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Lebegue-Bochner Spaces and Evolution Triples

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Abstract: The functional-analytic approach to the solution of partial differential equations requires knowledge of the properties of spaces of functions of one or several real variables. A large class of infinite dimensional dynamical systems (evolution systems) can be modeled as an abstract differential equation defined on a suitable Banach space or on a suitable manifold therein. The advantage of such an abstract formulation lies not only on its generality but also in the insight that can be gained about the many common unifying properties that tie together apparently diverse problems. It is clear that such a study relies on the knowledge of various spaces of vector valued functions (i.e., of Banach space valued functions). For this reason some facts about vector valued functions is introduced. We introduce the various notions of measurability for such functions and then based on them we define the different integrals corresponding to them. A function $f : \Omega \to X$ is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countable-valued, Σ-measurable functions. If X is separable and $f : \Omega \to X$, be a function then the following three properties are equivalent: (a) f is strongly measurable; (b) f is weakly measurable; (c) f is Borel measurable. A strongly measurable function $f : \Omega \to X$, is Bochner integrable if and only if the function $\omega \to \|f(\omega)\|_X$ is Lebesgue integrable (i.e., $\|f(.)\|_X \in L^1(\Omega)$). The emphasis of the project is on the so-called Bochner integral, which generalizes in a very natural way the classical Lebesgue integral to vector valued functions. We continue with vector valued functions and introduce the so-called Lebesgue-Bochner spaces, which extend to vector valued functions of the well known Lebesgue L^p-spaces. We also consider evolution triples and the function spaces associated with them. Evolution triples provide a suitable analytical framework for the study of a large class of linear and nonlinear evolution equations.

Keywords: Banach space, nonlinear, Lebesgue-Bochner, evolution triples.

1. Introduction

The functional-analytic approach to the solution of (partial) differential equations requires knowledge of the properties of spaces of functions of one or several real variables. A large class of infinite dimensional dynamical systems (evolution systems) can be modeled as an abstract differential equation defined on a suitable Banach space or on a suitable manifold therein. The advantage of such an abstract formulation lies not only on its generality but also in the insight that can be gained about the many common unifying properties that tie together apparently diverse problems. It is clear that such a study relies on the knowledge of various spaces of vector valued functions (i.e., of Banach space valued functions). For this reason some facts about vector valued functions is introduced. We introduce the various notions of measurability for such functions and then based on them we define the different integrals corresponding to them. The emphasis of the project is on the so-called Bochner integral, which generalizes in a very natural way the classical Lebesgue integral to vector valued functions. We continue with vector valued functions and introduce the so-called Lebesgue-Bochner spaces, which extend to vector valued functions of the well known Lebesgue L^p-spaces. We also consider evolution triples and the function spaces associated with them. Evolution triples provide a suitable analytical framework for the study of a large class of linear and nonlinear evolution equations.

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To have a good theory of integration, we need a reasonable notion of measurability of functions. In this direction the basic result is the Pettis measurability theorem, which was proved by Pettis (1938a). The main integral for vector-valued functions, which has a rich enough structure to have significant applications, is the Bochner integral. The Bochner integral can be traced in the works of Bochner (1933) and Dunford (1935) and for this reason is also known as\Dunford's first integral." Most of the properties of the Bochner integral follow from the corresponding properties of the classical Lebesgue integral. So some analysts say that the Bochner integral is the Lebesgue integral with the absolute value replaced by norms. The Pettis integral has much fewer applications, which require knowledge and use of sophisticated measure theoretic results. The theory of Pettis integration started with the work of Pettis and attracted renewed attention after the paper of Edgar (1977 A reflexive Banach space has the RNP, which was established by Phillips (1940), while the fact that a separable dual Banach space is an RNP space is due to Dunford & Pettis (1940). The Riesz Representation theorem for the Lebesgue Bochner spaces $L^p(\Omega; X)$, $p \in [1, +\infty))$ is essentially due to Bochner & Taylor (1938). Its proof can be found in Diestel & Uhl (1977, p. 97). Its extension (for $p = 1$) mentioned in Theorem (called Dinculeanu-Foias theorem) is due to Dinculeanu & Foias (1961) and its proof, based on \lifting theory," can be found in Ionescu-Tulcea & Ionescu-Tulcea (1969, p. 93). Absolute continuity of real valued functions was introduced by Vitali (1908), who established the fundamental fact that a real valued function on [0,1] is absolutely continuous if and only if it is the integral of its derivative (the fundamental theorem of Lebesgue calculus).

2. Measurable Vector-Valued Functions

In this we deal with functions which take values in Banach spaces. For such functions we define the various notions of measurability and different integrals corresponding to them. The domain of a function is a finite measure space (Ω, Σ, μ) and the range is a Banach space X. By X^* we denote the topological dual of X and by $\langle ., . \rangle$ the duality brackets for the pair (X^*, X). By $\mathcal{B}(X)$ we denote the Borel σ-field of X.

2.1. Definitions of Measurable Functions

Definition 2.1. *Let $f : \Omega \to X$ is a function.*

1. *Function f is said to be a simple functions, if it takes finite number of values, say $x_1, x_2, \ldots, x_N$ and E_k be an element of Σ (a σ-algebra) the formula $f(x) = \sum_{k=1}^{n} (x_k \chi_{E_k})$ is called the standard representation of f and χ_{E_k} is a characteristic function on E_k such that* $\chi_{E_k}(\omega) = \begin{cases} 1 & \text{if } \omega \in E_k \\ 0 & \text{otherwise} \end{cases}$.
2. *Function f is said to be strongly measurable (or Bochner measurable), if there exists a sequence $s_n : \Omega \to X$, $n \geq 1$, of simple functions, such that $s_n(\omega) \to f(\omega)$ for μ-almost all ω an element of Ω, where $\to$ denotes the convergence in the norm topology of X i.e. $\|f(\omega) - s_n(\omega)\| \to 0$ for μ-a.a $\omega \in \Omega$ (a. a means almost every where).*
3. *Function f is said to be weakly measurable, if for all $x^* \in X^*$, the function $\omega \longmapsto \langle x*, f(\omega) \rangle$ is Σ-measurable.*
4. *Function $f : \Omega \to X^*$ is said to be weak*-measurable, if for all $x \in X$ the function $\omega \longmapsto \langle f(\omega), x \rangle$ is Σ-measurable.*

