International **J**ournal of **M**athematics **A**nd Its **A**pplications
Vol.2 No.4 (2014), pp.49-57.
ISSN: 2347-1557(online)

Complete generators in 4-valued logic and Rousseau's results

M.A. Malkov[†,1]

[†] *Russian Research Center for Artificial Intelligence, Russia.*

Abstract : One of central problems of k-valued logic is identification and construction of complete generators (Sheffer functions). This problem is solved in 3-valued logic but is not solved in 4-valued logic. We prove Slupecki's theorem for functions with partial ranges and use it to construct complete generators of the functions. We use Rousseau's theorem to construct complete generators of functions with all ranges. For both cases we calculate the numbers of generators for every diagonal of the generators and give the minimal and maximal generators. We find that the number 9 of one-ary functions used by Rousseau can be decreased to 3. The number of generators of functions with ranges of cardinal 2 equals 41 760, the number of generators of functions with ranges of cardinal 3 equals 32 969 664, and the number for cardinal 4 equals 942 897 552.

Keywords : Multi-valued logics, Sheffer functions.

AMS Subject Classification: 03B50.

1 Introduction

There are very much multi-valued logics but the most fruitful of them is Post's [1,2]. We use this logic in our paper. In Post's logic, the negation, disjunction, and conjunction are presented by computable functions: $\bar{x} = x + 1 \pmod{k}$, $x_1 \vee x_2 = \max(x_1, x_2)$, and $x_1 \wedge x_2 = \min(x_1, x_2)$. One of central problems of k-valued logic is identification and construction of complete generators (Sheffer functions). Some classes of them were got for all k ([3]-[15]). But it is impossible to find all complete generators for all k since any complete generator must create

[1] `Corresponding author E-Mail: mamalkov@gmail.com (M.A. Malkov)`

all one-ary permutational functions but they belong to the symmetric groups, some of which can be sporadic. For $k = 3$ all complete generators have been constructed ([16],[17]). For $k = 4$ there are $4 \cdot 10^9$ complete generators and all of them we will construct in the paper. For $k = 5$ all complete generators can be constructed too, but it is very difficult since the number of two-ary functions equals $3 \cdot 10^{17}$ and all of them must be tested. For 6-valued logic this construction is impossible since the number of all 2-ary functions equals 10^{30}. After Post [18] we call a function $f(x_1, x_2)$ δ if $f(x, x) \neq x$ for all x. We call a function $f(x_1, x_2)$ *permutational* if it is δ and its range is {0,1,2,3}. We call a function $f(x)$ *permutational* if its range is {0,1,2,3}.

Below we call complete generators just generators. If we use functions with several ranges then we point out only the maximal range. For example, we use range {0,1,2} instead of ranges {}, {0},{1},{2}, {0,1}, {0,2}, {1,2}, {0,1,2} ({} means empty range and this range belongs to constants). If we use functions with only one range then we use definite article, otherwise articles are absent. The same rule is used for cardinals. We use ranges of functions and rages of their diagonals. We call ranges instead of ranges of functions. Except introduction the article has 3 sections. Section 2 is named "Classification of functions and generators". We present 4 levels of the classifications. The first level contains classes of functions such that the classes contain functions with ranges of the same cardinal. The second level contains additionally classes of functions with ranges of the same cardinal. The third level contains classes of functions which additionally have diagonals with the same range. And the last level contains classes of functions which have additionally the same diagonals. Generators have the same classification but classes of the first level do not exist for functions with ranges of the cardinals 0 and 1 since such functions do not generate anything except themselves.

