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Common Fixed Point Theorems for Nonself – Mapping in Metrically Convex Spaces

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Abstract : A common fixed point theorem for pair of nonself- mapping is proved in complete metrically convex metric space, which generalize earlier results due to M.Imdad [6] M.D. khan [4], MS Khan [5], Bianchini [12], Chatterjea [13], and others.

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1 Introduction

In many applications the mappings under examination may not always be self- mapping, therefore fixed point theorems for nonself-mapping from a natural subject for investigation. Assad and Kirk [8] initiated

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the study of fixed point for nonself –mappings in metrically convex spaces, noticed that with some kind of metric convexity, domain and range of mapping under examination can be considered of more varied type. In recent years this technique due to Assad and Kirk [8] has been utilized by many researchers of domain and by now there exists considerable literature on this topic. To mention a few, we cite [2, 3, 4, 5, 7, 8, 9, 10].

Assad [7] gave sufficient condition for nonself- mappings defined on a closed subset of complete metrically convex metric spaces satisfying Kannan type mappings [11] which have been generalized by M.S. Khan [5]. For the sake of completeness, we stat that main result of M.S. Khan [5].

Theorem 1.1. *Let $(X\ ,d)$be a complete metrically convex space and K a closed nonempty subset of X. Let $T : K \to X$be the mapping satisfying the inequality*

$$d\,(Tx,Ty) \leq C\ max\,\{d\,(x,Tx)\,,\ d(y,\ Ty)\}\ + C^{'}\{d\,(x,\ Tx) + d(y,Tx)\} \tag{1.1}$$

for every x , y in K, where C and C' are non-negative reals such that

$$\max\left\{\frac{C+C'}{1-C'}\ ,\ \frac{C'}{1-C-C'}\right\} = h\ \ > 0$$

$$\max\left\{\frac{1+C+C'}{1-C'}h\ ,\ \frac{1+C'}{1-C-C'}h\right\} = h'\ \ > 0$$

$$\max\{h,\ h'\} = h^{''} > 0$$

Further, for every x in ∂K , $Tx \in K$. Then T has a unique fixed point in K.

Currently M. Imdad and Ladlay Khan [6] generalized this result as,

Theorem 1.2. *Let $(X\ ,d)$be a complete metrically convex metric space and K a closed nonempty subset of X. if F is generalized T contraction mapping of K into X satisfying the following:*

1. $\delta K\ \subseteq TK\ \ ,\ \ FK \cap K \subseteq TK$
2. $Tx \in \delta K \Rightarrow Fx \in K$
3. (F,T) *is coincidentally commuting*
4. *TK is closed in X.*

Then F and T have a unique common fixed point.

Definition 1.3 ([6])**.** *Let (X,d) be a metric space and Ka nonempty subset of X.Let $F\ ,T : K \to X$be a pair of maps which satisfy the condition*

$$\phi\,(d\,(Fx,Fy)) \leq a\ max\left\{\frac{1}{2}\phi\,(d\,(Tx,Ty))\ \ ,\ \phi\,(d\,(Tx,Fx))\ \ ,\ \phi(d(Ty,Fy)\right\} + b\,\{\phi\,(d\,(Tx,Fy)) + \phi\,(d\,(Ty,Fx))\} \tag{1.2}$$

for all distinct x , $y\ \in K$, a , $b \geq 0$ such that $a + 4b < 1$and let $\phi : R^+ \to R^+$ be an increasing continuous function for which the following properties hold:

$$\phi(t) = 0 \iff t = 0, \qquad \phi(2t) \leq 2\phi(t)$$

Then F is called generalized T contraction mapping of K into X.

2 Preliminaries

We necessitate the following in the sequel

Definition 2.1 ([9]). *Let K be a nonempty subset of a metric space $(X\ ,d)$ and $T : K \to X$. The pair (F,T) is said to be weakly commuting if for every $x\ ,y \in K$ with $x = Fy$ and $Ty \in K$*

$$d(Tx, FTy) \leq d(Ty, Fy) \tag{2.1}$$

Definition 2.2 ([10]). *Let K be a nonempty subset of a metric space $(X\ ,d)$ and $F,T : K \to X$. The pair (F,T) is said to be compatible if every sequence $\{x_n\} \subset K$ and from the relation*

$$\lim_{n\to\infty} d(Fx_n\ ,Tx_n) = 0. \tag{2.2}$$

And $Tx_n \in K$ (for every) $n \in N$, it follows that

$$\lim_{n\to\infty} d(Ty_n\ ,FTx_n) = 0 \tag{2.3}$$

for every sequence $y_n \in K$ such that $y_n = Fx_n\ ,\ n \in N$

Definition 2.3 ([3]). *A pair of nonself –mapping (F,T) on a nonempty subset K of a metric space (X,d) is said to be coincidentally commuting if $Tx\ ,\ Fx \in K$ and $Tx = Fx \Longrightarrow FTx = TFx$*