Remark 2.2. *Evidently strong measurability of a function $f : \Omega \to X$ implies its weak measurability. Also strong measurability implies that for every $B \in \mathcal{B}(X)$, we have that $f^{-1}(B) \in \Sigma$ (i.e., f is Borel measurable). Moreover, adapting the proof of the classical result, which asserts that a measurable real-valued function is the μ-almost everywhere limit of a sequence of simple functions, we see that if X is separable, then $f : \Omega \to X$ is strongly measurable if and only if it is Borel* *measurable. In fact, in the next theorem, known as the Pettis measurability theorem, when X is separable, the situation simplifies considerably.*

Theorem 2.3 (Pettis Measurability Theorem)**.** *A function $f : \Omega \to X$ is strongly measurable if and only if it is weakly measurable and μ-almost separably valued (i.e., there exists a set $A \in \Sigma$ with $\mu(A) = 0$, such that $f(\Omega n A)$ is separable in X).*

Evidently $D_n \in \Sigma \cap C_0$ and $C_0 = \bigcup_{n=1}^{\infty} D_n$. Let $E_n = D_n \backslash \bigcup_{n=1}^{n-1} D_n$. Then $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma \cap C_0$ is a sequence of disjoint sets and $C_0 = \bigcup_{n=1}^{\infty} E_n$. We define $f_\epsilon(\omega) = \begin{cases} z_n & \text{if } \omega \in E_n, n \geq 1 \\ 0 & \text{if } \omega \in \Omega \backslash C_0 \end{cases}$. Clearly, $f_\epsilon : \Omega \to X$ is Σ-measurable, count ably valued (i.e takes countably many values) and $\|f(\omega) - f_\epsilon(\omega)\|_X < \epsilon$ for all $\omega \in \Omega$. Taking $\epsilon = \frac{1}{k}$, $k = 1$, we see that f is the uniform limit of a sequence $\{f_{1/k}\}$, $k = 1$ of countably-valued functions, hence f is strongly measurable. An interesting by product of the previous proof is the following result.

Corollary 2.4. *A function $f : \Omega \to X$ is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countable-valued, Σ-measurable functions.*

Proof. ($\Rightarrow$) Suppose $f : \Omega \to X$ is strongly measurable. Then there exists a sequence $\{f_n\}_{n=1}$ of X-valued of simple function such that $f_n(\omega) \to f(\omega)$ for $\mu - a.a\ \omega \in \Omega \Rightarrow \|f_n(\omega) - f(\omega)\|_X \to 0$. For $\mu - a.a\ \omega \in \Omega \Rightarrow \lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_X = 0$ for $\mu - a.a\ \omega \in \Omega$. Hence, for all $\in > 0$, there exists $N \in \mathbb{N}$ such that $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_X < \epsilon$ for all $n = 1$, for $\mu - a.a\ \omega \in \Omega \Rightarrow \lim_{n\to\infty} f_n(\omega) = f(\omega)$ for almost all $\omega \in \Omega$. Hence f is the uniform limit almost everywhere of sequence $\{f_n\}_{n=1}$ of countably-valued, Σ-measurable functions.
($\Rightarrow$) Suppose $f : \Omega \to X$ is the uniform limit almost everywhere of sequence $\{f_n\}_{n=1}$ of countably-valued, Σ-measurable functions. That is $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for almost all $\omega \in \Omega \Rightarrow \lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_X = 0$ for $\mu - a.a\ \omega \in \Omega \Rightarrow \|f_n(\omega) - f(\omega)\|_X \to 0$ for $\mu - a.a\ \omega \in \Omega$. Therefore, $f_n(\omega) \to f(\omega)$ for $\mu - a.a\ \omega \in \Omega$. Hence f is strongly measurable. □

Example 2.5. *If X is not separable, the weak measurability does not imply strongly measurability. Consider non separable Hilbert space $X = l^2([0,1])$ and let $\{e_t\}$, $t \in [0,1]$ be an orthonormal basis. Then the function $f : [0,1] \to l^2([0,1])$ is defined by $f(t) = e_t$ is weakly measurable. Since $(x^*, f(t)) = (x^*, e_t) = 0$ for all $x^* \in l^2([0,1])^* = l_2([0,1])$.*
On the other hand, if $A \subseteq [0,1]$, then $f([0,1]nA)$ is separable if and only if $[0,1]nA$ is countable and so we cannot have $\lambda^1(A) = 0$ therefore f is not strongly measurable (i.e. by pettis measurability theorem).

Definition 2.6. *A bounded function f on the domain E of finite measure is said to be lebesgue integral over E if the upper and lower integrals are equal. The common value is called the Lebesgue integral of f over E and denoted by $\int_a^b f d\mu$; then $\int_a^b f d\mu = \int_a^b f d\mu = \int_a^b f d\mu$.*

Note that: Anon-negative measurable function f on the E is said to be integrable over E if $\int_E f < +\infty$.

Example 2.7. *Let $f(x) = e^{-x}$ and $E = [0, \infty]$ then solve $\int_0^\infty f(x)dx$. Now we are ready to define the Bochner integral for measurable functions. To define the Bochner integral we need to extend the definition of Lebesgue integral to function that takes values in banach space, as a limit of integrals of simple functions.*

Definition 2.8.