Section 3 is "Generators of functions with partial ranges". We call functions *partial* if they have the ranges of the cardinals 2 and 3. Functions with the ranges of the cardinals 0 and 1 are partial too, but they are constants and functions with all variables to be fictitious. They need not investigation. We prove that a function is generator if the function generates all one-ary functions with partial ranges. We use this result to calculate numbers of generators for every ranges. But these numbers are the same for all ranges of the cardinal of diagonals. So we give numbers for ranges of the cardinals of diagonals only. The numbers are very large, therefore we present only least and greatest generators for every partial ranges. The last section is "Generators of functions with all ranges". We calculate numbers of generators of functions with the any range. There are several criteria for the calculation but only Rousseau's criterion is more quick for compare with the other criteria. We find that the number 9 of one-ary functions used by Rousseau's criterion can be decreased to 3. There are 6 ranges of diagonals of the cardinal 2, 4 ranges of the cardinal 3, and 1 range for the cardinal 4. The numbers are the same for all ranges of the cardinal 2 and the numbers are the same for all ranges of the cardinal 3. We give numbers for ranges of the cardinals of diagonals only. But we present least and greatest generators for every the ranges.

2 Classification of functions and generators

2.1 Classification of function

There is an hierarchy of classes. The first level of the of hierarchy is created by the next equivalent relation.

Definition 2.1. *Two functions are equivalent if they have ranges with the same cardinal.*

There are 5 classes of ranges with cardinal 4. The class with the cardinal 0 contains constants and the class with the cardinal 1 contains functions, all variables of which are fictitious. The classes 2, 3, and 4 contains the other functions. The first relation divides functions into classes and the next relation divides the classes into subclasses that are classes of the second level.

Definition 2.2. *Two functions are equivalent by the second equivalent relation if they have the same range.*

The classes with the cardinals 0, 1, 2 ,3 ,4 have 1,4, 6, 4, 1 subclasses respectively.

Definition 2.3. *Two function are equivalent by the third equivalent relation if they have diagonals with the same range.*

Functions of ranges with the cardinal 4 have 14 classes with the next diagonal ranges: {0}, {1}, {2}, {3}, {0,1}, {0,2}, {0,3}, {1,2}, {1,3}, {2,3},{0,1,2}, {0,1,3}, {0,2,3}, {1,2,3}, {0,1,2,3}.

Definition 2.4. *Two functions are equivalent by the forth equivalent relation if they have the same diagonal.*

For functions of ranges with the cardinal 4 there are 1 class of functions of diagonal range with the cardinal 1, 14 classes for the cardinal 2, 36 classes for the cardinal 3, and 104 classes for the cardinal 4.

2.2 Classification of generators

Generators are functions and they have the same classification. But the classes of the first level with the cardinals 0 and 1 have no generators since every function of the classes does not generate anything (except itself). And there are 4 classes of generators of the ranges with the cardinal 2, 12 classes of the ranges with the cardinal 3, and 9 classes of the range {0,1,2,3}. The total number of classes is 81. The numbers of generators for classes of all levels will be calculated in the next subsections. For that we must have a rule for the calculations. There are several rules of the calculations. By one of the rules, a function is a generator if it generates all two-ary functions. But this rule is not fulfilled for the classes since this takes very much time for computing. The next rule takes far less time. By the rule a function is a generator if the function generates all one-ary functions. This rule is true by Slupecki's theorem [19] for functions with all ranges. Further we prove the theorem whenever some ranges are absent. We denote generators of the third level by δ_2 for the cardinal 2 of diagonal ranges, by δ_3 for the cardinal 3 and δ_4 for the cardinal 4.

3 Generators of functions with partial ranges

3.1 Generators of functions with ranges of cardinal 2

We will prove Slupecki theorem for the functions and use it for calculation of generators. But it is enough to prove the theorem for functions with range {0,1}.

Theorem 3.1. *A subset of functions with range {0,1} is full if it contains any non-fictitious two-ary function f with the range {0,1} and all one-ary functions with ranges {0,1}.*

