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An Overview of Fixed Point Theorems in Dislocated Quasi Metric Spaces

N. Anu[1,*]

1 Department of Mathematics, Idhaya College For Women (Affiliated to Alagappa University), Karaikudi, Tamil Nadu, India.

Abstract: This paper proposes a new results in fixed point theorems for some contraction mapping conditions in dislocated quasi metric spaces. This results generalizes several well-known results in the literature.

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Keywords: Fixed point, dislocated quasi-metric spaces, dq-Cauchy, dq-converges, dq-limit, contraction mapping.

1. Introduction

Fixed point theory is the development of nonlinear analysis. In 1922 Banach proved a fixed point theorems for contractive mappings in complete metric spaces. It is a well-known Banach fixed point theorem. It has many applications in various branches of mathematics such as Differential equation, integral equation etc. Also many authors studies many contraction conditions and proved some new fixed point theorems. In 1968 Kannan proved a fixed point theorem for new types of contraction mappings called Kannan mappings in a complete metric spaces. In 1974 Lj.B.Ciric generalizes the Banach contraction principle in metric spaces. Some important generalizations of metric spaces are Dislocated metric spaces, Quasi metric spaces, Dislocated Quasi metric spaces. The notation of dislocated metric space where, the self distance for any point need not be equal to zero and generalized this Banach Contraction Principle in complete dislocated metric space. Dislocated metric spaces plays an essential role in topology. In 2005 F.M.Zeyada, G.H.Hassan and M.A.Ahmad [13] is a generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi metric spaces. The notation of dislocated quasi metric space was introduced by F.M.Zeyada, G.H.Hassan and M.A.Ahmad. In 2010 A.Isufati [5] proved some fixed point theorems for continuous contractive mappings with rational type expression of a dislocated quasi metric space. In 2014 muraliraj [6] generalizes some fixed point theorems in dq metric spaces, In 2014 muhammad sarwar, mujeeb ur rahmanand goharali [7,8] gave some fixed point result in dq metric spaces, In 2014 A.K.dubey, Reenashukla, and R.P.dubey [4] proved some fixed point results in dislocated quasi metric spaces. The purpose of this paper to establish a fixed point theorem for new contraction mappings in dislocated quasi metric spaces.

1.1. Preliminaries

In this section contains some basic definitions, lemmas and theorems in dislocated quasi metric spaces.

* *E-mail: anulaysi@gmail.com*

Definition 1.1. *A point x in a set X is called a fixed point of a mapping $T : X \to X$, if $Tx = x$.*

Definition 1.2. *Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function, called a distance function satisfies the following conditions:*

(1). $d(x, x) = 0$,

(2). $d(x, y) = d(y, x) = 0$, *then* $x = y$,

(3). $d(x, y) = d(y, x)$,

(4). $d(x, y) \leq d(x, z) + d(z, y)$,

(5). $d(x, y) \leq \max\{d(x, z), d(z, y)\}$, *for all* $x, y, z \in X$.

If d satisfies conditions (1) to (4) then it is called a metric on X. If it satisfies conditions (1), (2) and (4), then it is called a quasi-metric (or simply q - metric) on X. If it satisfies conditions (2), (3) and (4), then it is called a dislocated metric (or simply d - metric) on X. If it satisfies conditions (2) and (4), then it is called a dislocated quasi-metric (or simply dq - metric) on X. If a metric d satisfies the strong triangle inequality (5), then it is called an ultrametric on X. Clearly every metric space is a dislocated metric space but the converse is not necessarily true. Also every metric space is dislocated quasi - metric space but the converse is not true and every dislocated metric space is dislocated quasi - metric space but the converse is not true.

Definition 1.3. *A sequence $\{x_n\}$ in dq - metric space (X, d) is said to be a cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$, $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$, i.e $\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$.*

Definition 1.4. *A sequence $\{x_n\}$ in dq - metric space (X, d) is said to be a bi'cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$, $\max\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$.*

Note that every 'bi'cauchy sequence is a cauchy sequence.

