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Type II Truncated Frèchet Generated Family of Distributions

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Abstract: In this paper, we propose Type II truncated Frèchet (TIITF) generated family of distributions. Some statistical properties of the new family are derived. Type II truncated Frèchet-uniform, Type II truncated Frèchet-exponential; Type II truncated Frèchet-inverse Weibull and Type II truncated Frèchet-Rayleigh distributions are defined as sub-models. We proposed the parameters estimation of the model by using maximum likelihood (ML) method. Application of truncated Frèchet-inverse Weibull model is presented to show the importance of the suggested truncated family.

Keywords: Truncated Frèchet distribution, generated family of distributions, maximum likelihood method, Moments.

1. Introduction

Recently, generated families of distributions attract the attention of several authors. Some of the generators are the beta-G (Eugene et al. [5], Type II half logistic-G (Hassan et al. [8]), [0,1] truncated Fréchet-G (Abid and Abdulrazak [1]), odd Frèchet-G (Haq and Elgarhy [7]), among others. Abid and Abdulrazak [1] introduced truncated Fréchet (TF) distribution. The cdf and pdf of the TF distribution are

$$G(t) = e^{a} e^{-at^{-b}}, \tag{1}$$

and

$$g(t) = abe^{a} t^{-b-1} e^{-at^{-b}}, \quad t, a, b > 0, \tag{2}$$

where, a and b are scale and shape parameters respectively. We interest by putting $a = 1$, then we can rewrite the equations (1) and (2) as

$$G(t) = ee^{-t^{-b}}, \tag{3}$$

and

$$g(t) = bet^{-b-1} e^{-t^{-b}}, \quad t, a, b > 0, \tag{4}$$

This paper can be formed as follows: in Sections 2 and 3 define TIITF-G and investigate some of its general statistical properties. In Section 4, some new submodels of TIITF -G are considered. Sections 5 give the ML estimators. Real data example is presented in Section 6 and article ends with concluding remarks.

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2. Type II Truncated Fréchet-G Family

A new truncated family of distribution is introduced based on TF distribution in this section. Expansion of its pdf and cdf is obtained. Also, the quantile function is calculated. The cdf of the TIITF-G family is defined by using (3) and (4) as follows

$$F\left(x;b,\xi\right)=1-\int_{0}^{1-G(x;\xi)} b\ et^{-b-1}e^{-t^{-b}}dt=1-ee^{-(1-G(x;\xi))^{-b}}, \tag{5}$$

where ξ is the parameter vector of the $G(.)$ distribution. A random variable (rv) X has cdf (5) will be denoted by $X\sim TIITF-G$. The pdf corresponding to (5) is

$$f\left(x;b,\xi\right)=beg\left(x;\xi\right)G\left(x;\xi\right)\left(1-G\left(x;\xi\right)\right)^{-b-1}e^{-(1-G(x;\xi))^{-b}}. \tag{6}$$

The survival function; say $R\left(x;b,\xi\right)$, and hrf, say $h\left(x;b,\xi\right)$, are

$$R\left(x;b,\xi\right)=ee^{-(1-G(x;\boldsymbol{\xi}))^{-b}},$$

and

$$h\left(x;b,\xi\right)=bg\left(x;\xi\right)G\left(x;\xi\right)\left(1-G\left(x;\xi\right)\right)^{-b-1},$$

respectively. Expansion of pdf (6) is obtained by using the following exponential series

$$e^{-\alpha x}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\alpha^{j}}{j!}x^{j},\quad \alpha>0. \tag{7}$$

By using the following generalized binomial series

$$(1-Z)^{-\alpha}=\sum_{i=0}^{\infty}\binom{\alpha+i-1}{i}Z^{i},\quad \alpha>0\ \ and\ \ |Z|<1. \tag{8}$$

Employed (7) and (8) in (6), then the pdf of $TIITF-G$, where α is real, is

$$f\left(x;\alpha,\xi\right)=\sum_{i=0}^{\infty}\eta_{i}\ g\left(x;\boldsymbol{\xi}\right)G\left(x;\boldsymbol{\xi}\right)^{i+1}\ , \tag{9}$$

where, $\eta_i=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\ be\ (-1)^i\binom{b(j+1)+i}{i}$. Also, an expansion for $\left[F\left(x;\alpha,\xi\right)\right]^h$ is obtained, when h is an integer.

$$\left[F\left(x;\alpha,\xi\right)\right]^{h}=\sum_{k=0}^{\infty}S_{k}\ G(x;\boldsymbol{\xi})^{k}, \tag{10}$$

where,

$$S_k=\sum_{j=0}^{h}\sum_{i=0}^{\infty}\frac{(-1)^{j+i}j^{i}}{i!}\ e^{j}\binom{h}{j}\binom{bi+k-1}{k}.$$