Proof. There are b_1, b_2, b_3, b_4 such that $f(b_1, b_2) = 0$, $f(b_1, b_3) = f(b_2, b_4) = 1$ since otherwise f is fictitious. Then the function $R(x_1, x_2) = f\big(A_1(x_1), A_2(x_2)\big)$ equals 0 whenever $x_1 = x_2 = 0$ and equals 1 whenever $x_1 = 0 \wedge x_2 = 1$ or $x_1 = 1 \wedge x_2 = 0$, if $A_1 = (b_1, b_4, b_4, b_4)$ and $A_2 = (b_2, b_3, b_3, b_3)$. We will construct the function $S(x_1, x_2)$ which equals 0 if $x_1 = x_2 = 0$ and equals 1 for the other values of variables. Let the function B_1 have values (1,0,0,0) and B_2 have values (0,1,1,1). If $R(1,1) = 0$ then $S(x_1, x_2) = B_1\big(R(B_2(x_1), B_2(x_2))\big)$ and if $R(1,1) = 1$ then $S(x_1, x_2) = R\big(B_2(x_1), B_2(x_2)\big)$. Now we will construct the function $U(x_1, x_2, x_3, x_4, x_5)$ which equals x_1 for $x_2 = x_4 \wedge x_3 = x_5$ and equals 0 for the other values of variables. Let the function $E(x_1, x_2) = 0$ whenever $x_1 = x_2$ and $E(x_1, x_2) = 1$ whenever $x_1 \neq x_2$. Let the function $F(x_1, x_2) = x_1$ whenever $x_2 = 0$ and $F(x_1, x_2) = 0$ whenever $x_2 \neq 0$. Then $U(x_1, x_2, x_3, x_4, x_5) = F(x_1, S(E(x_2, x_4), E(x_3, x_5))$. The functions U and S were introduced by Lukasiewicz (cf. [19]). They allow to generate any function $f_0(x_1, x_2)$ with range $\{0, 1\}$:

$$f_0(x_1,x_2) = S\Big(U\big(f_0(0,0),0,0,x_1,x_2\big), S\Big(U\big(f_0(0,1),0,1,x_1,x_2\big),$$
$$S\Big(U\big(f_0(0,2),0,2,x_1,x_2\big), S\Big(U\big(f_0(0,3),0,3,x_1,x_2\big), ...$$
$$...S\Big(U\big(f_0(3,0),3,0,x_1,x_2\big), S\Big(U\big(f_0(3,1),3,1,x_1,x_2\big),$$
$$S\Big(U\big(f_0(3,2),3,2,x_1,x_2\big), S\Big(U\big(f_0(3,3),3,3,x_1,x_2\big)\Big)\Big)\Big)\Big)...\Big)\Big)\Big)\Big)$$

□

Corollary 3.2. *A function is a generator if it generates all one-ary functions with ranges {0,1}.*

Proof. If a function generates all one ary functions with range {0,1} then, by the theorem, the function is a generator. □

There are 6 the ranges of generator for diagonals of the cardinal 2: {01}, {0,2}, {0,3}, {1,2}, {1,3}, {2,3}. They have the same numbers of all functions, of δ functions and of generators. But the least generators are different and greatest generators are different. The total numbers of ranges and the least and greatest generators for every ranges are presented in table 1.

Table: 1

Numbers of δ functions and generators

Cardinals of diagonal ranges	All functions	Permutational δ functions	Generators	% of all δ functions	% of permutational functions
2	393 216	98 304	41 760	11	42

Boundaries of generators

Ranges	Least generators	Greatest generators
{0,1}	(1,0,0,0, 0,0,0,0, 0,0,1,0, 0,1,0,0)	(1,1,1,1, 1,0,1,1, 1,1,0,1, 1,0,1,1)
{0,2}	(2,0,0,0, 0,0,0,0, 0,2,0,0, 0,0,0,2)	(2,2,2,2, 2,2,2,2, 2,0,0,2, 2,2,2,0)
{0,3}	(3,0,0,0, 0,0,0,0, 0,0,0,3, 0,3,0,0)	(3,3,3,3, 3,3,3,3, 3,3,3,0, 3,0,3,0)
{1,2}	(1,1,1,1, 1,2,1,1, 2,1,1,1, 1,1,1,2)	(2,2,2,2, 2,2,2,2, 1,2,1,2, 2,2,2,1)
{1,3}	(1,1,1,1, 1,3,1,1, 1,1,1,3, 3,1,1,1)	(3,3,3,3, 3,3,3,3, 3,3,3,1, 1,3,3,1)
{2,3}	(2,2,2,2, 2,2,2,2, 2,2,3,3, 2,3,3,2)	(3,3,3,3, 3,3,3,3, 3,3,3,2, 3,2,2,2)
all	(1,0,0,0, 0,0,0,0, 0,0,1,0, 0,1,0,0)	(3,3,3,3, 3,3,3,3, 3,3,3,2, 3,2,2,2)