Definition 2.4 ([8]). *A metric space (X,d) is said to be metrically convex if for any distinct $x,y \in X$, there exists a point $z \in X$ with $x \neq z \neq y$ such that*

$$d(x,z) + d(z,y) = d(x,y) \tag{2.4}$$

Lemma 2.5 ([8]). *Let K be a nonempty closed subset of a metrically convex metric space X. if $x \in K$ and $\notin K$, then there exists a point $z \in \delta K$ (the boundary of K) such that*

$$d(x,z) + d(z,y) = d(x,y) \tag{2.5}$$

3 Our main result runs as follows

Theorem 3.1. *Let $(X\ ,d)$ be a complete metrically convex metric space and K a closed nonempty subset of X. Let $F,T : K \to X$ be the mappings satisfying the inequality*

$$d(Fx,Fy) \leq\ (a+b)\, d(Fx,Tx) + b\left[\max\{d(Tx,Fx)\,, d(Tx,Ty)\} + d(Ty,Fy)\right] \tag{3.1}$$

for every $x\ ,y\ \in K$, where a , b , and c are nonnegative reals such that

$$\frac{a+b+c}{1-c} = h > 0$$

and

(i) $\delta K \subseteq TK,\ FK \cap K \subseteq TK$

(ii) $Tx \in \delta K \Rightarrow Fx \in K$

(iii) (F, T) *is coincidentally commuting*

(iv) TK *is closed in* X

Then F *and* T *have a unique common fixed point.*

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way: Let $x \in \delta K$. Then (due to $\delta K \subset TK$) there exists a point $x_0 \in K$such that $x = Tx_0$. Since $Tx \in \delta K \Longrightarrow Fx \in K$, one concludes that $Fx_0 \in FK \cap K \subseteq TK$. Let $x_1 \in K$be such that $y_1 = Tx_1 = Fx_0$.
Let $y_2 = Fx_1$, if $y_2 \in K$, then $y_2 \in FK \cap K \subset TK$ which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. If $y_2 \notin K$, then there exists a point $t \in \delta K$ such that

$$d(Tx\ , t) + d(t\ , y_2) = d(Tx_1\ , y_2) \tag{3.2}$$

Since $t \in \delta K \subseteq TK$there exists a point $x_2 \in K$ such that $t = Tx_2$ so that

$$d(Tx_1\ , Tx_2) + d(Tx_2\ , y_2) = d(Tx_1\ , y_2) \tag{3.3}$$

Thus repeating the foregoing arguments one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(v) $y_{n+1} = Fx_n$

(vi) $y_n \in K\ \Rightarrow y_n = Tx_n$ *or* $y_n \notin K\ \Rightarrow Tx_n \in \delta K$

and

$$d(Tx_{n-1}\ , Tx_n) + d(Tx_n\ , y_n) = d(Tx_{n-1}\ , y_n)\,. \tag{3.4}$$

We denote

$$P = \{Tx_i \in \{Tx_n\}\ : Tx_i = y_i\ \}$$
$$Q = \{Tx_i \in \{Tx_n\}\ : Tx_i \neq y_i\ \}$$

Obviously, the two consecutive terms cannot lie in Q. Now, we distinguish the following three cases,
Case 1. If $Tx_n\ ,\ Tx_{n+1} \in P$ then