The quantile function $Q_{(u)}$ of X is given by

$$Q_{(u)}=G^{-1}\left\{1-\left[ln\left(\frac{e}{1-u}\right)\right]^{\frac{-1}{b}}\right\},$$

3. Main Properties of $TIITF-G$ Family

For a rv X, the PWMs, denoted by $\tau_{r,h}$, is given by

$$\tau_{r,h} = E\left(X^r F(x)^h\right) = \int_{-\infty}^{\infty} x^r f(x) F(x)^h dx. \tag{11}$$

The PWMs of $TIITF-G$ is obtained by inserting (9) and (10) into (11), as follows

$$\tau_{r,h} = \sum_{i,k=0}^{\infty} \eta_i S_k \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+k+1} dx.$$

Then,

$$\tau_{r,h} = \sum_{i,k=0}^{\infty} \eta_i S_k \tau_{r,i+k+1}.$$

The r$^{\text{th}}$ moment of $TIITF-G$ is

$$\grave{\mu}_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{i=0}^{\infty} \eta_i \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+1} dx.$$

Then,

$$\grave{\mu}_r = \sum_{i=0}^{\infty} \eta_i \tau_{r,i+1}.$$

The moment generating function(MGF) is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \grave{\mu}_r.$$

So, the MGF of TIITF-G is

$$M_X(t) = \sum_{i,r=0}^{\infty} \frac{t^r}{r!} \eta_i \tau_{r,i+1}.$$

The pdf of the r$^{\text{th}}$ order statistic, is

$$f_{x_{(r)}}(x) = \frac{f(x)}{B(r,n-r+1)} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} F(x)^{m+r-1}, \tag{12}$$

The pdf of the r$^{\text{th}}$ order statistic of $TIITF-G$ is calculated by substituting (5) and (6) in (12), replacing h with $m+r-1$

$$f_{x_{(r)}}(x;b,\xi) = \frac{g(x;\xi)}{B(r,n-r+1)} \sum_{m=0}^{n-r} \sum_{i,k=0}^{\infty} W^* G(x;\xi)^{i+k+1}, \tag{13}$$

where, $W^* = (-1)^m \binom{n-r}{m} \eta_i S_k$, $g(.)$ and $G(.)$ are the pdf and cdf of the $TIITF-G$ distribution, respectively. Further, the m$^{\text{th}}$ moment of the r$^{\text{th}}$ order statistics for $TIITF-G$ distribution is defined by:

$$E\left(X^m{}_{(r)}\right) = \int_{-\infty}^{\infty} x^m f_{x_{(r)}}(x;\alpha,\xi) dx. \tag{14}$$

By substituting (13) in (14), then

$$E\left(X^m{}_{(r)}\right) = \frac{1}{B(r,m-r+1)} \sum_{j=0}^{n-r} \sum_{i,k=0}^{\infty} W^* \int_{-\infty}^{\infty} x^m g(x;\xi) G(x;\xi)^{i+k+1} dx.$$

Then,

$$E\left(X^m{}_{(r)}\right) = \frac{1}{B(r,m-r+1)} \sum_{j=0}^{n-r} \sum_{i,t,k,l=0}^{\infty} W^* \tau_{m,i+k+1}.$$

4. Sub-Models

In this section, we define new four sub-models of the $TIITF-G$; namely, TIITF-uniform, TIITF-exponential, TIITF-inverse Weibull and TIITF-Rayleigh.

4.1. TIITF-Uniform Distribution

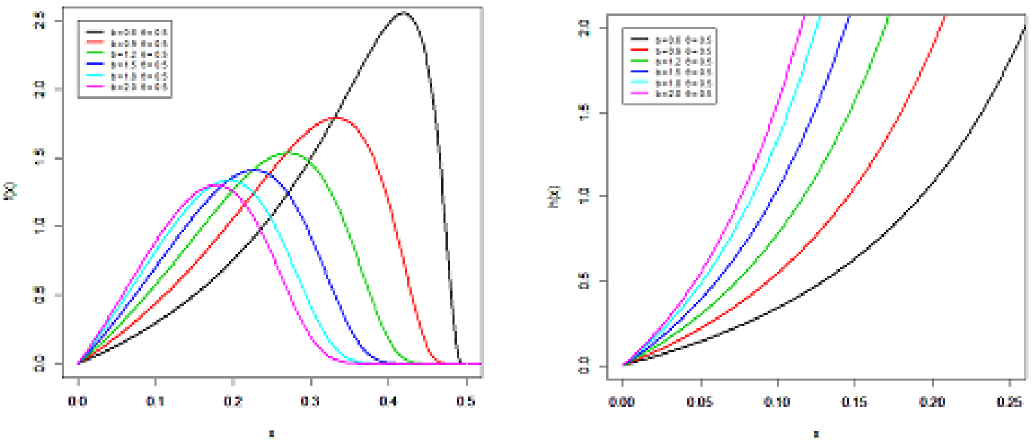
For $g(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$ and $G(x;\theta) = \frac{x}{\theta}$ the pdf of *TL-uniform* (TLU) is derived from (4) as the following

