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A Note on s-unitary Similarity of a Complex Matrices

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Abstract : In this work a new s-unitary similarity transformation of a normal matrix to complex symmetric form will be discussed. The classical Speech criterion for the s-unitary similarity between two complex $n \times n$ matrices is extended to the s-unitary similarity between two matrix sets of cardinally m where discussed.

Keywords : s-Orthogonal, s-Unitary Matrices, Similarity of Matrices, u-Orthogonal Similarity of Matrices.

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1 Introduction

Throughout this paper we use the following notation:

Notation 1.1. *Let F be a field and let $\mathcal{M}_n(F)$ be the algebra of $n \times n$ matrices.*

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Notation 1.2. *The secondary transpose (conjugate secondary transpose) of A is defined by $A^s = VA^TV$ ($A^\Theta = VA^*V$), where "V"is the fixed disjoint permutation matrix with units in its secondary diagonal.*

Definition 1.3. *[5] Let $A \in \mathcal{M}_n(F)$*

(i). The matrix A is called s-normal, if $AA^\Theta = A^\Theta A$

(ii). The matrix A is called s-orthogonal, if $AA^s = A^sA = I$. That is $A^TVA = V$

(iii). The matrix A is called s-unitary, if $AA^\Theta = A^\Theta A = I$. That is $A^*VA = V$

Definition 1.4. *[5] Let $A \in \mathcal{M}_n(F)$ and $B \in \mathcal{M}_n(F)$. A is said to be s-orthogonally similar (respectively s-unitarily) to B if there exists a s-orthogonal matrix $Q \in \mathcal{M}_n(F)$ such that $A = Q^sBQ(A = Q^\Theta BQ)$*

2 s-unitary Similarity of a Complex Matrices

Theorem 2.1. *Suppose A to be normal, having distinct singular values and $A = UBV^\Theta$, with U, V s-unitary and B a real upper bidiagonal matrix. Then the s-unitary similarity transformation $U^\Theta AU$ results in a normal complex symmetric matrix.*

Proof. Assume that the factorization $B = U^\Theta AV$, satisfying the constraints above, is known for algorithms computing this factorization). The s-unitary matrices U and V can be combinations of either Householder or Given transformations or of both, but are constructed such that the matrix B is real.

We will prove now that the s-unitary similar matrix $A_U = U^\Theta AU$ is normal complex symmetric. Only the complex symmetry $A_U^s = A_U$ needs a formal proof, since normality is preserved under similarity. The following relations hold for $A_UA_U^\Theta$:

$$A_UA_U^\Theta = (U^\Theta AU)(U^\Theta A^\Theta U) = U^\Theta AA^\Theta U = (U^\Theta AV)(V^\Theta A^\Theta U) = BB^\Theta. \tag{2.1}$$

The matrix B is real bidiagonal, implying that BB^Θ is real, symmetric and tridiagonal. Hence we have $BB^\Theta = \overline{BB^\Theta}$. This gives us for Equation (2.1)

$$A_UA_U^\Theta = BB^\Theta = B\bar{B}^\Theta = \bar{A}_UA_U^s \tag{2.2}$$

Assume that we have the following eigenvalue decomposition for $A_U = Q\Delta Q^\Theta$. Equation (2.2) leads us to the following two eigenvalue decompositions for the matrix product $A_UA_U^\Theta$:

$$Q|\Delta|^2Q^\Theta = A_UA_U^\Theta = \bar{A}_UA_U^s = \bar{Q}|\Delta|^2Q^s. \tag{2.3}$$

Since all singular values of A_U are distinct (remember that for normal matrices, $|\Delta| =$), the eigenvalue decompositions in Equation (2.3) are essentially unique and we obtain $QD = \bar{Q}$, with D a unitary diagonal matrix. This implies $A_U = Q\Delta Q^\Theta = Q\Delta\bar{Q}^s = Q\Delta(QD)^s = (QD^{1/2})\Delta(QD^{1/2})^s$, indicating that $A_U^s = A_U$ and hence the matrix A_U is complex symmetric.