3.2 Generators of functions with ranges of cardinal 3

We will prove Slupecki theorem for the functions and use it for calculation of generators. But it is enough to prove the theorem for functions with range {0,1,2}.

Theorem 3.3. *A subset of functions of range {0,1,2} is full, if it contains any non-fictitious two-ary function f with the range {0,1,2} and all one-ary functions with range {0,1,2}.*

Proof. The function S has been constructed in the previous proof. This function will be used. Now we will construct the function $S_1(x_1, x_2)$ which has values in {0,1,2} whenever $x_1 = 0 \wedge x_2 \in \{0, 1, 2\}$ or $x_1 \in \{0, 1, 2\} \wedge x_2 = 0$. There are b_1, b_2, b_3, b_4 such that $f(b_1, b_2) \neq f(b_1, b_4)$, $f(b_1, b_2) \neq f(b_3, b_4)$, and $f(b_1, b_4) \neq f(b_3, b_4)$ since otherwise f is fictitious. Let the function $A_1(x)$ have values (0,1,2) whenever x has values $\big(f(b_1, b_4), f(b_1, b_2), f(b_3, b_4)\big)$. Let the function $A_2(x)$ have values (b_1, b_3) whenever x has values (0,1). And let the function $A_3(x)$ have values (b_4, b_2) whenever x has values (0,1). Then there is a function $R(x_1, x_2) = A_1\big(f\big(A_2(x_1), A_3(x_2)\big)\big)$ such that $R(0, 0) = 0, R(0, 1) = 1, R(1, 0) = 2$. Let the functions $C_1(x)$ and $C_2(x)$ have values (0,0,1) and (0,1,0) respectively whenever x has values (0,1,2). And let the function $D(x)$ has values (0,1) whenever x has values (0,1). Then $S_1(x_1, x_2) = R\big(D\big(S(C_1(x_1)C_1(x_2)\big), D\big(S(C_2(x_1)C_2(x_2)\big)\big)$. The function $U(x_1, x_2, x_3, x_4, x_5)$ has been constructed in the previous proof. The functions U and S_1 allow to generate any function $f_0(x_1, x_2)$ with range $\{0, 1, 2\}$:

$$f_0(x_1, x_2) = S_1\Big(U\big(f_0(0, 0), 0, 0, x_1, x_2\big), S_1\Big(U\big(f_0(0, 1), 0, 1, x_1, x_2\big), S_1\Big(U\big(f_0(0, 2), 0, 2, x_1, x_2\big),$$
$$S_1\Big(U\big(f_0(0, 3), 0, 3, x_1, x_2\big), \ldots \;\; \ldots S_1\Big(U\big(f_0(3, 0), 3, 0, x_1, x_2\big), S_1\Big(U\big(f_0(3, 1), 3, 1, x_1, x_2\big),$$
$$S_1\Big(U\big(f_0(3, 2), 3, 2, x_1, x_2\big), S_1\Big(U\big(f_0(3, 3), 3, 3, x_1, x_2\big)\Big)\Big)\Big)\Big)\ldots\Big)\Big)\Big)\Big)$$

□

Corollary 3.4. *A function is a generator if it generates all one-ary functions with ranges {0,1,2}*

Proof. If a function generates all one ary functions with ranges {0,1,2} then, by the theorem, the function is a generator. □

There are 2-ary functions with the cardinals of ranges of diagonals 2 and 3. For the cardinal 2 there are 6 the ranges: {01}, {0,2}, {0,3}, {1,2}, {1,3}, {2,3}. Every of them has the same numbers of all functions, of δ functions, and of generators. For the cardinal 3 there are 4 the ranges of diagonals: {0,1,2}, {0,1,3}, {2,1,3}, {1,2,3}. Every of them also has the same numbers of all functions, of δ functions, and of generators. The number of all functions equals 172 186 884 for the any cardinal. The total numbers of generators and permutational δ functions of the cardinals 2 and 3 are presented in table 2.