(a). Let $s(\omega) = \sum\limits_{k=1}^{n} x_k \chi_{c_k}$ *be an X-valued simple function. The Bochner integral of s is defined by*

$$\int_{\Omega} s(\omega)\, d\mu(\omega) = \sum_{k=1}^{n} x_k \mu(c_k)$$

(b). A function $f : f \to x$ *is said to be Bochner integrable, if there exists a sequence* $\{s_n\}_{n=1}$ *of simple functions, such that*

$$\lim_{n\to\infty} \int_{\Omega} \|f(\omega) - s_n(\omega)\|\, d\mu = 0$$

If $A \in \Sigma$ *we define the Bochner integral of f on A as*

$$\int_{A} f(\omega)\, d\mu = \lim_{n\to\infty} \int_{\Omega} \chi_{A(\omega)} s_n(\omega) d\mu \tag{1}$$

Proposition 2.9. *A strongly measurable function* $f : \Omega \to X$*, is Bochner integrable if and only if the function* $\omega \to \|f(\omega)\|_X$ *is Lebesgue integrable (i.e.,* $\|f(.)\|_X \in L^1(\Omega)$*).*

Proof. $(\Rightarrow)$ Let $f : \Omega \to X$ is Bochner integrable. Then we can find a sequence $\{s_n\}_{n=1}$ of simple functions such that

$$\lim_{n\to\infty} \int_{\Omega} \|f(\omega) - s_n(\omega)\|\, d\mu = 0$$

Then for any $n = 1$; Fix $\epsilon = 1$ for all $\delta > 0$ there exists $N \in \mathbb{N}$ such that $\int_{\Omega} \|s_N(\omega)\|\, d\mu < 1 - \delta$. Then $\int_{\Omega} \|f(\omega)\|_X d\mu \leq \int_{\Omega} \left\|f(\omega) - s_n(\omega)\right\| d\mu + \int_{\Omega} \|s_n(\omega)\|\, d\mu < +\infty$ for $\mu - a.a\ \omega \in \Omega$. Therefore $\|f(.)\|_X \in L^1(\Omega)$.
$(\Leftarrow)$ Let $f : \Omega \to X$ is strongly measurable function and $\|f(.)\|_X \in L^1(\Omega)$, then there exists a sequence $\{s_n\}_{n=1}$ of simple functions such that $s_n(\omega) \to f(\omega)$ for all $\omega \in \Omega \backslash A$ with $\mu(A) = 0$

$$\lim_{n\to\infty} \|f(\omega) - s_n(\omega)\|_X = 0$$

Then $\lim\limits_{n\to\infty} \{s_n\}_{n\geq 1} = \|f(\omega)\|_X$ for $\mu - a.a\ \omega \in \Omega$. Let $h_n : \Omega \to X$ for all $n = 1$ be defined by $h_n(\omega) = \begin{cases} s_n(\omega), & \text{if } \|s_n(\omega)\|_X \leq 2\|f(\omega)\|_X \\ 0, & \text{otherwise} \end{cases}$. Evidently for every $n \geq 1$, h_n -is an X-valued simple function. Also $\lim\limits_{n\to\infty} \|f(\omega) - h_n(\omega)\|_X = 0$ for all $\omega \in \Omega \backslash A$ and $\|f(\omega) - h_n(\omega)\|_X \leq 3\|f(\omega)\|_X$. Therefore, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_{\Omega} \left\|f(\omega) - h_n(\omega)\right\| d\mu = 0$$

Therefore, f is Bochner integrable by definition. □

Corollary 2.10. *If* $f : \Omega \to X$ *is Bochner integrable function and* $A \in \Sigma$ *then* $\left\|\int_A f(\omega) d\mu\right\|_X \leq \int_A \|f(\omega) d\mu\|_X$.

Proof. Suppose $f : \Omega \to X$ is Bochner integrable function and $A \in \Sigma$. Then

$$\begin{aligned} \left\|\int_A f(\omega) d\mu\right\|_X &= \left\|\lim_{n\to+\infty} \sum_{k=1}^{n} x_k \mu(C_k \bigcap A)\right\|_X \\ &\leq \lim_{n\to+\infty} \sum_{k}^{n} \|x_k\|_X \mu(C_k \bigcap A) \\ &= \int_A \|f(\omega) d\omega\|_X \end{aligned}$$

□

Theorem 2.11 (Linearity of Bochner Integral)**.** *Let $f, g : \Omega \to X$ are two Bochner integrable functions, $A \in \Sigma$ and $a, b \in R$. Then $af + bg$ is Bochner integrable functions too, and $\int\limits_{\Omega}(af + bg)d\mu = a\int\limits_{A} fd\mu + b\int_A gd\mu$.*

Proof. Suppose $f, g : \Omega \to X$ are two Bochner integrable functions, then there is a sequences $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ of simple functions such that

$$\int_\Omega \|f(\omega) - f_n(\omega)\| \, d\mu = 0 \;\; and \;\; \int_\Omega \|g(\omega) - g_n(\omega)\| \, d\mu = 0$$

Then $\int_\Omega \|af(\omega) - af_n(\omega)\| \, d\mu = 0$ and $\int_\Omega \|bg(\omega) - bg_n(\omega)\| \, d\mu = 0$. For a given $\epsilon > 0$, there exists N_1 and $N_2 \in \mathbb{N}$ such that

$$\lim_{n\to+\infty}\int_\Omega \|af(\omega) - af_n(\omega)\| \, d\mu \prec \frac{\epsilon}{2};\; n \geq N_1 \;\; and$$
$$\lim_{n\to+\infty}\int_\Omega \|g(\omega) - g_n(\omega)\| \, d\mu \prec \frac{\epsilon}{2};\; n \geq N_2$$

Let $N = \max\{N_1; N_2\}$ then we have

$$\lim_{n\to+\infty}\int_\Omega \|[(af(\omega) + bg(\omega)) - (af_n(\omega) + bg_n(\omega)]\| \, d\mu \leq \lim_{n\to+\infty}\int\limits_\Omega \|af(\omega) - af_n(\omega)\| \, d\mu + \lim_{n\to+\infty}\int_\Omega \|g(\omega) - g_n(\omega)\| \, d\mu$$
$$\prec \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $af + bg$ is Bochner integrable function.

Claim: $\int_\Omega(af(\omega) + bg(\omega))d\mu = a\int_\Omega f(\omega)d\mu + b\int_\Omega g(\omega)d\mu$.

