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Existence of Best Proximity Points For (ψ, α, β)-Weakly Contractive Mappings in Generalized Metric Spaces

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Abstract: Isik and Turkoglu proved a common fixed theorem in a rectangular metric space by using three auxiliary functions. In this paper we extend the result for the existence of best proximity points for (ψ, α, β)-weakly contractive mappings in generalized metric space.

Keywords: Best proximity point, Rectangular metric space, p-property.

1. Introduction and Preliminaries

In 2012, Lakzian and Samet [1] proved a fixed point theorem of a self-mapping with certain conditions in the context of a rectangular metric space via two auxiliary functions. To generalize the main result [1]. Isik and Turkoglu [2] reported a common point result of two self-mappings in the setting of a rectangular metric space by using three auxiliary functions. The obtained results are inspired by the technique and ideas of [3-11]. Here in this paper we extend the result of N.Bilgili, E.Karapinar and D.Turkoglu [12].

Definition 1.1. *Let X be nonempty set and let $d : X \times X \to [0, \infty)$ resoectivelysatisfy the following conditions for all $x, y \in X$ and for all distinct points $u, v \in X$ each of which is different from x and y.*

(i). $d(x, y) = 0$ *iff* $x = y$

(ii). $d(x, y) = d(y, x)$

(iii). $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$

Then (X, d) is called the rectangular metric space also known as generalized metric space.

We recall the definitions of the following auxiliary functions. Let Γ be the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the condition $\psi(t) = 0$ iff $t = 0$. We denote ψ be the set of functions $\psi \in \Gamma$ such that ψ is continuous and nondecreasing. We reserve ϕ for the set of functions $\alpha \in \phi$ such that α is continuous. Finally we denote the set of functions $\beta \in \Gamma$ satisfying the following conditions: β is lower semi-continuous. Lakziand and Samet [1] proved the following fixed point theorem.

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Theorem 1.2 ([1])**.** *Let (X, d) be a Hausdorff and complete rectangular metric space and let $T : X \to X$ be a self mapping satisfying $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$ for all $x, y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$. Then T has unique fixed point.*

Definition 1.3. $A_0 = \{x \in A : d(x, y) = d(A, B)\}$, *for* $y \in B$; $B_0 = \{y \in B : d(x, y) = d(A, B)\}$, *for* $x \in A$, *where* $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Definition 1.4. *Let(A, B) be a pair of nonempty subsets of metric space (X, d) with $A_0 \neq 0$. Then the pair (A, B) is said to have p-property iff for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0, d(x_1, y_1) = d(A, B) = d(x_2, y_2)$.*

2. Main Results

Theorem 2.1. *Let (X, d) be a Hausdroff and complete Rectangular metric space and Let (A, B) be a pair of nonempty subsets of a metric space such that A_0 is nonempty. Let $T : A \to B$ be a mapping satisfying $T(A_0) \subset B_0$. Suppose*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y) - d(A, B)) - \phi(d(x, y) - d(A, B)) \tag{1}$$

for all $x \in A$, $y \in B$, where $\psi \in \Psi$ and $\phi \in \Phi$. Then T has best proximity point.

Proof. Choose $x_0 \in A$. Since $Tx_0 \in T(A_0) \subset B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Analogously, regarding the assumption, $Tx_1 \in T(A_0) \subset B_0$, we determine $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \;\; \forall \;\; n \in N \tag{2}$$

Claim: $d(x_n, x_{n+1}) \to 0$

If $x_N = x_{N+1}$, then x_N is best proximity point. By the p-property, we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$$

Hence we assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since $d(x_{n+1}, Tx_n) = d(A, B)$, from (2), we have for all $n \in N$.