Table: 2				
Numbers of δ functions and generators				
Cardinals of diagonal ranges	Permutational δ functions	Generators	% of all functions	% of Permutational δ functions
2	25 410 864	13 583 880	8	53
3	25 509 168	19 385 784	11	76
all	50 920 032	32 969 664	19	65

The least and greatest generators for every ranges are presented in table 3.

Table: 3		
Boundaries of generators		
Diagonal Ranges	Least generators	Greatest generators
generators δ_2		
{0,1}	(1,0,0,0, 2,0,0,0, 0,0,0,0, 0,0,0,1)	(1,3,3,3, 3,0,3,3, 3,3,1,3, 3,3,3,1)
{0,2}	(2,0,0,0, 0,0,0,0, 1,0,0,0, 0,0,0,2)	(2,3,3,3, 3,2,3,3, 3,3,0,3, 3,3,3,2)
{0,3}	(3,0,0,0, 0,0,0,0, 0,0,0,0, 1,0,1,0)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 2,0,3,0)
{1,2}	(1,0,0,0, 0,2,0,0, 0,0,1,0, 0,0,0,1)	(2,3,3,3, 3,2,3,3, 3,3,1,3, 3,3,3,2)
{1,3}	(1,0,0,0, 0,3,0,0, 0,0,1,0, 0,0 0 1)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,0,1,1)
{2,3}	(2,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,2)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,2,0,2)
all	(1,0,0,0, 0,2,0,0, 0,0,1,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,2,0,2)
generators δ_3		
{0,1,2}	(1,0,0,0, 0,2,0,0, 0,0,0,0, 0,0,0,1)	(2,2,2,2, 2,2,2,2, 2,0,1,2, 2,2,2,0)
{0,1,3}	(1,0,0,0, 0,3,0,0, 0,0,0,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,1,3, 1,3,3,0)
{0,2,3}	(2,0,0,0, 0,0,0,0, 0,0,3,0, 0,0,0,2)	(3,3,3,3, 3,3,3,3, 3,3,0,3, 3,2,3,2)
{1,2,3}	(1,1,1,1, 1,2,1,1, 1,1,3,1, 1,1,1,2)	(3,3,3,3, 3,3,3,3, 3,3,1,3, 3,2,3,2)
all	(1,0,0,0, 0,2,0,0, 0,0,0,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,1,3, 3,2,3,2)
all generators		
	(1,0,0,0, 0,2,0,0, 0,0,0,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,2,0,2)

4 Generators of functions of all ranges

4.1 Rousseau's theorem

The theorem gives criterion of finding generators. There are many other criteria but all of them take a very much time for computing. The time is far less if we use Rousseau's theorem . By the theorem a function $f(x_1, x_2)$ is a generator if it does not fulfill 3 conditions:

- the substitution rule (Martin [16],1954)

$$\exists\ D\ \forall x_1, x_2, x_3, x_4 \quad x_1 \sim x_3 \wedge x_2 \sim x_4(D) \to f(x_1, x_2) \sim f(x_3, x_4)(D)$$

  where D is a decomposition of $\{0, 1, 2, 3\}$ into disjoint subsets, $\sim$ means belonging to the same subset, the number of the decompositions is 16:

$$\{\{0\},\{1,2,3\}\},\{\{1\},\{0,2,3\}\},\{\{2\},\{0,1,3\}\},\{\{3\},\{0,1,2\}\},\{\{0,1\},\{2,3\}\},\{\{0,2\},\{3,4\}\},\{\{0,3\},\{1,2\}\},$$

$$\{\{1,2\},\{0,3\}\},\{\{1,3\},\{0,2\}\},\{\{2,3\},\{0,1\}\},\{\{0\},\{1\},\{2,3\}\},\{\{0\},\{2\},\{1,3\}\},$$

$$\{\{0\},\{3\},\{1,2\}\},\{\{1\},\{2\},\{0,3\}\},\{\{1\},\{3\},\{0,2\}\},\{\{2\},\{3\},\{0,1\}\},$$