$$f(x;b,\theta) = \frac{be}{\theta^2} x \left(1 - \frac{x}{\theta}\right)^{-b-1} e^{-\left(1-\frac{x}{\theta}\right)^{-b}}, \quad 0 < x < \theta.$$

and cdf is

$$F(x;b,\theta) = 1 - ee^{-\left(1-\frac{x}{\theta}\right)^{-b}}.$$

Plots of pdf and hrf for the TIITFU are displayed in Figure 1.



(a) (b)

Figure 1: Plots of (a) pdf and (b) hrf of the TIITFU distribution

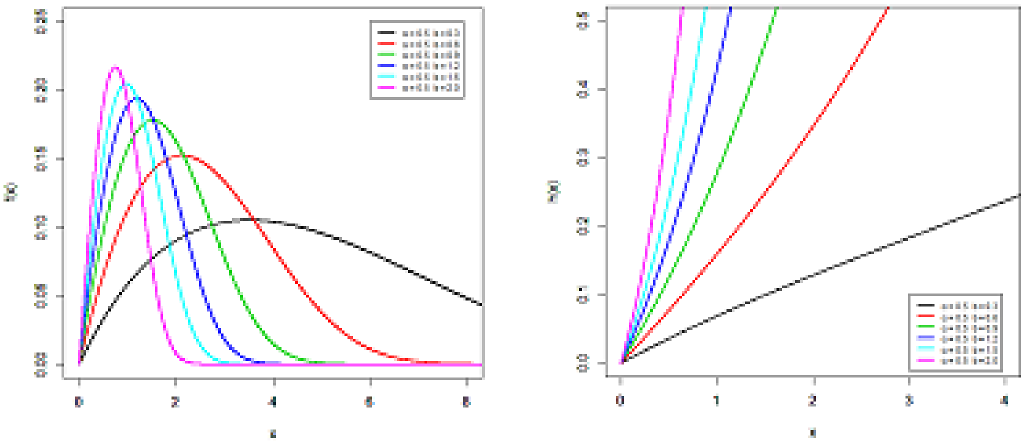
4.2. TIITF-Exponential Distribution

For $g(x;\alpha) = \alpha e^{-\alpha x}$, $x > 0$, $\alpha > 0$ and $G(x;\alpha) = 1 - e^{-\alpha x}$, we obtain the pdf and cdf of TIITF-exponential (TIITFE) as follows

$$f(x;b,\alpha) = be\alpha \left(1 - e^{-\alpha x}\right) e^{b\alpha x} e^{-e^{b\alpha x}}, \quad x > 0, \quad b, \alpha > 0.$$

and

$$F(x;b,\alpha) = \ 1 - ee^{-e^{b\alpha x}}.$$



(a) (b)