□

Theorem 2.2. *Let A be normal, having distinct singular values and $A = UBV^{\Theta}$, with U, V s-unitary and B a bidiagonal matrix. We have that $U^{\Theta}AU$ and $V^{\Theta}AV$ will be self-adjoint with respect to s-unitary diagonal matrices Ω_U and Ω_V respectively.*

Proof. Construct two s-unitary diagonal matrices D_U and D_V such that the new bidiagonal matrix $\hat{B}$ with $D_U^{\Theta}(U^{\Theta}AV)D_V = D_U^{\Theta}BD_V = \hat{B}$ is real. Let $A_U = U^{\Theta}AU$. Theorem 2.1 states that $D_U^{\Theta}A_UD_U$ is complex symmetric. This together with the s-unitarity of D_U implies $(D_U^{-1} = \bar{D}_U)$

$$D_U^{-1}A_UD_U = D_U^{\Theta}A_UD_U = (D_U^{\Theta}A_UD_U)^s = D_UA_U^sD_U^{-1}. \tag{2.4}$$

It remains to prove that there exists a Ω_U such that A_U is self-adjoint with respect to this matrix Ω_U. Take $\Omega_U = D_U^{-2}$, which is s-unitary diagonal. We obtain (use Equation (2.3) and (2.4)) $A_U^{\Theta} = \Omega_U^{-1}A_U^s\Omega_U = D_U^2A_U^sD_U^{-2} = A_U$, proving thereby the selfadjointness. The proof for A_V proceeds identically. □

Corollary 2.3. *Suppose A to be normal, having distinct singular values and $A = UBV^{\Theta}$, with U, V s-unitary and B an upper bidiagonal matrix. Then there exist s-unitary diagonal matrices D_U and D_V such that $D_U^{\Theta}(U^{\Theta}AU)D_U$ and $D_V^{\Theta}(V^{\Theta}AV)D_V$ are complex symmetric matrices.*

Corollary 2.4. *Under the conditions of Theorem 2.1 one can obtain A_U and A_V of complex symmetric form and hence self-adjoint for the standard scalar product. This means that the weight matrix Ω equals the identity.*

2.1 s-unitary Similarity of Matrix Families

Proposition 2.5. *For any two $n \times n$ matrices A and B, the following assertions are equivalent:*

(1). A and B are s-unitarily similar,

(2). the families $\{A, A^{\Theta}\}$ and $\{B, B^{\Theta}\}$ are unitarily similar,

(3). the familes $\{A, A^{\Theta}\}$ and $\{B, B^{\Theta}\}$ are similar.

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are obvious. Let us show that (3) implies (1). Let P be a nonsingular matrix such that

$$P^{-1}AP = B, \qquad P^{-1}A^{\Theta}P = B^{\Theta} \tag{2.5}$$

Then

$$U^{\Theta}AU = B \tag{2.6}$$

where U is the s-unitary factor in the polar decomposition $P = SU$. Indeed, equalities (1) imply that

$$PP^{\Theta}A = APP^{\Theta}.$$

Since the Hermitian factor S, which is a square root of PP^{Θ}, can be represented as a polynomial in PP^{Θ}, it follows from the relation above that $SA = AS$. This yields the desired result

$$B = P^{-1}AP = U^{\Theta}S^{-1}ASU = U^{\Theta}AU.$$

□

Proposition 2.6. *If two families of normal matrices are similar, then they are s-unitarily similar.*

Corollary 2.7. *If two families of s-normal matrices are similar, then they are s-unitarily similar.*

Proposition 2.8. *For any matrices A and B, the following assertions are equivalent:*

(1). A and B are s-unitarily similar,

(2). the pair $\{H_1, H_2\}$ and $\{G_1, G_2\}$ composed of the Hermitian components of A and B, respectively, are s-unitarily similar,

(3). the pairs composed of the Hermitian components of A and B are similar.

Proof. The validity of implication (3) ⇒ (2) follows from Proposition 2.6, because Hermitian matrices are normal. The fact that (2) implies (1) and (1) implies (3), follows from the definition of Hermitian Components. □

Theorem 2.9. *Completely reducible families α and β are similar if and only if*

$$tr\ \alpha(\omega) = tr\ \beta(\omega) \quad \forall\, \omega \in \langle X \rangle. \tag{2.7}$$

Proof. The proof is obtained by applying the following well-known Theorem from [1] to representation of monoid $\langle X \rangle$ if for two completely reducible representations of a semigroup, the trace of both matrices corresponding to an arbitrary element are the same, then these representations are similar. □