- the close rule (Martin [16], 1954):

$$\exists X\ \ X \subset \{0, 1, 2\} \wedge X \neq \varnothing \wedge \forall x_1, x_2 \quad x_1, x_2 \in X \to f(x_1, x_2) \in X$$

- the automorphism rule (Rousseau [20], 1967):

$$\forall s, x_1, x_2 \; f\big(s(x_1, s(x_2)\big) = s\big(f(x_1, x_2)\big)$$

where $s(x) \in \{s_1(x) = (1,0,3,2),\ s_2(x) = (2,3,0,1),\ s_3(x) = (3,2,1,0),\ s_4(x) = (0,1,3,2),\ s_5(x) = (0,2,1,3),$ $s_6(x) = (0,3,2,1),\ s_7(x) = (1,0,2,3),\ s_8(x) = (2,1,0,3),\ s_9(x) = (3,1,2,0)$. The functions s_4-s_9 will be excluded since they are not δ functions. So only 3 functions remain.

4.2 Generators of functions

There are 2-ary functions with the cardinals 2, 3, and 4 of ranges of diagonals. For the every cardinal the number of all functions is 4 294 967 296. For the cardinal 2 there are 6 the ranges of diagonals: {0,1}, {0,2}, {0,3}, {1,2}, {1,3}, {2.3}. Every of the ranges have the same numbers of all functions, of δ functions, and of generators. For the cardinal 3 we have 4 the ranges of diagonals: {0,1,2}, {0,1,3}, {2,1,3}, {1,2,3}. For every range the number of all functions, of δ functions, and of generators are the same. The last is the cardinal 4. There is only one the range of diagonals {0,1,2,3}. The least and greatest generators for every the cardinals and every the ranges of

Table: 4				
Numbers of δ functions and generators				
Cardinals of diagonal ranges	Permutational δ functions	Generators	% of all functions	% of permutational δ functions
2	377 242 320	221 809 968	5	59
3	779 797 200	592 308 864	14	76
4	150994 944	128 778 720	3	85
all	1 308 034 464	942 897 552	22	

the cardinals are presented in table 5. The results of function distributions by the conditions are given in table 6.

Table: 5		
Boundaries of generators		
Diagonal Ranges	Least generators	Greatest generators
generators δ_2		
{0,1}	(1,0,0,0, 2,0,0,0, 0,3,0,0, 0,0,0,0)	(1,3,3,3, 3,0,3,3, 3,3,1,3, 3,2,3,1)
{0,2}	(2,0,0,0, 0,0,0,0, 1,3,0,0, 0,0,0,0)	(2,3,3,3, 3,2,3,3, 3,3,0,3, 3,3,1,2)
{0,3}	(3,0,0,0, 0,0,0,0, 0,0,0,0, 1,2,0,0)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 2,3,1,0)
{1,2}	(1,0,0,0, 0,2,0,0, 0,3,1,0, 0,0,0,1)	(2,3,3,3, 3,2,3,3, 3,3,1,3, 3,3,0,2)
{1,3}	(1,0,0,0, 0,3,0,0, 0,0,1,0, 0,2,0,1)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,2,0,1)
{2,3}	(2,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,1,2)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,0,1,2)
all	(1,0,0,0, 0,2,0,0, 0,3,1,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,2,0,1)
generators δ_3		
{0,1,2}	(1,0,0,0, 0,2,0,0, 0,3,0,0, 0,0,0,0)	(2,3,3,3, 3,2,3,3, 3,3,1,3, 3,3,3,0)
{0,1,3}	(1,0,0,0, 0,3,0,0, 0,0,0,0, 0,2,0,0)	(3,3,3,3, 3,3,3,3, 3,3,1,3, 2,3,3,0)
{0,2,3}	(2,0,0,0, 0,0,0,0, 0,0,3,0, 0,0,1,0)	(3,3,3,3, 3,3,3,3, 3,3,0,3, 3,2,1,2)
{1,2,3}	(1,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,1,3, 3,2,0,2)
all	(1,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,1)	(3,3,3,3, 3,3,3,3, 3,3,0,3, 3,2,1,2)
generators δ_4		
{0,1,2,3}	(1,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,0)	(3,3,3,3, 3,2,3,3, 3,3,1,3, 2,3,3,0)
all generators		
	(1,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,0)	(3,3,3,3, 3,3,3,3, 3,3,3,3, 3,2,0,1)

