of $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$. That is*

$$F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \tag{5}$$

where,

$$\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(a_{ij}\eta^i\eta^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j) = Re\left\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\right\},$$

$$\beta(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(b_i\eta^i + b_i\bar{\eta}^i) = Re(b_i\eta^i), \tag{6}$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, $b_i = b_i(z)$. We denote

$$L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta}) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})). \tag{7}$$

Definition 2.3. *An ℝ-Complex Finsler space (M, F) is called Hermitian space, if the tensor $g_{ij} = 0$ and the Hermitian matric $g_{i\bar{j}}$ is invertible. An ℝ-Complex Finsler space (M, F) is called non-Hermitian space if the metric tensor $g_{i\bar{j}} = 0$ and the Hermitian matric g_{ij} is invertible. Where, g_{ij} and $g_{i\bar{j}}$ are the metric tensors of the space and are given by, $g_{ij} = \frac{\partial}{\partial \eta^i}\frac{\partial}{\partial \eta^j}L$ and $g_{i\bar{j}} = \frac{\partial}{\partial \eta^i}\frac{\partial}{\partial \bar{\eta}^j}L$.*

3. $\mathbb{R}$-Complex Matsumoto metrics.

The $\mathbb{R}$-complex Finsler space produce the tensor fields g_{ij} and $g_{i\bar{j}}$. The tensor field must $g_{i\bar{j}}$ be invertible in Hermitian geometry. These problems are about to Hermitian $\mathbb{R}$-complex Finsler spaces, if $\det(g_{i\bar{j}} \neq 0)$ and non-Hermitian $\mathbb{R}$-complex Finsler spaces, if $\det(g_{ij} \neq 0)$. In this section, we determine the fundamental tensor of complex Matsumoto metric and obtained condition for property of non-Hermitian $\mathbb{R}$-complex Finsler spaces. Consider $\mathbb{R}$-Complex Finsler space with Matsumoto metric,

$$L(\alpha,\beta) = \left(\frac{\alpha^2}{\alpha-\beta}\right)^2. \tag{8}$$

Then, it follows that $F = \frac{\alpha^2}{\alpha-\beta}$. Now, we find the following quantities on $\mathbb{R}$-complex Finsler spaces with Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$. From the equalities (2) and (3) with metric (8), we have

$$\alpha L_\alpha + \beta L_\beta = 2L, \qquad \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \tag{9}$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \qquad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \tag{10}$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \tag{11}$$

$$L_\alpha = \frac{2\alpha^3(\alpha-2\beta)}{(\alpha-\beta)^3}, \tag{12}$$

$$L_\beta = \frac{2\alpha^4}{(\alpha-\beta)^3}, \tag{13}$$

$$L_{\alpha\alpha} = 2\alpha^2\left\{\frac{\alpha^2-4\alpha\beta+6\beta^2}{(\alpha-\beta)^4}\right\}, \tag{14}$$

$$L_{\beta\beta} = \frac{6\alpha^4}{(\alpha-\beta)^4}, \tag{15}$$

$$L_{\alpha\beta} = \frac{2\alpha^3(\alpha-4\beta)}{(\alpha-\beta)^4}, \tag{16}$$

$$\begin{aligned} \alpha L_\alpha + \beta L_\beta &= \alpha\left[\frac{2\alpha^3(\alpha-2\beta)}{(\alpha-\beta)^3} + \beta\frac{2\alpha^4}{(\alpha-\beta)^3}\right], \\ &= \frac{2\alpha^5 - 2\alpha^4\beta}{(\alpha-\beta)^3} = \frac{2\alpha^4}{(\alpha-\beta)^2} = 2L, \end{aligned} \tag{17}$$

$$\begin{aligned} \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} &= \alpha\left[\frac{2\alpha^2(\alpha^2-4\alpha\beta+6\beta^2)}{(\alpha-\beta)^4}\right] + \beta\left[\frac{2\alpha^3(\alpha-4\beta)}{(\alpha-\beta)^4}\right] \\ &= \frac{2\alpha^5 - 3\alpha^4\beta + 4\alpha^3\beta^2}{(\alpha-\beta)^4} = \frac{2\alpha^3(\alpha-2\beta)}{(\alpha-\beta)^3} = 2L. \end{aligned} \tag{18}$$

We propose to determine the metric tensors of an $\mathbb{R}$-complex Finsler space with using the following equalities as:

$$g_{ij} = \frac{\partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})}{\partial\eta^i\partial\eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L(z,\eta,\bar{z},\lambda\bar{\eta})}{\partial\eta^i\partial\bar{\eta}^j}.$$