Definition 1.5. *A sequence $\{x_n\}$ in dq-metric space (X, d) is said to be a dq - convergent to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$. In this case x is called a dq-limit of $\{x_n\}$, we write $\lim_{n\to\infty} x_n = x$.*

Definition 1.6. *A dq - metric space (X, d) is said to be complete if every cauchy sequence in X convergent to a point of X.*

Definition 1.7. *Let (X, d_1) and (Y, d_2) be dq-metric spaces and let $T : X \to Y$ be a function. Then T is continuous if for each sequence (x_n) which is d_1q - convergent to x_0 in X, the sequence $(T(x_n))$ is d_2q - convergent to $T(x_0)$ in Y.*

Definition 1.8. *Let (X, d) be a dq-metric space. A mapping $T : X \to X$ is called contraction, if there exists $0 \leq \lambda <$ 1, $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$, where λ is called a contracting constant.*

Definition 1.9. *Let (X, d) be a dq-metric space, and let $T : X \to X$ be a self - mapping. Then T is called Kannan mapping. If $d(Tx, Ty) \leq \alpha\{d(x, Tx) + d(y, Ty)\}$ for all $x, y \in X$ and $0 \leq \alpha < 1/2$.*

Definition 1.10. *Let (X, d) be a dq - metric space, a self mapping $T : X \to X$ is called generalized contraction if and only if for all $x, y \in X$, there exist non-negative numbers α, β, γ and δ such that $\sup\{\alpha + \beta + \gamma + 2\delta\} < 1$ and*

$$d(Tx, Ty) \leq \alpha.d(x, y) + \beta.d(x, Tx) + \gamma.d(y, Ty) + \delta.[d(x, Ty) + d(y, Tx)]$$

Proposition 1.11. *Every convergent sequence in a dq - metric space is 'bi'cauchy.*

Lemma 1.12. *Every subsequence of dq - convergent sequence to a point x_0 is dq-convergent to x_0.*

Lemma 1.13. *Let (X,d) be a dq - metric space. If $T : X \to X$ is a contraction function, then $(T^n(x_0))$ is a cauchy sequence for each $x_0 \in X$.*

Lemma 1.14. *dq - limits in a dq - metric space are unique.*

Theorem 1.15. *Let (X,d) be a complete dq - metric space and let $T : X \to X$ be a continuous contraction function. Then T has a unique fixed point.*

Theorem 1.16. *Let (X,d) be a complete dq - metric space and let $T : X \to X$ be a continuous mapping satisfying the following condition*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta \left(\frac{d(x,Ty)\, d(y,Ty)}{d(x,y) + d(y,Ty)} \right)$$

for all $x, y \in X$, $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Then T has a unique fixed point.

2. Main Results

In this section contains two new contraction conditions of fixed point theorems in dislocated quasi metric spaces.

Theorem 2.1. *Let (X,d) be a complete dislocated quasi metric space and let $T : X \to X$ be a continuous mapping satisfying the condition*

$$\begin{aligned} d(Tx,Ty) \leq \alpha d(x,y) &+ \beta \left(\frac{d(x,Tx)\, d(y,Ty)}{d(x,y) + d(x,Tx)} \right) + \gamma \left(\frac{d(x,Tx)\, d(x,Ty)}{d(x,y) + d(y,Ty)} \right) \\ &+ \delta \left(\frac{d(x,Tx)\, d(x,Ty)}{d(x,y) + d(x,Tx)} \right) + \mu \left[d(x,Tx) + d(y,Ty) \right] \end{aligned}$$

for all $x, y \in X$, $\alpha, \beta, \gamma, \delta, \mu > 0$ and $\alpha + \frac{\beta}{2} + \gamma + 2\delta + 2\mu < 1$. Then T has a unique fixed point.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows. Let $x_0 \in X$, $T(x_0) = x_1, T(x_1) = x_2, \ldots, T(x_n) = x_{n+1}$, for all $n \in N$. Replace $x = x_{n-1}$ and $y = x_n$ in the given condition. Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta \left(\frac{d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_{n-1})} \right) \\ &+ \gamma \left(\frac{d(x_{n-1}, Tx_{n-1})\, d(x_{n-1}, Tx_n)}{d(x_{n-1}, x_n) + d(x_n, Tx_n)} \right) + \delta \left(\frac{d(x_{n-1}, Tx_{n-1})\, d(x_{n-1}, Tx_n)}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_{n-1})} \right) \\ &+ \mu \left[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) \right] \\ &\leq \alpha d(x_{n-1}, x_n) + \beta \left(\frac{d(x_{n-1}, x_n)\, d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_{n-1}, x_n)} \right) \\ &+ \gamma \left(\frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})} \right) + \delta \left(\frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_{n-1}, x_n)} \right) \\ &+ \mu \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right] \\ &\leq \alpha d(x_{n-1}, x_n) + \beta \left(\frac{d(x_{n-1}, x_n)\, d(x_n, x_{n+1})}{2d(x_{n-1}, x_n)} \right) \\ &+ \gamma \left(\frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_{n+1})} \right) + \delta \left(\frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{2d(x_{n-1}, x_n)} \right) \\ &+ \mu \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right] \\ &\leq \alpha d(x_{n-1}, x_n) + \frac{\beta}{2} d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) \end{aligned}$$