Since f and g are Bochner integrable functions, there exists sequences $\{f_n\}$ and $\{g_n\}$, $n \geq 1$ of simple functions such that $f_n(\omega) \to f(\omega)$ for $\mu - a.a\ \omega \in \Omega$ and $g_n(\omega) \to g(\omega)$ for $\mu - a.a\ \omega \in \Omega$. Thus $af_n(\omega) \to af(\omega)$ for $\mu - a.a\ \omega \in \Omega$ and $bg_n(\omega) \to bg(\omega)$ for $\mu - a.a\ \omega \in \Omega$ for any real a and b. Then $af_n(\omega) + bg_n(\omega) \to af(\omega) + bg(\omega)$ for $\mu - a.a\ \omega \in \Omega$. By Monotone Convergence theorem we have

$$\begin{aligned}\int_\Omega(af(\omega) + bg(\omega))d\mu &= \lim_{n\to\infty}[\int_\Omega(af_n(\omega) + bg_n(\omega))d\mu] \\ &= a\lim_{n\to\infty}\int_\Omega(af_n(\omega)d\mu + b\lim_{n\to\infty}\int_\Omega(g_n(\omega)d\mu \\ &= a\int_\Omega f(\omega)d\mu + b\int_\Omega g(\omega)d\mu.\end{aligned}$$

□

Proposition 2.12. *If $f : \Omega \to X$ is measurable function. $f_n : \Omega \to X$, $n \geq 1$ are Bochner integrable, $f_n(\omega) \to f(\omega)$ for $\mu - a.a\ \omega \in \Omega$ and there exists $h \in L^1(\Omega)$, such that $\|f_n(\omega)\|_X = h(\omega)$ for $\mu - a.a\ \omega \in \Omega$. And for all $n \geq 1$, then f is Bochner integrable and we have $\int_A f(\omega)\,d\mu = \lim\limits_{n\to\infty}\int_A f(\omega)d\mu$ for all $A \in \Sigma$.*

Proof. Since $f_n(\omega) \to f(\omega)$ for all $n \geq 1$ and $\|f_n(\omega)\|_X = h(\omega)$ for $\mu - a.a\ \omega \in \Omega$, we have $\|f(\omega)\|_X$ for $\mu - a.a\ \omega \in \Omega$ and the function $\omega \to \|f(\omega) - f_n(\omega)\|_X$ is Σ-measurable for every, $n \geq 1$. Since $\|f(\omega) - f_n(\omega)\|_X = 2h(\omega)$ for $\mu - a.a\ \omega \in \Omega$, we have that $\|f(.) - f_n(.)\|_X \in L^1(\Omega)$ for all $n \geq 1$. Thus by the lebesgue dominated convergence theorem for R-valued functions, we have

$$\int_A \|f(\omega) - f_n(\omega)\|_X d\mu \to 0. \tag{2}$$

By virtue of Definition 2.1 (b), for each $n \geq 1$, we can find an X-valued step function s_n, such that

$$\int_A \|f_n(\omega) - s_n(\omega)\|_X d\mu < \frac{1}{n}$$

We have

$$\begin{aligned}\int_A \|f(\omega) - s_n(\omega)\|_X d\mu &= \int_A \|f(\omega) - f_n(\omega)\|_X d\mu + \int_A \|f_n(\omega) - s_n(\omega)\|_X d\mu \\ &= \int_\Omega \|f(\omega) - s_n(\omega)\|_X d\mu \to 0 \text{ as } n \to +\infty, \text{ so f is Bochner integrable.}\end{aligned}$$

Moreover, from Corollary 2.5 and (2), we have

$$\begin{aligned}\left\|\int_A f(\omega)\, d\mu - \int_A s_n(\omega) d\mu\right\| &= \int_A \|f(\omega) - s_n(\omega)\|_X d\mu \\ &= \int_\Omega \|f(\omega) - s_n(\omega)\|_X d\mu \to 0 \quad as \quad n \to +\infty.\end{aligned}$$

Therefore $\int_A f(\omega)\, d\mu = \lim_{n\to\infty} \int_A s_n(\omega) d\mu$. □

Definition 2.13. *A set function $m : \Omega \to X$ is said to be a vector measure, if for all sequences $\{A_n\}_{n\geq 1} \subseteq \Sigma$ of pairwise disjoint sets, we have $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$, Where the series converges in the norm topology of X.*

The next proposition shows that the indefinite Bochner integral $A \to \int_A f d\mu$ of a Bochner integrable functions $f : \Omega \to X$ is a vector measure which is absolutely continuous with respect to μ (that is, $m \ll \mu$).

Proposition 2.14. *If $f : \Omega \to X$ is Bochner integrable function, then the set function $m : \Omega \to X$ defined by $m(A) = \int_\Omega f(\omega)\, d\mu$, for all $A \in \Sigma$ is a vector measure and m is absolutely continuous with respect to μ (that is $\lim_{\mu(A)\to 0} m(A) = 0$).*

Proof. Let $\{A_n\}_{n\geq 1} \subseteq \Sigma$ be a sequence of pair wise disjoint sets. Since $\left\|\int_{A_n} f(\omega)\, d\mu\right\|_X = \int_{A_n} \|f(\omega)\|_X d\mu$ for all $n \geq 1$, the series $\sum_{n=1}^{\infty} \int_{A_n} f(\omega)\, d\mu$ is dominated term-by-term by the convergent series of positive terms

$$\sum_{n=1}^{\infty} \int_{A_n} \|f(\omega)\|_X d\mu = \int_\Omega \|f(\omega)\|_X d\mu < +\infty \quad \text{(By Corollary 2.4)}$$

Therefore the series $\sum_{n=1}^{\infty} \int_{A_n} f(\omega)\, d\mu$ is absolutely continuous. Moreover, for all $k \geq 1$, we have

$$\begin{aligned}\left\|\int_{\bigcup_{n=1}^{\infty} A_n} f(\omega)\, d\mu - \sum_{n=1}^{k} \int_{A_n} f(\omega)\, d\mu\right\|_X &\leq \left\|\int_{\bigcup_{n=1}^{\infty} A_n} f(\omega)\, d\mu\right\|_X \\ &\leq \int_{\bigcup_{n=k+1}^{\infty} (A_n)} \|f(\omega)\|_X d\mu \to 0 \quad as \quad k \to +\infty\end{aligned}$$