Theorem 2.10 (Specht[2]& [3])**.** *Two complex matrices A and B are s-unitary similar if and only if the families*

$$\{\alpha(x_1) = A, \alpha(x_2) = A^{\Theta}\} \quad and \quad \{\beta(x_1) = B, \beta(x_2) = B^{\Theta}\} \tag{2.8}$$

satisfy the condition

$$tr\ \alpha(\omega) = tr\ \beta(\omega) \quad \forall\, \omega \in \langle \{x_1, x_2\} \rangle. \tag{2.9}$$

Theorem 2.11. *Two complex matrices A and B are s-unitarily similar if and only if the families*

$$\{\alpha(x_1) = H_1, \alpha(x_2) = H_2\} \quad and \quad \{\beta(x_1) = G_1, \beta(x_2) = G_2\} \tag{2.10}$$

composed of the Hermitian components of these matrices satisfy condition (2.9).

Proposition 2.12. *A family $\alpha : X \to M_n(\mathbb{C})$ is completely reducible if and only if it is similar sum of irreducible families.*

Proposition 2.13. *A family $\alpha : X \to M_n(\mathbb{C})$ is a normal family, then the space $\mathbb{C}^n$ can be decomposed into an orthogonal sum of $\alpha-$invarient subspaces that do not contain nontrivial $\alpha-$invarient subspaces.*

Proposition 2.14. *Assume that two normal families $\alpha : X \to M_n(\mathbb{C})$ and $\beta : X \to M_n(\mathbb{C})$ are similar. Then α and β are s-unitarily similar if and only if $\alpha + \beta$ is a normal family.*

Proof. Let U be a s-unitary matrix transforming α into β and f a polynomial such that $\alpha^{\Theta}(x) = \alpha(f)$. Then

$$\begin{aligned}\beta^{\Theta}(x) &= U^{\Theta}\alpha^{\Theta}(x)U \\ &= U^{\Theta}\alpha(f)U \\ &= \beta(f).\end{aligned}$$

Hence

$$(\alpha+\beta)^{\Theta}(x) = (\alpha+\beta)(f). \tag{2.11}$$

Since $x \in X$ is arbitrary, this proves the normality of the family $\alpha+\beta$.

Now, we assume that $\alpha+\beta$ is a normal family, that is for each $x \in X$, there exists $f \in \mathbb{C}[X]$ with property (2.11). Since

$$(\alpha+\beta)(f) = \alpha(f) \oplus \beta(f), \qquad (\alpha+\beta)^{\Theta}(x) = \alpha^{\Theta}(x) \oplus \beta^{\Theta}(x),$$

it follows from (2.11)that

$$\alpha^{\Theta}(x) = \alpha(f), \qquad \beta^{\Theta}(x) = \beta(f). \tag{2.12}$$

Let P transform α into β:

$$P^{-1}\alpha(x)P = \beta(x), \qquad x \in X.$$

Taking into account equalities (2.12), we can write

$$P^{-1}\alpha^{\Theta}(x)P = P^{-1}\alpha(f)P = \beta(f) = \beta^{\Theta}(x).$$

Thus, the matrices $\alpha(x)$ and $\beta(x)$ obey relations (2.5). Proceeding as in the proof of Proposition 2.5, we conclude that the s-unitary factor in the polar decomposition of P can be taken as a matrix transforming α in β. □

Corollary 2.15. *Assume that two s-normal families $\alpha : X \to M_n(\mathbb{C})$ and $\beta : X \to M_n(\mathbb{C})$ are similar. Then α and β are s-unitarily similar if and only if $\alpha+\beta$ is a s-normal family.*

Proposition 2.16. *If $\alpha_1 + ... + \alpha_s$ is a normal family, then any partial sum $\alpha_{i_1} + ... + \alpha_{i_t}$,$1 \le i_1 < ... < i_t \le s$, also is a normal family.*

Corollary 2.17. *If $\alpha_1 + ... + \alpha_s$ is a s-normal family, then any partial sum $\alpha_{i_1} + ... + \alpha_{i_t}$,$1 \le i_1 < ... < i_t \le s$, also is a s-normal family.*

Theorem 2.18. *Let $\alpha_i : X \to M_{n_i}(\mathbb{C})$, $i = 1, ..., l$, be irreducible pairwise dissimilar families. Then, for arbitrary matrices $C_1, ..., C_l$ of orders $n_1, ..., n_l$, respectively, there exists a polynomial $h \in \mathbb{C}[X]$ such that*

$$(\alpha_1 + ... + \alpha_l)(h) = C_1 \oplus ... \oplus C_l.$$

Proposition 2.19. *A family α is normal if and only if it is similar to a sum of irreducible families in which similar summands are s-unitarily similar.*