Figure 2: Plots of (a) pdf and (b) hrf of the TIITFE distribution

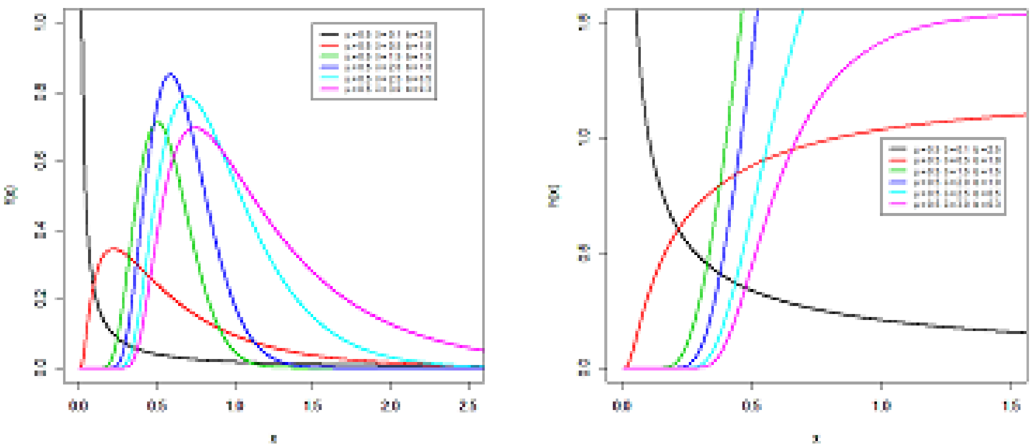
4.3. TIITF-Inverse Weibull Distribution

We consider the inverse Weibull distribution with pdf; $g(x;\mu,\delta)=\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}$, $x,\mu,\delta>0$ and cdf; $G(x;\mu,\delta)=e^{-\left(\frac{\mu}{x}\right)^{\delta}}$, hence the pdf and cdf of TIITF-inverse Weibull (TIITFIW) is

$$f(x;b,\mu,\delta)=be\delta\mu^{\delta}x^{-\delta-1}e^{-2\left(\frac{\mu}{x}\right)^{\delta}}\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-b-1}e^{-\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-b}},\quad x,\mu,\delta>0.$$

and

$$F(x;b,\mu,\delta)=1-ee^{-\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-b}},$$



(a) (b)

Figure 3: Plots of (a) pdf and (b) hrf of the TIITFIW distribution

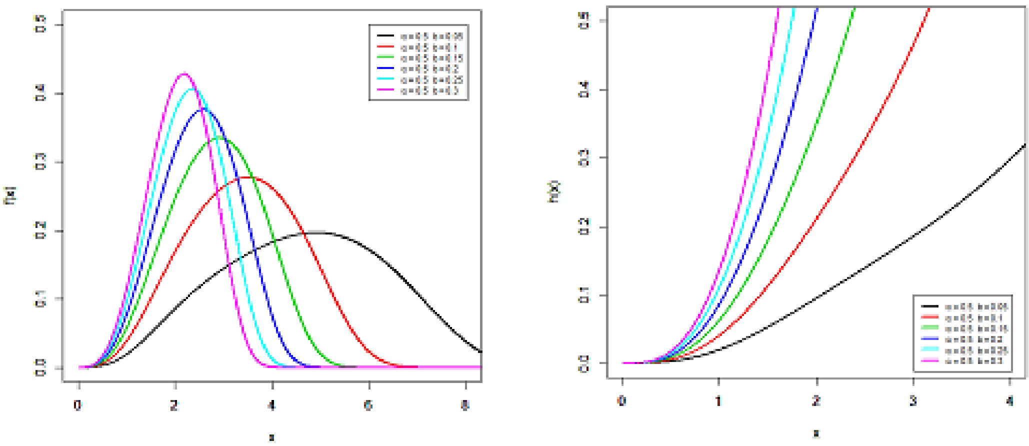
4.4. TIITF-Rayleigh Distribution

For $g(x;\alpha)=2\alpha xe^{-\alpha x^2}$, $x>0$, $\alpha>0$ and $G(x;\alpha)=1-e^{-\alpha x^2}$, we obtain the pdf and cdf of TIITF-Rayleigh (TIITFR) as follows

$$f(x;b,\alpha)=2be\alpha x\left(1-e^{-\alpha x^2}\right)e^{b\alpha x^2}e^{-e^{b\alpha x^2}},\quad x>0,b,\alpha>0.$$

and

$$F\,(x;b,\alpha) =\ 1 - ee^{-e^{b\alpha x^2}}.$$



(a) (b)

Figure 4: (a) pdf and (b) hrf of the TIITFR distribution

5. ML Estimator

Let $X_1, X_2, \ldots, X_n$ be the observed values from the $TIITF-G$ family with set of parameter $\Phi = (b,\xi)^T$. The log-likelihood function for parameter vector $\Phi = (\alpha,\xi)^T$ is obtained as follows

$$ln(L,\Phi) = nlnb + n + \sum_{n=1}^{n} ln\ g(x_i;\xi) + \sum_{n=1}^{n} ln\ G(x_i;\xi) - (b+1)\sum_{i=1}^{n} ln[1-G(x_i;\xi)] - \sum_{i=1}^{n}[1-G(x_i;\xi)]^{-b}.$$

The partial derivatives of the log-likelihood function with respect to b and ξ components of the score vector $U_\Phi = (U_b, U_{\xi_k})^T$ can be obtained as follows

$$U_b = \frac{n}{b} - \sum_{i=1}^{n} ln[1-G(x_i;\xi)] + \sum_{i=1}^{n}[1-G(x_i;\xi)]^{-b} ln[1+G(x_i;\xi)]$$

and

$$U_{\xi_k} = \sum_{i=1}^{n}\frac{g'_k(x_i;\xi)}{g(x_i;\xi)} + \sum_{i=1}^{n}\frac{G'_k(x_i;\xi)}{G(x_i;\xi)} + (b+1)\sum_{i=1}^{n}\frac{G'_k(x_i;\xi)}{1-G(x_i;\xi)} + b\sum_{i=1}^{n} G'_k(x_i;\xi)[1-G(x_i;\xi)]^{-b-1}.$$

where, $g'_k\,(x_i;\xi) = \frac{\partial g(x_i;\xi)}{\partial \xi_k}$ and $G'_k\,(x_i,\xi) = \frac{\partial G(x_i;\xi)}{\partial \xi_k}$.