So $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$, that is m is a vector measure. Since $\|f(.)\|_X \in L^1(\Omega)$, from the absolute continuity of the Lebesgue integral, we have

$$\lim_{\mu(A)\to 0} \int_\Omega \|f(\omega)\|_X d\mu = 0$$

From Corollary 2.4, we have $\lim_{\mu(A)\to 0} \|m(A)\|_X = \lim_{\mu(A)\to 0} \left\|\int_A f(\omega)\, d\mu\right\|_X = \lim_{\mu(A)\to 0} \int_A \|f(\omega)\|_X d\mu = 0$ and so $m \ll \mu$. Thus far the theory of the Bochner integration is a straightforward extension of the theory of Lebesgue integration, with the absolute values replaced by norms. The next theorem exhibits a strong property of the Bochner integral that has no counterpart in the theory of Lebesgue integration. □

2.2. Weaker integrals for Banach space valued functions

Definition 2.15. *Let $f : \Omega \to X$ be a function for all $A \in \Sigma$.*

Suppose that $f : \Omega \to X$ is weakly measurable. We say that f is Pettis integrable if for every $A \in \Sigma$, there exists $x_A \in X$ such that $\langle x^*; x_A\rangle_X = \int_A \langle x^*; f(\omega)\rangle_X d\mu \ \ \forall x^* \in X^*$. Then we write $x_A = (P) - \int_A f(\omega)d\mu$.

Suppose that $f : \Omega \to X$ is weakly measurable. We say that f is Dunford integrable if for every $A \in \Sigma$, there exists $x_A^{**} \in X^{**}$, such that $\langle x_A^{**}; x^*\rangle_{X^*} = \int_A \langle f(\omega), x^*\rangle_X d\mu$ for all $x^* \in X^*$. Then we write $x_A^{**} = (D) - \int_A f(\omega)d\mu$.

Suppose that $f : \Omega \to X^*$ is w^*-measurable. We say that f is Gelfand integrable if for every $A \in \Sigma$, there exists ${x_A}^* \in X^{**}$, such that $\langle {x_A}^*; x\rangle_{X^*} = \int_A \langle f(\omega), x\rangle_X d\mu$ for all $x^* \in X^*$. Then we write ${x_A}^{**} = (G) - \int_A f(\omega)d\mu$.

Remark 2.16. *Clearly, we have that Bochner integrability implies Pettis integrability and Pettis integrability implies Dunford integrability. The reverse implications need not be true. Of course, if X is reflexive, then the Pettis and Dunford integrals coincide. Finally note that the Gelfand integral is actually the Pettis integral for X^*-valued functions.*

For a Pettis integrable function $f : \Omega \to X$, we consider the set-valued function $A \longmapsto (P) - \int_A f d\mu$. We want to know if this is a μ-continuous vector measure, as was the case with the Bochner integral (by Proposition 2.3).

To answer this we need some preparation.

Definition 2.17. *Let $\sum_{n=1}^{\infty} x_n$ be a series of elements of X.*

(a). We say that the series $\sum\limits_{n=1}^{\infty} x_n$ is unconditionally convergent to x, if for all permutations π of N, the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges to x.

(b). We say that the series $\sum\limits_{n=1}^{\infty} x_n$ is weakly subseries convergent to x, if for every strictly increasing sequence $\{x_k\}$ of integers, the series $\sum\limits_{n=1}^{\infty} x_{n_k}$ is weakly convergent.

Remark 2.18. *If $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent, it is unconditionally convergent equivalent to subseries convergent.*

3. Lebesgue-Bochner Spaces and Evolution Triples

3.1. Lebesgue-Bochner Spaces

Using the Bochner integral introduced in the previous section, we can intro- duce generalizations of the classical Lebesgue spaces to Banach space valued functions. As in the previous section (Ω, Σ, μ) is a finite measure space and X is a Banach space. Additional hypotheses will be introduced as needed.

Definition 3.1. *Let $p \in [1, +\infty]$. By $L^p(\Omega, X)$ we denote the space of equivalence classes of measurable functions $f : \Omega \to X$, such that $\|f(.)\|_X \in L^p(\Omega)$. Also, introduce their respective norms by $\|f\|_p = \left(\int \|f(\omega)\|_X^p\right) 1/p$ for $1 \leq p < +\infty$ And $\|f\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X$ for $p = +\infty$.*

Remark 3.2. *The equivalence relation $f \sim g$ if and only if $f(\omega) = g(\omega) \to 0$ for $\mu - a.a\ \omega \in \Omega$.*

Proposition 3.3. *$(L^p(\Omega, X), \|\|_p)$ is a Banach space, for $p \in [1, +\infty]$.*

- *If $p \in (1; +\infty)$.Σ is countably generated and X is separable, then $L^p(\Omega, X)$ is separable.*
- *If $p \in (1; +\infty)$ and X is reflexive, then $L^p(\Omega, X)$ is reflexive.*
- *If X is a Hilbert space, then $L^2(\Omega, X)$ is a Hilbert space too with inner product $(f, g)_2 = \Omega^{\int (f(\omega); g(\omega)) X d\mu}$.*