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \psi(d(x_n, x_{n+1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})) - d(A, B)) \\ &- \phi(d(x_n, x_{n+1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})) - d(A, B)) \\ &\leq \psi(d(x_n, x_{n+1}) - d(A, B)) - \phi(d(x_n, x_{n+1}) - d(A, B)) \end{aligned} \tag{3}$$

We get $d(x_n, x_{n+1}) = d(A, B)$ and follows $d(x_n, x_{n+1}) = 0$ a contradiction. From (3) we get that $\psi(d(x_n, x_{n+1})) = 0$ and $d(x_n, x_{n+1}) = 0$ contradicting our assumption. Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for any $n \in N$ and hence $\{d(x_n, x_{n+1})\}$ is monotone decreasing sequence of nonnegative real numbers, hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. In the view of the fact from (2), for any $n \in N$, we have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$$

Taking the limit as $n \to \infty$ in the above inequality, and using the conditions of ψ and ϕ we have $\psi(r) \leq \psi(r) - \phi(r)$ which implies $\phi(r) = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{4}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.

If otherwise there exists $\varepsilon > 0$, for which we can find two sub sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for all positive integers $m_k > n_k > k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and $d(x_{m_k}, x_{n_{k-1}}) < 1$. Now $\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k})$. That is $\varepsilon \leq d(x_{m_k}, x_{n_k}) < \varepsilon + d(x_{n_{k-1}}, x_{n_k})$. Taking the limit as $k \to \infty$ in the above inequality and using (4) we have

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon \tag{5}$$

Again $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$. Taking the limit as $k \to \infty$ in the above inequalities and using (4) and (5) we have

$$\lim_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \varepsilon \tag{6}$$

Again

$$\begin{aligned} d(x_{m_{k+1}}, x_{n_{k+1}}) &\leq d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}) \end{aligned}$$

Letting $k \to \infty$ in the above inequalities and using (4) and (5) we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) = \varepsilon \tag{7}$$

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \varepsilon \tag{8}$$

For $x = x_{m_k}$, $y = y_{m_k}$ we have

$$\begin{aligned} d(x_{m_k}, Tx_{m_k}) - d(A, B) &\leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, Tx_{n_k}) - d(A, B) \\ &= d(x_{m_k}, x_{m_{k+1}}) \end{aligned}$$

Similarly $d(x_{n_k}, Tx_{n_k}) - d(A, B) = d(x_{m_k}, x_{n_{k+1}})$ and $d(x_{n_k}, Tx_{m_k}) - d(A, B) = d(x_{n_k}, x_{m_{k+1}})$. From (1) we have

$$\begin{aligned} \psi(d(x_{m_{k+1}}, x_{n_{k+1}})) &= \psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq \psi((d(x_{m_k}, x_{n_k}) + d(x_{m_k}, Tx_{m_k}) + d(x_{n_k}, Tx_{n_k})) - d(A, B)) - \phi((d(x_{m_k}, x_{n_k}) + d(x_{m_k}, Tx_{m_k}) \\ &+ d(x_{n_k}, Tx_{n_k})) - d(A, B)) \\ &\leq \psi((d(x_{m_k}, x_{n_k}) + d(x_{m_k}, x_{m_{k+1}}) + d(x_{n_k}, x_{n_{k+1}})) - d(A, B)) - \phi((d(x_{m_k}, x_{n_k}) + d(x_{m_k}, x_{m_{k+1}}) \\ &+ d(x_{n_k}, x_{n_{k+1}})) - d(A, B)) \end{aligned}$$

It follows that

$$\begin{aligned} \psi(d(Tx_{m_k}, Tx_{n_k})) \leq \psi((d(x_{m_k}, x_{n_k}) + d(x_{n_k}, Tx_{n_{k+1}}) + d(x_{m_k}, Tx_{m_{k+1}})) - d(A, B)) \\ - \phi((d(x_{m_k}, x_{n_k}) + d(x_{n_k}, Tx_{n_{k+1}}) + d(x_{m_k}, Tx_{m_{k+1}})) - d(A, B)) \end{aligned}$$