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(y_n\ ,\ y_{n+1}) \\ &= d(Fx_{n-1}\ , Fx_n) \\ &\leq\ (a+c)\, d(Fx_{n-1} Tx_{n-1}) + b\,[max\,\{d(Tx_{n-1}\ , Fx_{n-1})\,, d(Tx_{n-1} Tx_n)\} + d(Tx_n\ , Fx_n)] \\ &= (a+c)\, d(y_n Tx_{n-1}) + b\,[max\,\{d(Tx_{n-1}\ , y_n)\,, d(Tx_{n-1} Tx_n)\} + d(Tx_n\ , y_{n+1})] \\ &= (a+c)\, d(Tx_n\ , Tx_{n-1}) + b\,[max\,\{d(Tx_{n-1}\ , Tx_n)\,, d(Tx_{n-1}\ , Tx_n)\}] + b\ d(Tx_n\ , Tx_{n+1}) \end{aligned}$$

$$(1-b)d\left(Tx_n\ ,Tx_{n+1}\right) \leq\ (a+c)\,d\left(Tx_n\,Tx_{n-1}\right)+b\ d(Tx_{n-1}\ ,Tx_n)$$

$$(1-b)\ d\left(Tx_n\ ,Tx_{n+1}\right) \leq\ (a+c+b)\,d(Tx_{n-1}\ ,Tx_n)$$

$$d\left(Tx_n\,Tx_{n+1}\right) \leq \frac{(a+c+b)}{1-b}d\left(Tx_{n-1}\,Tx_n\right) \tag{3.5}$$

Case 2. If $Tx_n \in P\ ,Tx_{n+1} \in Q$ then

$$d\left(Tx_n,Tx_{n+1}\right)+d\left(Tx_{n+1}\ ,\ y_{n+1}\right)=d(Tx_n\ ,\ y_{n+1})$$

$$d\left(Tx_n,Tx_{n+1}\right) \leq\ d\left(Tx_n\ ,\ y_{n+1}\right)=d\left(y_n\ ,y_{n+1}\right) \tag{3.6}$$

$$\begin{aligned}
d(y_n\ ,\ y_{n+1}) &= d(Fx_{n-1}\ ,Fx_n)\\
&\leq\ (a+c)\,d\left(Fx_{n-1}\,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,Fx_{n-1}\right),d\left(Tx_{n-1}\ ,Tx_n\right)\right\}+d(Tx_n\,Fx_n)\right]\\
&=(a+c)\,d\left(y_n\,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,y_n\right),d\left(Tx_{n-1}\,Tx_n\right)\right\}+d(Tx_n\ ,y_{n+1})\right]\\
&=(a+c)\,d\left(Tx_n\ ,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\,Tx_n\right),d\left(Tx_{n-1}\,Tx_n\right)\right\}\right]+b\ d(Tx_n\,Tx_{n+1})\\
&=(a+c)\,d\left(Tx_n\,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,Tx_n\right),d\left(Tx_{n-1}\ ,Tx_n\right)\right\}\right]+b\ d(y_n\,y_{n+1})
\end{aligned}$$

$$(1-b)d\left(y_n\ ,y_{n+1}\right) \leq\ (a+c)\,d\left(Tx_n\ ,Tx_{n-1}\right)+b\ d(Tx_{n-1}\ ,Tx_n)$$

$$(1-b)d\left(y_n\ ,y_{n+1}\right) \leq\ (a+c+b)\,d\left(Tx_n\ ,Tx_{n-1}\right)$$

$$d\left(y_n\ ,y_{n+1}\right) \leq \frac{(a+c+b)}{1-b}d\left(Tx_{n-1}\ ,Tx_n\right) \tag{3.7}$$

Since $d\left(Tx_n\ ,Tx_{n+1}\right) \leq d\left(y_n\,y_{n+1}\right)$ from(3.6). Therefore,

$$d\left(Tx_n\ ,Tx_{n+1}\right) \leq \frac{(a+c+b)}{1-b}d\left(Tx_{n-1}\ ,Tx_n\right) \tag{3.8}$$

Case 3. If $Tx_n \in Q$, $Tx_{n+1} \in P$ then $Tx_{n-1} \in P$

Since Tx_n is the convex linear combination of Tx_n and y_n, it follows that

$$d\left(Tx_n,Tx_{n+1}\right) \leq\ \max\left\{d\left(Tx_{n-1}\ ,\ Tx_{n+1}\right)\ ,d\left(y_n\ ,\ Tx_{n+1}\right)\right\}. \tag{3.9}$$