Proof. Let $\{f_n\}$ be a sequence in $L^p(\Omega, X)$ such that $\sum_{n=1}^{\infty} \|f_n\|_p \leq +\infty$, for $1 \leq p < \infty$. Put $M = \sum_{n=1}^{\infty} \|f_n\|_p = +\infty$. Let $g_n = \sum_{k=1}^{n} |f_k|$. Then $\|g_n\| = \sum_{k=1}^{n} \|f_k\| = \sum_{n=1}^{\infty} \|f_n\|_p = +\infty$, thus $g_n \in L^p(\Omega, X)$. Put $f = \sum_{k=1}^{\infty} f_k$. Then $\|g_n - f\|_p = \left\| \sum_{n=k+1}^{\infty} f_n \right\| = \sum_{n=k+1}^{\infty} \|f_n\|_p \leq +\infty$. Hence for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|g_n - f\|_p \leq \epsilon/2$ for all $n \leq N$. Therefore $\lim_{n\to\infty} \|g_n\|_p = \|f\|_p$. Then $\|g_n - g_m\|_p = \|g_n - f + f - g_m\|_p = \|g_n - f\|_p + \|f - g_m\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $L^p(\Omega, X)$ is Banach space for $1 \leq p < \infty$. Let $p = \infty$. Let $\{f_n\}$, $n \geq 1$, be a Cauchy sequence in $L^\infty(\Omega, X)$. It follows that $|f_n - f_m| = \|f_n - f_m\|_{L^\infty(\Omega)}$ Almost everywhere and hence $f_n(\omega) \to f(\omega)$ almost everwhere, where f is measurable and essentially bounded.

Choose $\epsilon > 0$ and $N(\epsilon)$ such that $\|f_n - f_m\|_{L^\infty(\Omega)} < \Omega$ for all $n, m \geq N(\epsilon)$. Since

$$|f(\omega) - f_n(\omega)| = \lim_{n\to\infty} \|f_n - f_m\|_{L^\infty(\Omega)} < \epsilon,$$

we have $\|f - f_n\|_{L^\infty(\Omega)} < \epsilon$ for all $n \geq N(\epsilon)$. So that $\|f_n - f\|_{L^\infty(\Omega;X)} \to 0$. Therefore $\lim_{n\to\infty} \|f_n\|_\infty = \|f\|_\infty$ and $L^\infty(\Omega; X)$ is Banach space. That is $L^p(\Omega, X)$ is Banach space for $p \in [1; +\infty]$. □

Remark 3.4.

(a). The σ-field Σ is countably generated if there exists a countable subfamily T, such that $\Sigma = \sigma T$. If Ω is an open or closed subset of R^n, then Borel σ-field $\mathcal{B}(X)$ is countably generated.

(b). Simple function is dense in $L^p(\Omega, X)$ and if $Z \subseteq R^n$ is a bounded open set then $Z \subseteq C^\infty(\overline{Z}; X)$ is dense in $L^p(\Omega, X)$ for $p \in [1; +\infty)$.

3.2. Bounded Variation and Radon-Nikodym Property (RNP)

Definition 3.5. *Let $m : \Sigma \to X$ be a vector measure. We say that m is of bounded variation; if $|m|(\Omega) < +\infty$, where* $|m|(\Omega) = \sup$

$limits_{T_A} \Sigma_{C \in T_A} \|m(C)\|_X$ for all $A \in \Sigma$ with T_A running through the set of all finite Σ-partitions of A.

The quantity $|m| : \Sigma \to \mathbb{R}_+$ is called the variation of m and a measure. A Banach space X is said to have the Radon-Nikodym Property (RNP); if for every probability space (Ω, Σ, μ) and every vector measure $m : \Sigma \to X \to X$ of bounded such that $m \ll \mu$ (i.e. if $\mu(A) = 0$ then $m(A) = 0$); there exists $f \in L^1(\Omega, X)$ such that $m(A) = \int_A f(\omega) d\mu$ for all $A \in \Sigma$.

Remark 3.6. *The Radon-Nikodym Property is not a property that every Banach space has. To see this suppose that $X_1 = C_0$ On $([0; 1], \mathcal{B}([0; 1]), \lambda^1)$. $\mathcal{B}([0; 1])$ is the Borel δ-field of [0,1] and λ^1 is the Lebesgue measure. Consider a vector measure $m : \mathcal{B}([0; 1]) \to C_0$ defined by $m(A) = \left\{ \int_A \cos ntdt \right\}_{n\geq 1}$ for all $A \in \mathcal{B}([0; 1])$. By Riemann-Lebesgue Lemma $m(A) \in C_0$ for all $A \in \mathcal{B}([0; 1])$. Also $m \ll \lambda^1$. However, m cannot have Radon-Nikodym derivative in $\epsilon L^1([0; 1]; C_0)$. Since $\{\cos nt\}_{n\geq 1} \notin C_0$ For all $t \in [0; 1]$. Therefore C_0 Lacks the Radon-Nikodym Property.*

Proposition 3.7. *If X is Reflexive or it is a separable dual space, then X has RNP.*

Theorem 3.8 (Riesz-Representation Theorem for the lebesgue-Bochner spaces)**.** *If $p \in [1; +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $(L^p(\Omega, X))^* = L^q(\Omega, X^*)$ if and only if X^* has Radon-Nikodym Property and the duality pairing is given by $\langle f; g \rangle_{L^p(\Omega, X)} = \int \langle f(\omega); g(\omega) \rangle_X d\mu$ for all $f \in L^p(\Omega, X)$ and $g \in L^q(\Omega, X^*)$.*

Note that: What can we say if X^* does not have Radon-Nikodym Property (for example when $C([0; 1])$)?

Definition 3.9. *By $L^{\infty}(\Omega; X^{*}_{w^{*}})$ we denote the space of all w^{*}-measurable functions $g : \Omega \to X^{*}$, such that there exists $c > 0$ with*

$$|\langle g(\omega); x\rangle_X| =\leq c\|x\|_X \quad \text{for } \mu - a.a\ \omega \in \Omega \tag{3}$$

(The exceptional μ-null set may depend on x). Two functions $g; h$ are equivalent in $L^{\infty}(\Omega; X^{*}_{w^{*}})$ and denoted by $g \approx h$ if $\langle g(\omega); x\rangle_X = \langle h(\omega); x\rangle_X$ for $\mu - a.a\ \omega \in \Omega$ and all $x \in X$.