From (4), (5), (6) and (7) and letting $k \to \infty$ in the above inequalities and using the conditions of ψ and ϕ, we have $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$ which is contradiction by virtue of property ϕ. Hence $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X, d), there exists x^* in A such that$x_n \to x^*$. Putting $x = x_n$ and $y = x^*$ and since $d(x_n, Tx^*) \leq d(x_n, x^*) + d(x^*, Tx_n)$ and $d(x^*, Tx_n) \leq d(x^*, Tx^*) + d(Tx^*, Tx_n)$. We have

$$\begin{aligned}\psi(d(x_{n+1}, Tx^*) - d(A, B)) &\leq \psi(d(Tx_n, Tx^*) - d(A, B)) \\ &\leq \psi((d(x_n, x^*) + d(x_n, Tx_n) + d(x^*, Tx^*)) - d(A, B)) \\ &- \phi((d(x_n, x^*) + d(x_n, Tx_n) + d(x^*, Tx^*)) - d(A, B))\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequalities and using the conditions of ψ and ϕ, we have

$$\psi((d(x^*, Tx^*) - d(A, B)) \leq \psi((d(x^*, Tx^*) - d(A, B)) - \phi((d(x^*, Tx^*) - d(A, B))$$

This implies that $d(x^*, Tx^*) = d(A, B)$. Hence x^* is a best proximity point of T.

For the uniqueness, let p and q be two best proximity point and suppose that $p \neq q$, then putting $x = p$ and $y = q$ in (1) we obtain

$$\psi(d(T_p, T_q)) \leq \psi((d(p, q) + d(p, T_p) + d(q, T_q) - d(A, B)) - \phi((d(p, q) + d(p, T_p) + d(q, T_q) - d(A, B))$$

That is $\psi(d(p, q)) \leq \psi(d(p, q)) - \phi(d(p, q))$. Contradiction by virtue of a property ϕ. Therefore $p = q$. This completes the proof. □

Theorem 2.2. *Let (X, d) be a Hausdroff and complete Rectangular metric space and Let (A, B) be a pair of nonempty subsets of a metric space such that A_0 is nonempty. Let $T : A \to B$ be a mapping satisfying $T(A_0) \subset B_0$. Suppose*

$$\psi(d(Tx, Ty)) \leq \alpha(d(x, y) - d(A, B)) - \beta(d(x, y) - d(A, B)) \tag{9}$$

for all $x \in A, y \in B$, where $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma$ and these mappings satisfy the condition

$$\psi(t) - \alpha(t) + \beta(t) > 0 \ \ \forall \ \ t > 0 \tag{10}$$

Then T has best proximity point.

Note: since the proof is the mimic of the proof of Theorem 1.1, we say that the above theorem is equivalent to Theorem 2.1.

Theorem 2.3. *Theorem 2.2 is a consequence of Theorem 2.1.*

Proof. Taking $\alpha = \psi$ in Theorem 2.2, we obtain immediately Theorem 2.1. Indeed let $T : A \to B$ be a mapping satisfying (9) with $\psi \in \Psi$, $\alpha \in \Phi, \beta \in \Gamma$ and let these mappings satisfy conditions (10). From (9), for all $x \in A, y \in B$, we have

$$\begin{aligned}\psi(d(Tx, Ty)) &\leq \alpha(d(x, y) - d(A, B)) - \beta(d(x, y) - d(A, B)) \\ &= \psi(d(x, y) - d(A, B)) - [\beta(d(x, y) - d(A, B)) - \alpha(d(x, y) - d(A, B)) + \psi(d(x, y) - d(A, B))]\end{aligned} \tag{11}$$

Define $\theta : [0, \infty) \to [0, \infty)$ by $\theta(t) = \beta(t) - \alpha(t) + \psi(t), t \geq 0$. Then we have

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y) - d(A, B)) - \theta(d(x, y) - d(A, B)) \tag{12}$$

for all $x \in A, y \in B$. Due to the definition of θ, we observe that $\theta \in \Gamma$. Now Theorem 2.2 follows immediately from Theorem 2.1. □

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