If $d\left(Tx_{n-1},Tx_{n+1}\right) \leq\ d\left(y_n\ ,Tx_{n+1}\right)$. Then

$$\begin{aligned}
d\left(Tx_n,Tx_{n+1}\right) &\leq d\left(y_n\ ,\ Tx_{n+1}\right)\\
&= d(y_n\ ,\ y_{n+1})\\
&= d(Fx_{n-1}\ ,Fx_n)\\
&\leq\ (a+c)\,d\left(Fx_{n-1}\ ,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,Fx_{n-1}\right),d\left(Tx_{n-1}\,Tx_n\right)\right\}+d(Tx_n\ ,Fx_n)\right]\\
&=(a+c)\,d\left(y_n\ ,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,y_n\right),d\left(Tx_{n-1}\ ,Tx_n\right)\right\}+d(Tx_n\ ,y_{n+1})\right]\\
&=(a+c)\,d\left(y_n\ ,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,y_n\right),d\left(Tx_{n-1}\ ,Tx_n\right)\right\}\right]+b\ d(Tx_n\ ,Tx_{n+1})
\end{aligned}$$

$$(1-b)\,d\left(Tx_n\ ,Tx_{n+1}\right) \leq\ (a+c)\,d\left(y_n\ ,Tx_{n-1}\right)+b\left[max\left\{d\left(Tx_{n-1}\ ,y_n\right),d\left(Tx_{n-1}\,Tx_n\right)\right\}\right] \tag{3.10}$$

Since

$$d\left(Tx_{n-1},y_n\right)+d\left(y_n\ ,Tx_n\right)=d\left(Tx_{n-1},Tx_n\right)$$

$$d\left(Tx_{n-1},y_n\right) \leq d\left(Tx_{n-1},Tx_n\right)$$

$$\begin{aligned}(1-b)\,d\,(Tx_n\ ,Tx_{n+1}) &\leq\ (a+c)\,d\,(y_n\ ,Tx_{n-1})+b\ d\,(Tx_{n-1}\ ,Tx_n)\\ &\leq (a+c)d\,(Tx_{n-1}\ ,Tx_n)+b\ d\,(Tx_{n-1}\ ,Tx_n)\\ &=(a+c+b)\ d\,(Tx_{n-1}\ ,Tx_n)\end{aligned}$$

$$d\,(Tx_n\ ,Tx_{n+1})\leq\frac{(a+c+b)}{1-b}d\,(Tx_{n-1}\ ,Tx_n)\,. \tag{3.11}$$

On the other hand if $d(y_n\ ,Tx_{n+1})\leq d(Tx_{n-1}\ ,Tx_{n+1})$.
Then $d\,(Tx_n\ ,Tx_{n+1})\leq d\,(Tx_{n-1}\,Tx_{n+1})=d(y_{n-1}\ ,y_{n+1})$ from (3.9).

$$\begin{aligned}d\,(Tx_n\ ,Tx_{n+1}) &\leq d(y_{n-1}\ ,y_{n+1})\\ &=d(Fx_{n-2}\ ,Fx_n)\\ &\leq\ (a+c)\,d\,(Fx_{n-2}\ ,Tx_{n-2})+b\,[max\,\{d\,(Tx_{n-2}\ ,Fx_{n-2})\,,d\,(Tx_{n-2}\,Tx_n)\}+d(Tx_n\ ,Fx_n)]\\ &=(a+c)\,d\,(y_{n-1}\ ,Tx_{n-2})+b\,[max\,\{d\,(Tx_{n-2}\ ,y_{n-1})\,,d\,(Tx_{n-2}\ ,Tx_n)\}+d(Tx_n\ ,y_{n+1})]\\ &=(a+c)\,d\,(y_{n-1}\ ,Tx_{n-2})+b\,[max\,\{d\,(Tx_{n-2}\ ,y_{n-1})\,,d\,(Tx_{n-2}\,Tx_n)\}]+b\ d(Tx_n\ ,Tx_{n+1})\end{aligned}$$

$$(1-b)\,d\,(Tx_n\ ,Tx_{n+1})\leq\ (a+c)\,d\,(Tx_{n-1}\ ,Tx_{n-2})+b\,[max\,\{d\,(Tx_{n-2}\ ,Tx_{n-1})\,,d\,(Tx_{n-2}\ ,Tx_n)\}]\,. \tag{3.12}$$

Since

$$\begin{aligned}d\,(Tx_{n-2}\ ,Tx_n)+d\,(Tx_n\ ,\ Tx_{n-1}) &=d(Tx_{n-2}\ ,Tx_{n-1})\\ d\,(Tx_{n-2}\ ,Tx_n) &\leq d(Tx_{n-2}\ ,Tx_{n-1}).\end{aligned}$$

