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Fuzzy PWI-Ideals of Lattice Pseudo-Wajsberg Algebras

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Abstract: In this paper, we introduce the notions of fuzzy pseudo-Wajsberg implicative ideal (fuzzy PWI-ideal) and fuzzy pseudo lattice ideal in lattice pseudo-Wajsberg algebra. Also, we obtain some of their related properties. Moreover, we investigate the properties of fuzzy PWI-ideal related to homomorphism and kernel.

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1. Introduction

The concept of fuzzy subset and various operations on it were first introduced by Zadeh [12] in 1965. A fuzzy subset can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy subset. Individuals may belong in the fuzzy subset to a greater (or) lesser degree as indicated by a larger or smaller membership grade. This membership represents by real-number values ranging in the closed interval between 0 and 1. Since then, fuzzy subsets have been applied to diverse field. The study of fuzzy subset and their application to mathematical contexts has reached to what is now commonly called fuzzy Mathematics. Fuzzy algebra is an important branch of fuzzy Mathematics. The study of fuzzy algebra structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld [1]. Since then these ideas have been applied to other algebraic structures such as semi groups, ring, ideals, module and vector spaces.

Mordchaj Wajsberg [11] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrenns [2]. Also, they [2] defined lattice structure of Wajsberg algebras. Further, they [2] introduced the notion of an implicative filter of lattice Wajsberg algebras and discussed some properties. Pseudo-Wajsberg algebras are generalizations of Wajsberg algebras. Pseudo-Wajsberg algebras were introduced by Rodica Ceterchi [9] in 2001. Rodica Ceterchi [10] introduced the lattice structure of pseudo-Wajsberg algebras and discussed some results in generalized pseudo-Wajsberg algebras. The authors [6] introduce the notions of PWI-ideal and pseudo lattice ideal and investigated some properties of lattice pseudo-Wajsberg algebra.

In the present paper, we introduce the notions of fuzzy pseudo-Wajsberg implicative ideal (fuzzy PWI-ideal) and fuzzy pseudo lattice ideal in lattice pseudo-Wajsberg algebra and discuss some of their properties with illustrations. Finally, we investigate the properties of fuzzy PWI-ideal related to homomorphism and kernel.

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2. Preliminaries

In this section, we recall some basic definitions and its properties that are needful for developing the main results.

Definition 2.1 ([2]). *An algebra $(A, \longrightarrow, -, 1)$ with a binary operation "$\longrightarrow$" and a quasi-complement "$-$" is called a Wajsberg algebra if it satisfies the following axioms for all $x, y, z \in A$*

(1). $1 \longrightarrow x = x$.

(2). $(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x$.

(3). $(x \longrightarrow y) \longrightarrow ((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) = 1$.

(4). $\left(x^- \longrightarrow y^-\right) \longrightarrow (y \longrightarrow x) = 1$.

Definition 2.2 ([9]). *An algebra $(A, \longrightarrow, \rightsquigarrow, -, \sim, 1)$ with the binary operations "$\longrightarrow$", "$\rightsquigarrow$" and the quasi complements "$-$", "$\sim$" is called a pseudo-Wajsberg algebra if it satisfies the following axioms for all $x, y, z \in A$,*

(1). (a). $1 \longrightarrow x = x$.

(b). $1 \rightsquigarrow x = x$.

(2). $(x \rightsquigarrow y) \longrightarrow y = (y \rightsquigarrow x) \longrightarrow x = (y \longrightarrow x) \rightsquigarrow x = (x \longrightarrow y) \rightsquigarrow y$.

(3). (a). $(x \longrightarrow y) \longrightarrow ((y \longrightarrow z) \rightsquigarrow (x \longrightarrow z)) = 1$.

(b). $(x \rightsquigarrow y) \rightsquigarrow ((y \rightsquigarrow z) \longrightarrow (x \rightsquigarrow z)) = 1$.

(4). $1^- = 1^\sim = 0$.

(5). (a). $\left(x^- \rightsquigarrow y^-\right) \longrightarrow (y \longrightarrow x) = 1$.

(b). $(x^\sim \longrightarrow y^\sim) \rightsquigarrow (y \rightsquigarrow x) = 1$.

(6). $(x \longrightarrow y^-)^\sim = (y \rightsquigarrow x^\sim)^-$.