$$+\delta d\left(x_{n-1},x_{n+1}\right)+\mu d\left(x_{n-1},x_{n}\right)+\mu d\left(x_{n},x_{n+1}\right)$$
$$\leq \alpha d\left(x_{n-1},x_{n}\right)+\frac{\beta}{2}d\left(x_{n},x_{n+1}\right)+\gamma d\left(x_{n-1},x_{n}\right)+\delta d\left(x_{n-1},x_{n}\right)$$
$$+\delta d\left(x_{n},x_{n+1}\right)+\mu d\left(x_{n-1},x_{n}\right)+\mu d\left(x_{n},x_{n+1}\right)$$
$$d\left(x_{n},x_{n+1}\right)\leq\left(\alpha+\gamma+\delta+\mu\right)d\left(x_{n-1},x_{n}\right)+\left(\frac{\beta}{2}+\delta+\mu\right)d\left(x_{n},x_{n+1}\right)$$

Therefore,

$$d(x_n,x_{n+1})\leq\left(\frac{\alpha+\gamma+\delta+\mu}{1-\left(\frac{\beta}{2}+\delta+\mu\right)}\right)d(x_{n-1},x_n)$$
$$=\lambda d(x_{n-1},x_n),\quad where\quad \lambda=\left(\frac{\alpha+\gamma+\delta+\mu}{1-\left(\frac{\beta}{2}+\delta+\mu\right)}\right)<1$$

In the same way, we have $d(x_{n-1},x_n)\leq\lambda d(x_{n-2},x_{n-1})$ and $d(x_n,x_{n+1})\leq\lambda^2 d(x_{n-2},x_{n-1})$. Continue this process, in general $d(x_n,x_{n+1})\leq\lambda^n d(x_0,x_1)$. Since $0\leq\lambda<1$ as $n\to\infty$, $\lambda^n\to 0$. Hence $\{x_n\}$ is a dq - cauchy sequence in X. Thus $\{x_n\}$ dislocated quasi converges to some z in X. Since T is continuous we have $T(z)=\lim_{n\to\infty}T(x_n)=\lim_{n\to\infty}x_{n+1}=z$. Thus z is a fixed point T.

Uniqueness: Let $x\in X$ is a fixed point. Then by the given condition,

$$d\left(x,x\right)=d(Tx,Tx)$$
$$\leq\alpha d\left(x,x\right)+\beta\frac{d\left(x,x\right)d\left(x,x\right)}{\left[d\left(x,x\right)+d\left(x,x\right)\right]}+\gamma\frac{d\left(x,x\right)d\left(x,x\right)}{\left[d\left(x,x\right)+d\left(x,x\right)\right]}+\delta\frac{d\left(x,x\right)d\left(x,x\right)}{\left[d\left(x,x\right)+d\left(x,x\right)\right]}+\mu\left[d\left(x,x\right)+d\left(x,x\right)\right]$$
$$d(x,x)\leq\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+2\mu\right)d(x,x)$$

which is true only if $d(x,x)=0$, since $0\leq\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+2\mu\right)<1$ and $d(x,x)\geq 0$. Thus $d(x,x)=0$ if x is fixed point of T. Let x, y be fixed point, (i.e.) $Tx=x$, $Ty=y$. Then by the given condition we have,

$$d\left(x,y\right)=d\left(Tx,Ty\right)$$
$$\leq\alpha d(x,y)+\beta\frac{d\left(x,x\right)d(y,y)}{d\left(x,y\right)+d(x,x)}+\gamma\frac{d\left(x,x\right)d(x,y)}{d\left(x,y\right)+d(y,y)}+\delta\frac{d\left(x,x\right)d(x,y)}{d\left(x,y\right)+d(x,x)}+\mu\left[d\left(x,x\right)+d\left(y,y\right)\right]$$
$$d(x,y)\leq\alpha d(x,y)$$

Similarly $d(y,x)\leq\alpha d(y,x)$. Hence $|d(x,y)-d(y,x)|\leq\alpha|d(x,y)-d(y,x)|$, which implies $d(x,y)=d(y,x)$. Since $0\leq\alpha<1$. Again from the given condition, $d(x,y)\leq\alpha d(x,y)$, which gives, $d(x,y)=0$. Since $0\leq\alpha<1$. Further $d(x,y)=d(y,x)=0\Rightarrow x=y$. Hence fixed point of T is unique. □

Corollary 2.2. *Let (X,d) be a complete dislocated quasi metric space and let $T:X\to X$ be a continuous mapping satisfying the condition*

$$d\left(Tx,Ty\right)\leq\alpha d\left(x,y\right)+\mu\left[d\left(x,Tx\right)+d\left(y,Ty\right)\right]$$

for all $x,y\in X$, $\alpha,\mu>0$ and $\alpha+2\mu<1$. Then T has a unique fixed point.