Therefore,

$$\begin{aligned}(1-b)\,d\,(Tx_n\ ,Tx_{n+1}) &\leq\ (a+c)\,d\,(Tx_{n-1}\ ,Tx_{n-2})+bd\,(Tx_{n-2}\ ,Tx_{n-1})\\ &=(a+c+b)\,d\,(Tx_{n-1}\ ,Tx_{n-2})\end{aligned}$$

$$d\,(Tx_n\ ,Tx_{n+1})\leq\frac{(a+c+b)}{1-b}d\,(Tx_{n-1}\ ,Tx_{n-2})\,. \tag{3.13}$$

Thus in the all case we have $d\,(Tx_n\ ,Tx_{n+1})\ \leq\ A\max\ \{d\,(Tx_{n-1}\ ,Tx_n)\,,d(Tx_{n-1}\ ,Tx_{n-2})\}$, where $A=\frac{(a+c+b)}{1-b}$.
It can be easily shown that by induction that for $n\geq 1$we have

$$d\,(Tx_n\ ,Tx_{n+1})\leq A^n\max\{d\,(Tx_0\ ,Tx_1)\,,d\,(Tx_1\ ,Tx_2)\}.$$

Now for any positive integer r we have

$$\begin{aligned}d\,(Tx_n\ ,Tx_{n+r}) &\leq d\,(Tx_n\ ,Tx_{n+1})+d\,(Tx_{n+1}\ ,Tx_{n+2})+...+d\,(Tx_{n+r-1}\ ,Tx_{n+r})\\ &\leq\ \left\{1+A+A^2+A^3+...+A^{p-1}\right\}A^n max\{d\,(Tx_0,\ Tx_1)\,,\ d(Tx_1,\ Tx_2)\} \qquad (3.14)\\ &\leq\frac{1}{1+A}\ A^n\max\{d\,(Tx_0,\ Tx_1)\,,\ d\,(Tx_1,\ Tx_2)\}\end{aligned}$$

This implies that $d\,(Tx_n,\ Tx_{n+1})\rightarrow 0\ as\ n\rightarrow\infty$. So that $\{\ Tx_n\}$ is a Cauchy sequence and hence converge to a point z in X. We assume that a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ contained in P and TK is a closed subspace of X. Since $\{Tx_{n_k}\}$ is Cauchy in TK, it converse to a point, $z \in TK$. Let $u \in T^{-1}z$ then $Tu = z$. Here one also needs to note that $\{Fx_{n_k-1}\}$ will also converge to z.

$$d(Fx_{n_k-1}, Fu) \leq (a+c)\, d(Fx_{n_k-1} Tx_{n_k-1}) + b\, [max\{d(Tx_{n_k-1}, Fx_{n_k-1}), d(Tx_{n_k-1}, Tu)\} + d(Tu, Fu)]$$

Which , on letting,$k \to \infty$ reduces to

$$d(z, Fu) \leq (a+c)\, d(z, z) + b\ max\{d(z,z), d(z, Tu)\} + bd(Tu, Fu)$$
$$d(Fu, z) \leq bd(Tu, Fu)$$

$$d(Fu, z) \leq bd(Fu, Tu) \tag{3.15}$$

Yielding there by $Tu = Fu$ which shows that $u$is a point of coincidence for F and T. Since the pair (F, T) is coincidentally commuting, therefore

$$z = Tu = Fu \ \Rightarrow Fz = FTu = TFu = Tz \tag{3.16}$$

To prove that z is fixed point of,F consider $d(Fz, z) = d(Fz, Fu)$

$$d(Fz, z) \leq (a+b)\, d(Fz, Tz) + b\, [max\{d(Tz, Fz),\ d(Tz, Tu)\} + d(Tu, Fu)]$$
$$d(Fz, z) \leq (a+b)\, d(Fz, Tz) + bmax\{d(Tz, Fz),\ d(Tz, Tu)\} + bd(Tu, Fu)$$
$$d(Fz, z) \leq b\ d(Fz, z)$$

Which shows that z is a common fixed point of F and T. The proof goes on similar lines in case we assume subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ contained in Q. hence it is omitted. The uniqueness of fixed point follows easily this completes the proof. □

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