As it follows from the table, non-generators satisfy only closed and substitution conditions. This means that the conditions are needed for identification of generators. But the automorphic condition is superfluous for generators δ_3.

Table: 6		
Distribution of functions by conditions		
Fulfilled conditions	Numbers	The least functions
Functions for cardinal 1 of ranges of diagonals		
substitution	0	
closing	59 616 648	(0,0,0,0, 0,0,0,0, 0,0,0,0, 1,2,3,0)
substitution and closing	7 492 216	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0)
automorphism	0	
automorphism and substitution	0	
automorphism and closing	0	
all 3	0	
no one	0	
total	67 108 864	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0)
Functions for cardinal 2 of ranges of diagonals		
substitution	529 104	(1,0,0,0, 0,2,0,0, 0,3,1,1, 0,2,1,1)
closing	1 075 792 032	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,2,3,1)
substitution and closing	111 056 736	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,1)
automorphism	23 808	(1,0,0,0, 0,2,0,0, 3,3,1,3, 3,3,3,2)
automorphism and substitution	768	(1,0,0,1, 1,3,0,2, 2,3,3,2, 2,0,3,1)
automorphism and closing	62 112	(0,0,0,0, 0,0,0,2, 1,3,3,3, 3,3,3,3)
all 3	11 616	(0,0,0,0, 0,0,0,0, 2,2,2,2, 2,2,2,2)
no one	221 809 968	(1,0,0,0, 0,2,1,3, 0,2,1,3, 3,3,3,2)
total	1 409 286 144	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,1)
Functions for cardinal 3 of ranges of diagonals		
substitution	857 472	(1,0,0,0, 0,2,0,0, 0,3,0,0, 0,2,0,0)
closing	1 689 461 736	(0,0,0,0, 0,0,0,0, 0,0,1,0, 0,2,3,2)
substitution and closing	133 291 032	(0,0,0,0, 0,0,0,0, 0,0,1,0, 0,0,0,2)
automorphism	0	
automorphism and substitution	0	
automorphism and closing	0	
all 3	0	
no one	592 308 864	(1,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,1)
total	2 415 919 104	(0,0,0,0, 0,0,0,0, 0,0,1,0, 0,0,0,2)
Functions for cardinal 4 of ranges of diagonals		
substitution	150 240	(1,0,0,1, 0,3,0,0, 0,0,0,0, 1,0,0,2)
closing	258 471 948	(0,0,0,0, 0,1,0,0, 0,0,2,0, 1,2,3,3)
substitution and closing	15 154 484	(0,0,0,0, 0,1,0,0, 0,0,2,0, 0,0,0,3)
automorphism	42 558	(1,0,0,0, 0,2,0,0, 2,2,3,2, 2,2,2,0)
automorphism and substitution	3 330	(1,0,0,1, 0,3,0,0, 3,3,0,3 2,3,3,2)
automorphism and closing	43 178	(0,0,0,0, 0,1,0,2, 1,3,2,3, 3,3,3,3)
all 3	8 726	(0,0,0,0, 0,1,0,0, 2,2,2,2, 2,2,2,3)
no one	128 778 720	(1,0,0,0, 0,2,0,0, 0,0,3,0, 0,0,0,0)
total	402 653 184	(0,0,0,0, 0,1,0,0, 0,0,2,0, 0,0,0,3)
All functions		
substitution	1 536 816	(1,0,0,0, 0,2,0,0, 0,3,0,0, 0,2,0,0)
closing	3 083 342 364	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,2,3,1)
substitution and closing	266 994 468	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0)
automorphism	66 366	(1,0,0,0, 0,2,0,0, 2,2,3,2, 2,2,2,0)
automorphism and substitution	4 098	(1,0,0,0, 0,0,0,0, 0,0,0,2, 0,0,3,0)
automorphism and closing	105 290	(0,0,0,0, 0,0,0,2, 1,3,3,3, 3,3,3,3)
all	20 342	(0,0,0,0, 0,0,0,0, 2,2,2,2, 2,2,2,2)
no one	942 897 552	(1,0,0,0, 2,0,0,0, 0,3,0,0, 0,0,0,0)
total	4 294 967 256	(0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0)