Proof. Put $\beta=\gamma=\delta=0$ in the above Theorem 2.1., it can be proved easily. □

Theorem 2.3. *Let (X,d) be a complete dq-metric space and let $T:X\to X$ be a continuous mapping satisfying the following condition*

$$d(Tx,Ty)\leq\alpha d(x,y)+\beta\frac{d\left(x,Tx\right)d\left(y,Ty\right)}{d\left(x,y\right)}+\gamma\frac{d\left(y,Ty\right)\left[1+d\left(x,y\right)\right]}{\left[1+d\left(x,Tx\right)\right]}$$

for all $x,y\in X$, $\alpha,\beta,\gamma>0$, $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows. Let $x_0 \in X$, $T(x_0) = x_1, T(x_1) = x_2, \ldots, T(x_n) = x_{n+1}$, for all $n \in N$. Replace $x = x_{n-1}$ and $y = x_n$ in the given condition. Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n)\, d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \gamma \frac{d(x_n, x_{n+1})\,[1 + d(x_{n-1}, x_n)]}{[1 + d(x_{n-1}, x_n)]} \\ &\leq \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_n, x_{n+1}) \\ &\leq \alpha d(x_{n-1}, x_n) + (\beta + \gamma) d(x_n, x_{n+1}) \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\alpha}{1 - (\beta + \gamma)} d(x_{n-1}, x_n) \\ &\leq \lambda d(x_{n-1}, x_n), \quad where \ \ \lambda \leq \frac{\alpha}{1 - (\beta + \gamma)}, \quad 0 \leq \lambda < 1 \end{aligned}$$

Similarly we have $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$, and $d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1})$. In this way, we get $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$. Since $0 \leq \lambda < 1$, as $n \to \infty$, $\lambda^n \to 0$. Similarly we show that $d(x_n, x_{n+1}) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi-metric space X. So there is a point such that $z \in X$ such that $x_n \to z$. Since T is continuous we have $T(z) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = z$. Thus $T(z) = z$. Thus T has a fixed point.

Uniqueness: Let x be a fixed point of T. Then by the given condition

$$\begin{aligned} d(x, x) &= d(Tx, Tx) \\ &\leq \alpha d(x, x) + \beta \frac{d(x, x)\, d(x, x)}{d(x, x)} + \gamma \frac{d(x, x)\,[1 + d(x, x)]}{[1 + d(x, x)]} \\ d(x, x) &\leq (\alpha + \beta + \gamma) d(x, x) \end{aligned}$$

which is true only if $d(x, x) = 0$, since $0 \leq (\alpha + \beta + \gamma) < 1$ and $d(x, x) \geq 0$. Thus $d(x, x) = 0$, if x is a fixed point of T. Now, let $x, y \in X$ be fixed point of T. That is, $Tx = x$, $Ty = y$. Then by the given condition, we have

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\leq \alpha d(x, y) + \beta \frac{d(x, x)\, d(y, y)}{d(x, y)} + \gamma \frac{d(y, y)\,[1 + d(x, x)]}{[1 + d(x, y)]} \\ d(x, y) &\leq \alpha d(x, y) \end{aligned}$$

Similarly $d(y, x) \leq \alpha d(y, x)$. Hence $|d(x, y) - d(y, x)| \leq \alpha |d(x, y) - d(y, x)|$, which implies $d(x, y) = d(y, x)$. Since $0 \leq \alpha < 1$. Again from the given condition we have, $d(x, y) \leq \alpha d(x, y)$ which gives, $d(x, y) = 0$. Since $0 \leq \alpha < 1$. Further $d(x, y) = d(y, x) = 0 \Rightarrow x = y$. Hence fixed point of T is unique. □

Corollary 2.4. *Let (X, d) be a complete dq-metric space and let $T : X \to X$ be a continuous mapping satisfying the following condition*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)}$$

for all $x, y \in X$, $\alpha, \beta > 0$, $\alpha + \beta < 1$. Then T has a unique fixed point.

Proof. Put $\gamma = 0$ in the above Theorem 2.3., it can be proved easily. □

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