Each of these being of interest in the following :

We consider,

$$\begin{aligned} \frac{\partial\alpha}{\partial\eta^i} &= \frac{1}{2\alpha}(a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j) = \frac{1}{2\alpha l_i}, \quad \frac{\partial\beta}{\partial\eta^i} = \frac{1}{2}b_i. \\ \frac{\partial\alpha}{\partial\bar{\eta}^i} &= \frac{1}{2\alpha}(a_{\bar{i}\bar{j}}\bar{\eta}^j + a_{i\bar{j}}\eta^j) = \frac{\partial\beta}{\partial\bar{\eta}^i} = \frac{1}{2}b_{\bar{i}}, \end{aligned}$$

where, $l_i = (a_{ij}\eta^j + a_{i\bar{j}}\eta^{\bar{j}})$, $l_{\bar{j}} = a_{\bar{i}\bar{j}}\bar{\eta}^i + a_{i\bar{j}}\eta^i$. We find immediately, $l_i\eta^i + l_{\bar{j}}\bar{\eta}^j = 2\alpha^2$. We denote:

$$\eta^i = \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i}F^2 = 2F\frac{\partial}{\partial \eta^i}\left(\frac{\alpha^2}{\alpha-\beta}\right),$$

$$\eta_i = \rho_0 l_i + \rho_1 b_i,$$

where

$$\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha, \tag{19}$$

and

$$\rho_1 = \frac{1}{2}L_\beta. \tag{20}$$

Differentiating ρ_0 and ρ_1 with respect to η^j and $\bar{\eta}^j$ respectively, which yields:

$$\frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2}l_j + \rho_{-1}b_j,$$

and

$$\frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2}l_{\bar{j}} + \rho_{-1}b_{\bar{j}}.$$

Similarly $\frac{\partial \rho_1}{\partial \eta^i} = \eta_{-1}l_i + \mu_0 b_i$, $\frac{\partial \rho_1}{\partial \bar{\eta}^i} = \rho_{-1}l_{\bar{i}} + \mu_0 b_{\bar{i}}$, where,

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha - L_\alpha}}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \mu_0 = \frac{L_{\beta\beta}}{4}. \tag{21}$$

By direct computation, using (19), (20), (20) and (21), we obtain the following result.

Theorem 3.1. *The invariants of ℝ-complex Finsler space with Matsumoto metric: $\tilde{\rho}_0$, $\tilde{\rho}_1$, $\tilde{\rho}_{-2}$, $\tilde{\rho}_{-1}$ and μ_0 are given by:*

$$\tilde{\rho}_0 = \frac{1}{2}\alpha^{-1}L_\alpha = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3},$$

$$\tilde{\rho}_1 = \frac{1}{2}L_\beta = \frac{\alpha^4}{(\alpha-\beta)^3},$$

$$\tilde{\rho}_{-2} = \frac{\beta(\alpha-4\beta)}{(\alpha-\beta)^4},$$

$$\tilde{\rho}_{-1} = \frac{\alpha^2(\alpha-4\beta)}{2(\alpha-\beta)^4}$$

$$\tilde{\mu}_0 = \frac{L_{\beta\beta}}{4} = \frac{3\alpha^4}{2(\alpha-\beta)^4},$$

subscripts -2, -1, 0, 1 gives us the degree of homogeneity of these invariants.

3.1. Fundamental tensor of ℝ-Complex Finsler space with Matsumoto metric

The fundamental metric tensors of ℝ-complex Finsler space with (α, β) metric are given by[14]:

$$g_{ij} = \rho_0 a_{ij} + \rho_{-2}l_i l_j + \mu_0 b_i b_j + \rho_{-1}(b_j l_i + b_i l_j) \tag{22}$$

From Theorem 3.1, we have

$$\tilde{g}_{ij} = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}a_{ij} + \frac{\beta(\alpha-4\beta)}{(\alpha-\beta)^4}l_i l_j + \frac{3\alpha^4}{2(\alpha-\beta)^4}b_i b_j + \frac{\alpha^2(\alpha-4\beta)}{2(\alpha-\beta)^4}(b_j l_i + b_i l_j). \tag{23}$$

$$\tilde{g}_{i\bar{j}} = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}a_{i\bar{j}} + \frac{\beta(\alpha-4\beta)}{(\alpha-\beta)^4}l_i l\bar{j} + \frac{3\alpha^4}{2(\alpha-\beta)^4}b_i b\bar{j} + \frac{\alpha^2(\alpha-4\beta)}{2(\alpha-\beta)^4}(b_{\bar{j}}l_i + b_i l_{\bar{j}}). \tag{24}$$