References

[1] E.L. Post, *Determination of all closed systems of truth tables*, Bull. Amer. Math. Soc., 26(1920), 437-437.

[2] E.L. Post, *Introduction to a general theory of elementary propositions*, Amer. J. Math., 43(4)(1921), 163-185.

[3] D.L. Webb, *Definition of Post's generalized negative and maximum in terms of one binary operation*, Amer. J. Math., 58(1936), 193-194.

[4] N.M. Martin, *Some analogues of Sheffer stroke functions in n-valued logic*, Indagenous Mathematicae, 12(1950), 373-400.

[5] N.M. Martin, *A note on Sheffer functions in n-valued logic*, Methodos 3(1951), 240-242.

[6] J.D. Swift, *Algebraic properties of n-valued propositional calculi*, Amer. Math. Monthly, 59(1952), 612-621.

[7] E. Götlind, *Some Sheffer functions in n-valued locic*, Port. Math., 12(1952), 141-149.

[8] T. Evans and L. Hardy, *Sheffer stroke unctions in many-valued logics*, Port. Math., 53(1957), 83-93.

[9] R.L. Graham, *On n-values functionally complete truth functions*, J. Symbolic Logic, 32(1967).

[10] A. Salomaa, *Some completeness criteria for sets of functions over a finite domain*, Ann. Univ. Turku, A (63)(1963), 1-19.

[11] F.R. Singer, *Some Sheffer functions for m-valued logics*, Scripta Math., 28(1967), 21-27.

[12] O. Davies Roy, *On m-valued Sheffer functions*, Preprint, University of Leicester, (1968), 201-211.

[13] J.C. Muzio, *Some large classes of n-valued Sheffer functions*, Proc. Camb. Phil. Soc., 74(1973), 190-195.

[14] N.M. Martin, *Direct analogues of the Sheffer stroke functions in m-valued logic*, Notre Dame J. of formal logic, XVII(3)(1976), 415-420.

[15] I.G. Rosenberg, *On generating large classes of Sheffer functions*, Aequationes Mathematicae, 17(1978), 164–181.

[16] N.M. Martin, *Sheffer functions of 3-valued logic*, J. Symbolic logic, 19 (1)(1954), 45-51.

[17] M.A. Malkov, *Complete generators in 3-valued logic and wrong Wheeler's results*, Int. J. of Math. And its Applications, 4(2)(2014).

[18] E.L. Post, *Two-valued iterative systems of mathematical logic*, Princeton Univ. Press, Princeton, (1941).

[19] J. Slupecki, *Kryterium pelnosci wielowartesciowych systemow logiki zdan*, Comptes Rendus des Seances de la Societe des Sciences et des Lettres de Varsovie, III(32)(1939), 102-109 (Polish). *A criterion of functional fullness of many-valued systems of propositional logic*, Studia Logica, XXX(1972), 153-157 (English).

[20] G. Rousseau, *Completeness in finite algebras with a single operation*, Proc. Amer. Math. Soc., 18(1967), 1009-1013.