Corollary 2.20. *A family α is s-normal if and only if it is similar to a sum of irreducible families in which similar summands are s-unitarily similar.*

Theorem 2.21. *Normal familes $\alpha : X \to M_n(\mathbb{C})$ and $\beta : X \to M_n(\mathbb{C})$ are s-unitarily similar if and only if*

$$tr\ \alpha^{\Theta}(v)\alpha(w) = tr\ \beta^{\Theta}(v)\beta(w), \qquad \forall\ v, w \in \langle X\rangle. \tag{2.13}$$

Before proving the theorem, we make the following observation. Since trace is a linear function, the validity of equalities (2.13) implies that these equalities remain true when the words v and w are replaced by arbitrary polynomials $f, g \in \mathbb{C}[X]$. The function

$$(f, g) = tr\ \ \alpha^{\Theta}(g)\alpha(f)$$

defines a scalar product on the complex linear space $\mathbb{C}[X]$. Thus, Theorem 2.21 asserts that the matrix families α and β are s-unitarily similar if they define the same scalar product on $\mathbb{C}[X]$.

Proof. The fact that conditions (2.13) are necessary for s-unitary similarity is trivial. Let us prove that these conditions are sufficient. Conditions (2.13)contain conditions (2.7) in the particular case $v = e$. In view of Theorem 2.9 and the fact, noted above, that normal families are completely reducible, these conditions are ensure that α and β are similar.

To prove that α and β are s-unitarily similar, we first consider the case in which α is an irreducible family. By Burnside's Theorem, there exists words $w_1, ..., w_{n^2}$ such that the matrices

$$\alpha(w_1), ..., \alpha(w_{n^2})$$

are linearly independent. Denote by $\alpha'(w_i)$ the column vector of dimension n^2 whose first n components from the first column in $\alpha(w_i)$, the next n components from the second column, and so on. Consider the matrix

$$G = [\alpha'(w_1), ..., \alpha'(w_{n^2})]$$

of order n^2. It is obvious that G is a nonsingular matrix. By a direct calculation, one can verify that

$$G^{\Theta} n\alpha(w)G = \|tr\ \alpha^{\Theta}(w_i)\alpha(w)\alpha(w_j)\|$$

for an arbitrary word $w \in \langle X\rangle$. Here $n\alpha(w)$ is the direct sum of n copies of $\alpha(w)$. Define

$$H = [\beta'(w_1), ..., \beta'(w_{n^2})].$$

For an arbitrary word $w \in \langle X\rangle$, condition (2.13)imply the equality

$$G^{\Theta} n\alpha(w)G = H^{\Theta} n\beta(w)H. \tag{2.14}$$

In particular, when $w = e$, we have $G^{\Theta}G = H^{\Theta}H$. The last equality implies that H is a nonsingular matrix, while $U = GH^{-1}$ is s-unitary. Then it follows from (2.14) that

$$U^{\Theta} n\alpha(w)U = n\beta(w).$$

Thus, the families $n\alpha$ and $n\beta$ are s-unitarily similar moreover, they are normal by construction. According to Proposition 2.14, the family $n\alpha + n\beta$ also is normal. Then the family $\alpha + \beta$ is normal as well. As already said, equality (2.13) ensures that α and β are similar. Again applying Proposition 2.14, this time to the families α and β, we conclude that α is s-unitarily similar to β.

Now, assume that α and β are arbitrary normal families of matrices. By Proposition 2.19, each family can be transformed by a s-unitary similarity into a sum of irreducible families any two of which are either dissimilar or s-unitary similar. Therefore, it suffices to gives a proof for the families

$$\alpha = \alpha_1 + ... + \alpha_s \qquad and \beta = \beta_1 + ... + \beta_t, \tag{2.15}$$

which are exactly such sums.