Or, equivalently,

$$\tilde{g}_{ij} = \rho_0\left[a_{ij} + pl_il_j + qb_ib_j + r\eta_i\eta_j\right], \tag{25}$$

$$\tilde{g}_{i\bar{j}} = \rho_0\left[a_{i\bar{j}} + pl_il_{\bar{j}} + qb_ib_{\bar{j}} + r\eta_i\eta_{\bar{j}}\right], \tag{26}$$

where, $\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha$.

$$p = \frac{\beta(\alpha-4\beta)}{2\alpha^2(\alpha-\beta)(\alpha-2\beta)}, \tag{27}$$

$$q = \frac{3\alpha^2}{2(\alpha-\beta)(\alpha-2\beta)}, \tag{28}$$

$$r = \frac{(\alpha-4\beta)}{2(\alpha-\beta)(\alpha-2\beta)}. \tag{29}$$

The next objectives is to obtain the determinant and the inverse of the tensor field $\tilde{g}_{ij}$. The solution of the non-singular non-Hermitian metric $\tilde{Q}_{ij}$ as follows. The following proposition is proved by [6].

Proposition 3.2. *Suppose:*

- (Q_{ij}) *is a non-singular* $n\times n$ *complex matrix with inverse* Q^{ji}*;*
- C_i *and* $C_{\bar{i}} = \bar{C}_i, i = 1, \ldots\ldots..n$ *are complex numbers;*
- $C^i := Q^{ji}C_j$ *and its conjugates;* $C^2 := C^iC_i = \bar{C}^iC_{\bar{i}}$*;* $H_{ij} := Q_{ij} \pm C_iC_j$.

Then,

(i). $det(H_{ij}) = (1 \pm C^2)det(Q_{ij})$,

(ii). Whenever $(1 \pm C^2) \neq 0$, *the matrix* (H_{ij}) *is invertible and in this case its inverse is* $H^{ij} = Q^{ji} \pm \frac{1}{1\pm C^2}C^iC^j$.

Using the above proposition we prove the following theorem:

Theorem 3.3. *For a non-Hermitian* $\mathbb{R}$*-Complex Finsler space with Matsumoto metric* $F = \frac{\alpha^2}{\alpha-\beta}$, *then they have the following:*

(i). The contravarient tensor $\tilde{g}^{ij}$ *of the fundamental tensor* $\tilde{g}_{ij}$ *is:*

$$\tilde{g}^{ji} = \frac{(\alpha-\beta)^3}{\alpha^2(\alpha-2\beta)}a^{ji} + \left[\frac{p}{1+p\gamma} + \frac{p^2q\epsilon^2}{\tau(1+p\gamma)^2}\right]\eta^i\eta^j + \frac{qb^ib^j}{\tau} + \frac{pq\epsilon}{\tau(1+p\gamma)}(b^i\eta^j + b^j\eta^i) + \frac{M^2\eta^i\eta^j + MN(\eta^ib^j + \eta^jb^i + N^2b^ib^j)}{1+(M\gamma+N\epsilon)\sqrt{r}},$$

where,

$$M = \left[1 + \left(\frac{p}{1+p\gamma} + \frac{p^2q\epsilon^2}{\tau(1+p\gamma)^2}\right)\right]\gamma + \frac{pq\epsilon}{\tau(1+p\gamma)^3} \quad \textit{and} \quad N = \frac{q}{\tau} + \frac{pq\epsilon\gamma}{\tau(1+p\gamma)},$$

(ii). $det(a_{ij} + pl_il_j + qb_ib_j + r\eta_i\eta_j) = [1 + (M\gamma + N\epsilon)\sqrt{r}]\left[1 + \omega + \frac{p\epsilon^2}{1+p\gamma}\right](1+p\gamma)det(a_{ij})$, *where,* $r = \frac{(\alpha-4\beta)}{2(\alpha-\beta)(\alpha-2\beta)}$.