Definition 2.3 ([10]). *An algebra $(A, \wedge, \vee, \longrightarrow, \rightsquigarrow, -, \sim, 1)$ is called a lattice pseudo-Wajsberg algebras if it satisfies the following axioms for all $x, y \in A$,*

(1). A partial ordering "$\leq$" on a lattice pseudo-Wajsberg algebra A such that $x \leq y$ if and only if $x \longrightarrow y = 1 \Leftrightarrow x \rightsquigarrow y = 1$.

(2). $x \vee y = (x \longrightarrow y) \rightsquigarrow y = (y \longrightarrow x) \rightsquigarrow x = (x \rightsquigarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x$.

(3).
$$\begin{aligned} x \wedge y &= (x \rightsquigarrow (x \longrightarrow y)^\sim)^- = \left((x \longrightarrow y) \longrightarrow x^-\right)^\sim \\ &= \left(y \longrightarrow (y \rightsquigarrow x)^-\right)^\sim = ((y \rightsquigarrow x) \rightsquigarrow y^\sim)^- \\ &= (y \rightsquigarrow (y \longrightarrow x)^\sim)^- = \left((y \longrightarrow x) \longrightarrow y^-\right)^\sim \\ &= (x \longrightarrow (x \rightsquigarrow y)^-)^\sim = ((x \rightsquigarrow y) \rightsquigarrow x^\sim)^-. \end{aligned}$$

Proposition 2.4 ([10]). *In a lattice pseudo-Wajsberg algebra $(A, \longrightarrow, \rightsquigarrow, -, \sim, 1)$ which satisfies the following axioms for all $x, y, z \in A$,*

(1). (a). $x \longrightarrow x = 1$

(b). $x \rightsquigarrow x = 1$

(2). (a). If $x \longrightarrow y = 1$ *and* $y \longrightarrow x = 1$*, then* $x = y$*.*

(b). If $x \rightsquigarrow y = 1$ *and* $y \rightsquigarrow x = 1$*, then* $x = y$*.*

(c). If $x \longrightarrow y = 1$ *and* $y \rightsquigarrow x = 1$*, then* $x = y$*.*

(3). (a). $(x \longrightarrow 1) \rightsquigarrow 1 = 1$*.*

(b). $(x \rightsquigarrow 1) \longrightarrow 1 = 1$*.*

(4). (a). $x \longrightarrow 1 = 1$*.*

(b). $x \rightsquigarrow 1 = 1$*.*

(5). (a). If $x \longrightarrow y = 1$ *and* $y \longrightarrow z = 1$*, then* $x \longrightarrow z = 1$*.*

(b). If $x \rightsquigarrow y = 1$ *and* $y \rightsquigarrow z = 1$*, then* $x \rightsquigarrow z = 1$*.*

(6). (a). $x \longrightarrow (y \rightsquigarrow x) = 1$*.*

(b). $x \rightsquigarrow (y \longrightarrow x) = 1$*.*

(7). $x \longrightarrow (y \rightsquigarrow z) = 1 \Leftrightarrow y \rightsquigarrow (x \longrightarrow z) = 1$*.*

(8). (a). $(x \longrightarrow y) \rightsquigarrow ((z \longrightarrow x) \longrightarrow (z \longrightarrow y)) = 1$*.*

(b). $(x \rightsquigarrow y) \longrightarrow ((z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)) = 1$*.*

(9). $x \longrightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \longrightarrow z)$*.*

Proposition 2.5 ([10])**.** *In a lattice pseudo-Wajsberg algebra* $(A, \longrightarrow, \rightsquigarrow, -, \sim, 1)$ *which satisfies the following axioms for all* $x, y \in A$*,*

(1). (a). $(x^{-} \rightsquigarrow 0) \longrightarrow x = 1$*.*

(b). $(x^{\sim} \longrightarrow 0) \rightsquigarrow x = 1$*.*

(2). $0 \longrightarrow x = 1 = 0 \rightsquigarrow x$*.*

(3). (a). $x \longrightarrow 0 = x^{-}$*.*

(b). $x \rightsquigarrow 0 = x^{\sim}$*.*

(4). $(x^{-})^{\sim} = x = (x^{\sim})^{-}$*.*

(5). (a). $x^{\sim} \longrightarrow y^{\sim} = y \rightsquigarrow x$*.*

(b). $x^{-} \rightsquigarrow y^{-} = y \longrightarrow x$*.*

(6). $x^{\sim} \longrightarrow y = y^{-} \rightsquigarrow x$*.*

(7). $x \leq y \Longleftrightarrow y^{-} \leq x^{-} \Longleftrightarrow y^{\sim} \leq x^{\sim}$*.*