As already noted, equality (2.13) implies the similarity between α and β. The proof of a theorem in [1] concerning the role of the trace of a completely reducible representation shows that this similarity can be described in more exact terms, namely, one necessarily has $s = t$ in expression (2.15), and there exists a permutation σ of the integers $\{1, ..., s\}$ such that α_i is similar to β_{σ_i}. It will be convenient to assume that $\alpha_1, ..., \alpha_l$, $l \leq s$, is the complete list of pairwise dissimilar summands. This can be achieved by a permutation of diagonal blocks, which is a similarity transformation performed by a permutation matrix, that is s-unitary similarity transformation.

Further, assume that m_i, $1 \leq i \leq l$, is he number of times that a summand similar to α_i enters the family α. Then equalities (2.13) can be rewritten as

$$\sum_{i=1}^{l} m_i \; tr \; \alpha_i^{\Theta}(v)\alpha_i(w) = \sum_{i=1}^{l} m_i \; tr \; \beta_{\sigma(i)}^{\Theta}(v)\alpha_{\sigma(i)}(w), \quad \forall \, v, w \in \langle X \rangle. \tag{2.16}$$

Note that, because of the linearity of the trace, equalities (2.16) remain true if the words v and w are replaced by arbitrary polynomials.

Fix an arbitrary α_k, $1 \leq k \leq l$, and a word $w \in \langle X \rangle$. According to Theorem 2.18, there exists a polynomial $h \in \mathbb{C}[X]$ such that

$$\alpha_k(h) = \alpha_k(w), \tag{2.17}$$

$$\alpha_i(h) = 0 \qquad \forall \, i \neq k. \tag{2.18}$$

Let P_k be a matrix that transforms the family α_k into $\beta_{\sigma(k)}$. In particular,

$$P_k^{-1}\alpha_k(h)P_k = \beta_{\sigma(k)}(h), \qquad P_k^{-1}\alpha_k(w)P_k = \beta_{\sigma(k)}(w).$$

In view of (2.17), it follows that

$$\beta_{\sigma(k)}(h) = \beta_{\sigma(k)}(w). \tag{2.19}$$

Taking (2.18) into account, we have

$$\beta_{\sigma(i)}(h) = P_i^{-1}\alpha_i(h)P_i = 0 \quad \forall \, i \neq k. \tag{2.20}$$

Replacing w in (2.16) by a polynomial h, taking relations (2.17)–(2.20) into account, and dividing both sides by m_k, we arrive at the equality

$$tr\ \alpha_k^{\Theta}(v)\alpha_k(w) = tr\ \beta_{\sigma(k)}^{\Theta}(v)\beta_{\sigma(k)}(w).$$

Recall that no restrictions were placed on the words v and w. According to the first part of the proof, the irreducible families α_k and $\beta_{\sigma(k)}$ are s-unitarily similar. Let U_k be a s-unitary matrix transforming α_k and $\beta_{\sigma(k)}$, and let F be a permutation matrix such that

$$F^{\Theta}(\beta_1(x) \oplus ... \oplus \beta_s(x))F = \beta_{\sigma(1)}(x) \oplus ... \oplus \beta_{\sigma(s)}(x) \quad \forall x \in X.$$

It is easy to see that the s-unitary matrix

$$(U_1 \oplus ... \oplus U_s)F^{\Theta}$$

transforms the family α into β. □

2.2 Finite Criteria for s-unitary Similarity

Specht's theorem was refined by Pearcy [4, 3], who converted t into a finite criterion by showing that the verification of equality (2.9) can be limited to words of length at most $2n^2$. This proves the existence of a finite complete system of s-unitary invariants. We show that Theorem 2.21 can also be transformed into a finite criterion. Denote by X^k the set composed of words of length at most k and by $\bar{k}$ the number of such words. Define

$$d_k = \dim span\{\alpha(p),\ p \in X^k\}.$$

It is obvious that

$$1 = d_0 \leq d_1 \leq ...$$

Moreover, if

$$1 = d_0 < d_1 < ... < d_l = d_{l+1}, \tag{2.21}$$

then

$$d_l = d_{l+1} = ... \quad and \quad \dim \mathcal{A} = d_l. \tag{2.22}$$

The least number l such that $d_l = d_{l+1}$ will be called the length of the family α.