Proof. **Step 1:** We claim of this theorem proved by following three steps:

We write $\tilde{g}_{ij}$ from (25) in the form.

$$\tilde{g}_{ij} = \rho_0[a_{ij} + pl_i l_j + ql_i l_j + r\eta_i\eta_j]. \tag{30}$$

We take $\tilde{Q}_{ij} = a_{ij}$ and $\tilde{C}_i = \sqrt{p}l_i$. By applying the Proposition 3.2 we obtain $\tilde{Q}^{ij} = a^{ji}$, $\tilde{C}^2 = \tilde{C}_i\tilde{C}^i = \sqrt{p}l_i \times \tilde{Q}^{ji} \times \tilde{C}_j = \sqrt{p}l_i \times a^{ji} \times \sqrt{p}l_j = p \times l_i a^{ij} l_j = p\gamma$, and $1 + \tilde{C}^2 = (1 + p\gamma)$. So, the matrix $\tilde{H}_{ij} = a_{ij} - pl_i l_j$, is invertible with

$$\tilde{H}^{ij} = a^{ji} + \frac{1}{1+p\gamma}\eta^i\eta^j,$$
$$det(a_{ij} + pl_i l_j) = (1 + p\gamma) = det(a_{ij}).$$

Step 2: Now, we consider $\tilde{Q}_{ij} = a_{ij} + pl_i l_j$, and $\tilde{C}_i = \sqrt{q}b_i$, By applying the Proposition 3.2 we have

$$\tilde{Q}^{ji} = a^{ji} + \frac{p\eta^i\eta^j}{1+p\gamma},$$
$$\tilde{C}^2 = \tilde{C}_i\tilde{C}^i = \tilde{Q}^{ji} \times \tilde{C}_j = \sqrt{q}b_i\left[a^{ij} + \frac{p\eta^i\eta^j}{1+p\gamma}\sqrt{q}b^j\right],$$
$$\tilde{c}^2 = q\left[\omega + \frac{p\epsilon^2}{1+p}\right].$$

Therefore,

$$1 + \tilde{C}^2 = 1 + q\left[\omega + \frac{p\epsilon^2}{1+p\gamma}\right] \neq 0,$$

where, $\in = b_j\eta^j$, $\omega = b_j b^j$. It results that the inverse of $\tilde{H}_{ij} = a_{ij} + pl_i p_j + qb_i b_j$ exists and it is

$$\tilde{H}^{ji} = Q^{ji} + \frac{1}{1+C^2}C^i C^j,$$
$$tildeH^{ji} = a^{ji} + \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q\left[b^i + \frac{p\epsilon\eta^i}{1+p\gamma}\right]\left[b^j + \frac{p\epsilon\eta^j}{1+p\gamma}\right]}{\tau} \tag{31}$$
$$\tilde{H}^{ji} = a^{ji} + \left(\frac{p}{1+p\gamma} + \frac{qp^2\epsilon^2}{\tau(1+p\gamma)^2}\right)\eta^i\eta^j + \frac{pq\epsilon}{1+p\gamma}(b^i\eta^j + b^j\eta^i) + \frac{q}{\tau}b^i b^j, \tag{32}$$

where,

$$\tau = 1 + q\left[\omega + \frac{p\epsilon^2}{1+p\gamma}\right].$$

and,

$$det\left[a_{ij} + pl_i l_j + qb_i b_j\right] = \left[1 + q\left(\omega + \frac{p\epsilon^2}{1+p\gamma}\right)\right](1 + p\gamma)det(a_{ij}). \tag{33}$$

Step 3: We put

$$\tilde{Q}_{ji} = a_{ij} + pl_i l_j + qb_i b_j, \tag{34}$$

and $\tilde{C}_i = \sqrt{r}\eta_i$, clearly observe that and obtain

$$\tilde{Q}^{ji} = a^{ji} + \left(\frac{p}{1+p\gamma} + \frac{qp^2\epsilon^2}{\tau(1+p\gamma)}\right)\eta^i\eta^j + \frac{pq\epsilon}{1+p\gamma}(b^i\eta^j + b^j\eta^i) + \frac{q}{1+p\gamma}b^i b^j, \tag{35}$$

and $\tilde{C}_i = M\eta^i + Nb^j$, where

$$M = \left[1 + \left(\frac{p}{1+p\gamma} + \frac{p^2q\epsilon^2}{\tau(1+p\gamma)^2}\right)\right]\gamma + \frac{pq\epsilon}{\tau(1+p\gamma)^3}, \tag{36}$$

$$N = \frac{q}{\tau} + \frac{pq\epsilon\gamma}{\tau(1+p\gamma)}. \tag{37}$$

And

$$\tilde{C}^2 = (M\gamma + N\epsilon)\sqrt{r},$$
$$1 + \tilde{C}^2 = 1 + (M\gamma + N\epsilon)\sqrt{r} \neq 0,$$

clearly, the matrix $\tilde{H}_{ij}$ is invertible.