3.3. Absolutely continuous

Definition 3.10. *A function $f : T = [0; b] \to X$ is said to be absolutely continuous, if for every $\epsilon > 0$, we can find $\delta(\epsilon) > 0$ $\sum_{n=1}^{\infty}(b_n; a_n) < \delta$, We have $\sum_{n=1}^{\infty}\|f(b_n) - f(a_n)\|_X < \epsilon$. Also for a function $f : T = [0; b] \to X$ and a partition $P : 0 = x_0 < x_1 < \cdots < x_n = b$ of T, We define $V(f; p) = \sum_{k=1}^{m}\|f(x_k) - f(x_{k-1})\|_X$. The variation of f on T is defined by $V(f)(b) = \sup\{V(f; p) :\ p \text{ is a partition of } T\}$. When $V(f)(b)$ is finite, we say that f is of bounded variation.*

Remark 3.11. *Clearly the function $t \longmapsto V(f)(t)$ is an increasing function and if $f : T = [0; b] \to X$ is absolutely continuous, then it is of bounded variation. The converse is not true. It is well known that a R-valued, continuous function is almost everywhere differentiable on T and it is the indefinite integral of its derivative. The result is no longer true for X-valued in general.*

Example 3.12. *Let $X = L^1[0, 1]$ and consider the function $f : [0, 1] \to X$, defined be $f(t) = \chi_{[0,t]}\ \ \forall\ \ t \in [0, 1]$. It is easy to see that f is absolutely continuous. However, f is nowhere differentiable on [0, 1]. Indeed, if f is differentiable at $t = t_0 \in [0, 1]$, then for every $g \in L^{\infty}[0, 1] = (L^1[0, 1])^{*}$, the function $t \longmapsto v(t) = \langle g, f(t)\rangle_{L^1[0,1]} = \int_0^1 g(s) f(t)(s)\, ds = \int_0^t g(s)\, ds$ is differentiable at $t = t_0$. Let $g(s) = \begin{cases} 1 \ \ 1 & \text{if } s = t_0 \\ 0 \ \ -1 & \text{if } s > t_0 \end{cases}$, we have $v(t) = \begin{cases} t \qquad 1 & \text{if } s = t_0 \\ 2t_0 - t \ \ -1 & \text{if } s > t_0 \end{cases}$ and v clearly is not differentiable at $t = t_0$. Not that in this example $X = L^1[0, 1]$ does not have the RNP.*

Remark 3.13. *The result is more generally true if we assume that X has the RNP. This follows from the fact the RNP is passed to closed linear subspaces of X and if X is a separable Banach space with the RNP, then it has the separable dual (see Diestel & Uhl (1977)). So a careful reading of the previous proof reveals that it remains valid if instead we assume only that X has the RNP. The next result is an extension of the so-called "Lagrange lemma" and \DuBois-Reymond lemma" (see Denkowski, Migorski & Papageorgiou (2003b)) to Banach space valued functional.*

Proposition 3.14. *If $f, g \in L^1(T; X)$ (with $T = [0; b]$) then the following conditions are equivalent:*

(a). $f(t) = u + \int_0^t g(s)ds$; $u \in X$, for a.a $\forall\ t \in T$,

(b). $\int_0^b f(t) v'(t)dt = -\int_0^b g(t) v(t)dt$ for all $v \in C^{\infty}((0; b))$.

(c). for every $x^{*} \in X^{*}$, $\frac{d}{dt}\langle x^{*}; f()\rangle_X = \langle x^{*}; g()\rangle_X$ in the distributional sense on $(0; b)$.

Proof. $(a) \Rightarrow (b)$: Suppose $f(t) = u + \int_0^t g(s)ds$; $u \in X$, for a.a $\forall\ t \in T$. Then $f'(t) = g(s) + u \in X$, since $L^1(T; X)$ is separable, X is separable. Let $\{x_n\}_{n\geq 1}$ be dense in X.

$$\begin{aligned}\int_0^t \langle x_n, f(t)\rangle_X v'(t)\, dt &= \int_0^t \left\langle x_n, u + \int_0^t g(s)\, ds\right\rangle_X v'(t)\, dt \\ &= \int_0^t \langle x_n, u + g(s)\rangle_X v(t)\, dt\end{aligned}$$

$(a) \Rightarrow (c)$: Suppose $f(t) = u + \int_0^t g(s)ds$; $u \in X$, for a.a $\forall\, t \in T$, then for every $x^* \in X^*$;

$$\begin{aligned}
\frac{d}{dt} f(t) &= g(s) \\
\Rightarrow \frac{d}{dt}\langle x^*, f(t)\rangle &= \left\langle x^*, \frac{d}{dt} f(t)\right\rangle \\
&= \left\langle x^*, \frac{d}{dt}\left(u + \int_0^t g(s)\,ds\right)\right\rangle \\
&= \langle x^*, 0 + g(t)\rangle \\
&= \langle x^*, g(t)\rangle
\end{aligned}$$

$(c) \Rightarrow (b)$: From the definition of distributional derivative, for all $v \in C_c^\infty((0,b))$ and all $x^* \in X^*$, we have

$$\int_0^t \langle x^*, f(t)\rangle_X v'(t)dt = -\int_0^t \frac{d}{dt}\langle x^*, f(t)\rangle_X v(t)dt = \int_0^t \langle x^*, g(t)\rangle_X v(t)dt,$$

so

$$\int_0^t \langle x^*, v'(t) f(t) + v(t) g(t)\rangle_X dt = \left\langle x^*, \int_0^t (v'(t) f(t) + v(t) g(t))dt\right\rangle_X = 0,\ \forall\, x^* \in X^*$$

Thus $\int_0^b f(t)\, v'(t)dt = -\int_0^b g(t)\, v(t)dt$ for all $v \in C^\infty((0;b))$.