It will be convenient to assume that the set $\langle X \rangle$ is indexed ($\langle X \rangle = \{p_0, p_1, ...\}$) so that a shorter word has a smaller index than a longer one. In particular, $p_0 = e$. For the system of vectors $\alpha(p_0)$, $\alpha(p_1), ..., \alpha(p_{\bar{k}-1})$, we form the new Gram matrix

$$G_k = \|tr\ \alpha^{\Theta}(p_i)\alpha(p_j)\|.$$

It rank is equal to the maximum number of linearly independent vectors in the system; hence we have

$$rank\ G_k = d_k.$$

This equality implies

Proposition 2.22. *The length $\alpha : X \to M_n(\mathbb{C})$ is equal to the least number l with the property that*

$$rank\ G_l = rank\ G_{l+1}.$$

Proposition 2.23. *The functional $tr\ \alpha^{\Theta}(v)\alpha(w)$ is defined for arbitrary $v, w \in \langle X\rangle$, if one knows its values on words $v \in X^l$, $w \in X^{l+1}$, where l is the length of the family α.*

Proof. Write the matrix G_k as

$$G_k = H_k^{\Theta} H_k, \qquad where \quad H_k = [\alpha'(p_0), ..., \alpha'(p_{\bar{k}-1})].$$

Define an auxiliary matrix family $\hat{\alpha} : X \to M_{\bar{l}}(\mathbb{C})$ from the equations

$$n\alpha(x)H_l = H_l\hat{\alpha}(x), \quad x \in X. \tag{2.23}$$

Since

$$n\alpha(x)H_l = [\alpha'(x), \alpha'(xp_1)...\alpha'(xp_{\bar{l}-1})], \quad x \in X,$$

and in view of properties (2.21) and (2.22) of the number l, Equation (2.23) are indeed salvable. The multiplication of (2.23) by H_l^{Θ} yields the system

$$H_l^{\Theta} n\alpha(x)H_l = G_l\hat{\alpha}(x), \quad x \in X. \tag{2.24}$$

Systems (2.23) and (2.24) are equivalent, because the first system is solvable [?]. Since

$$H_l^{\Theta} n\alpha(x)H_l = \|tr\ \alpha^{\Theta}(p_i)\alpha(xp_j)\|,$$

we can assert that the family $\hat{\alpha}$ is completely determined by the numbers

$$tr\ \alpha^{\Theta}(v)\alpha(w), \quad v \in X^l,\ w \in X^{l+1},$$

where l is the length of α.

It is clear that equalities (2.23) remain true if x is replaced by an arbitrary word w. Multiplying the equalities

$$n\alpha(w)H_l = H_l\hat{\alpha}(w) \quad H_l^{\Theta} n\alpha^{\Theta}(v) = \hat{\alpha}^{\Theta}(v)H_l^{\Theta},$$

we obtain

$$H_l^{\Theta} n\alpha^{\Theta}(v)n\alpha(w)H_l = \hat{\alpha}^{\Theta}(v)H_l^{\Theta}H_l\hat{\alpha}(w),$$

or

$$\|tr\ \alpha^{\Theta}(vp_i)\alpha(wp_j)\| = \hat{\alpha}^{\Theta}(v)G_l\hat{\alpha}(w). \tag{2.25}$$

Note that the entry (1, 1) of the matrix in (2.25) is equal to $tr\ \alpha^{\Theta}(v)\alpha(w)$.

Thus, given the matrix G_l and the matrix family $\hat{\alpha}$ are determined by the values indicated in the formulation of Proposition 2.23. □

Now, it is easy to prove a finite criterion for the s-unitary similarity between the normal matrix families.

Theorem 2.24. *Let $\alpha : X \to M_n\mathbb{C}$ be a normal family of length l. It is s-unitarily similar to the families $\beta : X \to M_n\mathbb{C}$ if*

$$tr\ \alpha^{\Theta}(v)\alpha(w) = tr\ \beta^{\Theta}(v)\beta(w) \quad \forall v, w \in X^{l+1}. \tag{2.26}$$

Proof. Equalities (2.26) ensure that α and β have the same new Gram matrix, that is

$$G_k(\alpha) = G_k(\beta) \qquad k = 1, ..., l, l+1.$$

Hence by Proposition 2.22 the length of the family α and β is equal to l. Now applying Proposition 2.23, we conclude that equalities (2.26) are fulfilled for all $v, w \in \langle X\rangle$. Then, by Theorem 2.21, α and β are s-unitarily similar. □

Corollary 2.25. *For matrices A and B to be s-unitarily similar, it suffices that*

$$tr\ \alpha(w) = tr\ \beta(w) \quad \forall w \in X^{2l+2},$$

where l is the length of the family α.

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