$$\tilde{C}^i = a^{ji} + \left\{ \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q\left[b^i + \frac{p\epsilon\eta^i}{1+p\gamma}\right]\left[b^j + \frac{p\epsilon\eta^j}{1+p\gamma}\right]}{\tau} \right\}\eta_j,$$

and

$$\tilde{C}^j = a^{ji} + \left\{ \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q\left[b^i + \frac{p\epsilon\eta^i}{1+p\gamma}\right]\left[b^j + \frac{p\epsilon\eta^j}{1+p\gamma}\right]}{\tau} \right\}\eta_i,$$

where $\tilde{C}^i\tilde{C}^j = M^2\eta^i\eta^j + MN(\eta^i b^j + \eta^j b^i) + N^2 b^i b^j$. Again by applying Proposition 3.2 we obtain the inverse of $\tilde{H}_{ij}$ as:

$$\tilde{H}^{ji} = a^{ji} + \left[\frac{p}{1+p\gamma} + \frac{p^2 q\epsilon^2}{\tau(1+p\gamma)^2}\right]\eta^i\eta^j + \frac{qb^ib^j}{\tau} + \frac{pq\epsilon}{\tau(1+p\gamma)}(b^i\eta^j + b^j\eta^i)$$
$$+ \frac{M^2\eta^i\eta^j + MN(\eta^i b^j + \eta^j b^i) + N^2 b^i b^j}{1 + (M\gamma + N\epsilon)\sqrt{r}}. \tag{38}$$

$$det(a_{ij} + pl_il_j + qb_ib_j + r\eta_i\eta_j) = \left[1 + (M\gamma + N\epsilon)\sqrt{r}\right]\left[1 + \omega + \frac{p\epsilon^2}{1+p\gamma}\right](1+p\gamma)det(a_{ij}). \tag{39}$$

But $\tilde{g}_{ij} = \rho_0\tilde{H}_{ij}$, with $\tilde{H}_{ij}$ from last step. Thus

$$\tilde{g}^{ji} = \frac{1}{\rho_0}\tilde{H}^{ij}. \tag{40}$$

Therefore, from equation (38) in (40) and the equation 39, then we obtained claims (i) and (ii). Where, $\tilde{g_{ij}}$ and $\tilde{g}_{i\bar{j}}$ represents the $\mathbb{R}$-complex Finsler space with Matsumoto metric. □

We observed the terms of γ, ϵ , and δ from above Theorem 3.1, immediately we state:

Proposition 3.4. *In a non-Hermitian $\mathbb{R}$-complex Finsler space with Matsumoto metric then have the following properties.*

$$\gamma + \bar{\gamma} = l_i\eta^i + l_{\bar{j}}\eta^{\bar{j}} = a_{ij}\eta^j\eta^i + a_{\bar{j}\bar{k}}\eta^{\bar{k}}\eta^{\bar{j}} = 2\alpha^2, \tag{41}$$
$$\epsilon + \bar{\epsilon} = b_j\eta^j + b_{\bar{j}}\eta^{\bar{j}} = 2\beta, \quad \delta = \epsilon, \tag{42}$$

where

$$l_i = a_{ij}\eta^j, \quad \eta_i = \frac{\alpha^2(\alpha - 2\beta)}{(\alpha-\beta)^3}a_{ij}\eta^i + \frac{\alpha^4}{(\alpha-\beta)^3}b_i, \quad \gamma = a_{jk}\eta^j\eta^k = l_k\eta^k, \quad \epsilon = b_j\eta^j,$$
$$b^k = a^{jk}b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk}\eta^j b^k = l_k b^k, \quad l_j = a^{jl}l_i = \eta^j.$$

4. Conclusion

The $\mathbb{R}$-complex Finsler space is an important quantities in complex Finsler geometry and it has well known interrelation with the other quantities like $\mathbb{R}$-complex Finsler space with class of (α, β)-metrics. In this paper we determined the fundamental metric tensors $\tilde{g}_{ij}$ and $\tilde{g}_{i\bar{j}}$ of $\mathbb{R}$-complex Finsler space with Matsumoto metrics and also find their determinants. Finally, we studied the property of non-Hermitian $\mathbb{R}$-complex Matsumoto metric.

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