$(b) \Rightarrow (a)$: Suppose $\hat{f}(t) = \int_0^b g(s)\,ds \quad \forall\, t \in T$. Evidently $\hat{f}$ is absolutely continuous and $\widehat{f'}(t) = g(t)$ for almost all $t \in T$. Let h=$f - \hat{f}$. We have $\int_0^b h(t)\, v(t)dt = 0$ for all $v \in C^\infty((0;b))$. So using the above concepts and theorems we have $h(t) = u \in X \quad \forall\, t \in T$ and $f(t) = u + \int_0^t g(s)ds$; $u \in X$, for a.a $\forall\, t \in T$. □

Corollary 3.15. *If $f, g \in L^1(T;X)$ (with $T = [0;b]$) and one of the equivalent statements (a), (b) or (c) in Proposition above holds, then f is almost everywhere equal to an absolutely continuous function $f_1 : T \to X$.*

Definition 3.16. *Let $f, g \in L^1(T;X)$ (with $T = [0;b]$). We say that g is the distributional (weak) derivative of f, if $\int_0^b f(t)\, v'(t)dt = -\int_0^b g(t)\, v(t)dt$ for all $v \in C^\infty((0;b))$.*

We denote this derivative of f by Df. Let $p \in [1;+\infty]$ and $T = [0;b]$. We define: $W^{1,p}((0;b);X) = \{f \in L^p(T,X) : Df \in L^p(T,X)\}$.

Let $p \in [1;+\infty]$ and $T = [0;b]$ We define $AC^{1,p}(T,X) = \{f : T \to X : f$ is absolutely continuous, differentiable almost every where with derivative $f' \in L^p(T,X)\}$.

4. Evolution Triple

Now we are about to introduce a notion that plays a central role in the study of evolution equations. The modern strategy in studying parabolic equations is to make use of many different function spaces. The concept of evolution triple, which we define next, provides an appropriate analytical framework to realize this strategy.

4.1. Definition of Evolution triples

Definition 4.1. *A triple of spaces $(X; H; X^*)$ is said to be an evolution triple, if the following are true:*

(a). X is a separable, reflexive Banach space;

(b). H is a separable Hilbert space;

(c). The embedding $X \subseteq H$ is continuous and dense.

Remark 4.2. *By virtue of Lemma 2.2.27(b), the embedding $H^* \subseteq X^*$ is continuous and dense. Since by the Riesz-Fréchet representation theorem (see, e.g., Denkowski, Migórski & Papageorgiou (2003a)) assume that $H = H^*$ then we have that all embeddings $X \subseteq H \subseteq X^*$ are continuous and dense. For all $h \in H$ and all $x \in X$, we have $\langle h, x\rangle_X = (h; x)_H$ i.e. $\langle \cdot, \cdot\rangle_{X|H\times X} = (\cdot; \cdot)_H$. Also for $x^* \in X^*$, we have $\langle x^*, x\rangle_X = \lim_{\substack{\|\cdot\|_{X^*} \\ h\to x^*}} (h; x)_H$ (since H is dense in X^*). Therefore, if X is a Hilbert space too, we do not represent the elements of X^* using the inner product of X (the Riesz-Frechet Theorem), but using the inner product of H.*

Example 4.3. *If $Z \subseteq R^N$ is a bounded set with smooth boundary and $p \in [2; +\infty)$, then as we shall see in Section 2.4, the spaces $X = W^{1,p}(T)$, $H = L^2(Z)$ and $X^* = (W^{1,p}(Z))$. Form an evolution triple. For the evolution triple $(X; H; X^*)$, we can consider the reflexive Banach space $W_{pp'}(T) = \left\{u \in vL^p(T; X) : u' \in L^{p'}(T; X^*)\right\}$ With $\frac{1}{p} + \frac{1}{p'} = 1$, introduced earlier. In the next proposition we establish a regularity property for the elements of $W_{pp'}(T)$ and also derive an "integration by parts formula," which is crucial in the treatment of evolution equations.*

5. Concluding Remarks

To have a good theory of integration, we need a reasonable notion of measurability of functions. In this direction the basic result is the Pettis measurability theorem (see Theorem 1.1.3), which was proved by Pettis (1938a). The main integral for vector-valued functions, which has a rich enough structure to have significant applications, is the Bochner integral. The Bochner integral can be traced in the works of Bochner (1933) and Dunford (1935) and for this reason is also known as\Dunford's first integral." Most of the properties of the Bochner integral follow from the corresponding properties of the classical Lebesgue integral, by virtue of Corollary 1.1.4. So some analysts say that the Bochner integral is the Lebesgue integral with the absolute value replaced by norms. The Pettis integral has much fewer applications, which require knowledge and use of sophisticated measure theoretic results. The theory of Pettis integration started with the work of Pettis and attracted renewed attention after the paper of Edgar (1977).

A detailed study of the Pettis integral with applications can be found in the monograph of Talagrand (1984a). On the subject of vector-valued functions and their integration, the reader can consult the books of Diestel & Uhl (1977), Dunford & Schwartz (1958) and Hille & Phillips (1957). The proof of the Orlicz-Pettis theorem can be found in Diestel & Uhl (1977, p. 22).

A reference to Lebesgue-Bochner spaces can be found in every book dealing with infinite dimensional dynamical systems. They are a natural generalization of the classical Lebesgue spaces using the notion of Bochner integral. Vector measures were already considered by Pettis (1938b). However, the real expansion on the subject occurred in the late 60s and during the 70s, when there was a systematic study of the geometry of Banach spaces. That is when RNP spaces were introduced and studied in detail. That a re-flexive Banach space has the RNP, which was established by Phillips (1940), while the fact that a separable dual Banach space is an RNP space is due to Dunford & Pettis (1940). The proof of Proposition 2.2.3 can be found in Diestel & Uhl (1977). Theorem 2.2.4 (the Riesz Representation theorem for the Lebesgue Bochner spa, $p^2[1; +\infty)$) is essentially due to Bochner & Taylor (1938). Its proof can be found in Diestel & Uhl (1977). Its extension (for $p = 1$) mentioned in Theorem 2.2.12 (called Dinculeanu-Foias theorem) is due to Dinculeanu & Foias (1961) and its proof, based on \lifting theory," can be found in Ionescu-Tulcea & Ionescu-Tulcea (1969).

Absolute continuity of real valued functions (see Definition 2.3.1) was introduced by Vitali (1908), who established the fundamental fact that a real valued function on [0;1] is absolutely continuous if and only if it is the integral of its derivative (the fundamental theorem of Lebesgue calculus).

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