6. Data Analysis

In current section, we study an application to evaluate the flexibility of the TIITFIW model. In order to compare the TIITFIW model with other fitted distributions. We compare the fits of the TIITFIW distribution with the beta transmuted Weibull (BTW) (Afify et al., [2]), McDonald Weibull (McW) (Cordeiro et al., [4]) and new modified Weibull (NMW) (Almalki and Yuan [3]) distributions.

The data set is taken from Gross and Clark [6]. The ML estimates and their standard errors (SEs) of the model parameters are mentioned in Table 1. In the Table 2, the measures Anderson Darling statistic (A^*), Cramér-von Mises statistic (W^*), Akaike information criterion (AIC) and Hannan-Quinn information criterion (HQIC) are presented.

Model	MLE and SE				
TIITFIW $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta})$	1.081	8.906	0.138	-	-
	(0.074)	(3.197)	(0.053)		
NMW $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\theta})$	0.12	2.78	8.227×10^{-5}	0.0003	2.79
	(0.07)	(20.37)	(1.512×10^{-3})	(0.025)	(0.43)
McW $(\boldsymbol{\alpha}, \boldsymbol{\beta}, a, b, c)$	2.77	0.38	79.11	17.898	3.006
	(6.38)	(0.188)	(119.131)	(39.511)	(13.968)
BTW $(\boldsymbol{\alpha}, \boldsymbol{\beta}, a, b, \boldsymbol{\lambda})$	5.62	0.53	53.34	3.57	-0.77
	(9.35)	(0.15)	(111.45)	(4.27)	(3.89)

Table 1: MLEs and their SEs (in parentheses) for the data set.

Model	AIC	HQIC	A*	W*
TIITFIW	43.172	43.76	0.369	0.039
NMW	51.17	52.15	1.068	0.176
McW	43.91	44.88	0.469	0.080
BTW	43.05	44.02	0.398	0.069

Table 2: Measures of goodness-of-fit statistics for the data set

The Table 2 show that the TIITFIW model has the lowest values for AIC, $HQIC$, A^* and W^*. So, it could be chosen as the best model. Figure 5 shows the estimated pdf and cdf for the TIITFIW model.

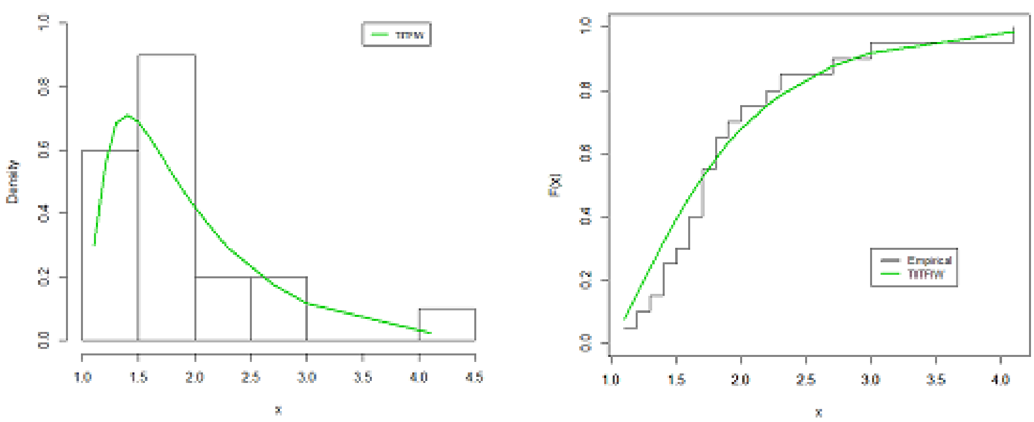


Figure 5: The estimated pdf and cdf plots of the TIITFIW model

7. Summary and Conclusion

In this article, we studied a new generated family of distributions called the TIITF-G family. Four new sub-models are proposed. We discussed several properties of the TIITF-G family. We estimate the parameters using ML method. Application prove the importance and potentiality of the TIITF-G family.

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