(8). (a). $(x \longrightarrow y)^{\sim} = (y^{\sim} \rightsquigarrow x^{\sim})^{-}$*.*

(b). $(x \rightsquigarrow y)^{-} = (y^{-} \longrightarrow x^{-})^{\sim}$*.*

Proposition 2.6 ([10])**.** *In a lattice pseudo-Wajsberg algebra* $(A, \longrightarrow, \rightsquigarrow, -, \sim, 1)$ *which satisfies the following axioms for all* $x, y, z \in A$*,* *(1). (a).* $(x \vee y)^{-} = x^{-} \wedge y^{-}$; $(x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}$.

(b). $(x \wedge y)^{-} = x^{-} \vee y^{-}$; $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$.

(2). (a). $(x \vee y) \longrightarrow z = (x \longrightarrow z) \wedge (y \longrightarrow z)$.

(b). $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$.

(3). (a). $z \longrightarrow (x \wedge y) = (z \longrightarrow x) \wedge (z \longrightarrow y)$.

(b). $z \rightsquigarrow (x \wedge y) = (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)$.

(4). (a). $(x \vee y) \longrightarrow y = x \longrightarrow y$.

(b). $(x \vee y) \rightsquigarrow y = x \rightsquigarrow y$.

(5). (a). $x \longrightarrow (x \wedge y) = x \longrightarrow y$.

(b). $x \rightsquigarrow (x \wedge y) = x \rightsquigarrow y$.

(6). (a). $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$.

(b). $(x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

(7). (a). $x \longrightarrow (y \vee z) = (x \longrightarrow y) \vee (x \longrightarrow z)$.

(b). $x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$.

(8). (a). $(x \wedge y) \longrightarrow z = (x \longrightarrow z) \vee (y \longrightarrow z)$.

(b). $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$.

(9). $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

(10). (a). $(x \wedge y) \longrightarrow z = (x \longrightarrow y) \longrightarrow (x \longrightarrow z) = (y \longrightarrow x) \longrightarrow (y \longrightarrow z)$.

(b). $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z)$.

(11). (a). $z \longrightarrow (x \vee y) = (x \longrightarrow y) \rightsquigarrow (z \longrightarrow y) = (y \longrightarrow x) \rightsquigarrow (z \longrightarrow x)$.

(b). $z \rightsquigarrow (x \vee y) = (x \rightsquigarrow y) \longrightarrow (z \rightsquigarrow y) = (y \rightsquigarrow x) \longrightarrow (z \rightsquigarrow x)$.

Definition 2.7 ([4])**.** *A non-empty subset I of Wajsberg algebra A is called an ideal if it satisfies the following axioms for all $x, y \in A$,*

(1). $0 \in I$.

(2). $x \in I$ *and* $y \leq x$ *imply* $y \in I$.

(3). $x, y \in I$ *imply* $x^{-} \longrightarrow y \in I$.

Definition 2.8 ([5])**.** *Let A be a lattice Wajsberg algebra. Let I be a nonempty subset of A. Then, I is called a WI-ideal of A satisfies the following axioms for all $x, y \in A$,*

(1). $0 \in I$.

(2). $(x \longrightarrow y)^{-} \in I$ *and* $y \in I$ *imply* $x \in I$.

Definition 2.9 ([6])**.** *Let A be a lattice pseudo-Wajsberg algebra. Let F be a non-empty subset of A is called a PWI-ideal of A if it satisfies the following axioms for all $x, y \in A$,* *(1).* $0 \in F$.

(2). $y \in F$ *and* $(x \longrightarrow y)^{-} \in F$ *imply* $x \in F$.

(3). $y \in F$ *and* $(x \rightsquigarrow y)^{\sim} \in F$ *imply* $x \in F$.

Definition 2.10 ([6])**.** *Let A be a lattice pseudo-Wajsberg algebra. A PWI-ideal F of A is a nonempty subset of A is called a pseudo lattice ideal if it satisfies the following axioms for all $x, y \in A$,*

(1). $0 \in F$.

(2). $x \in F$ *and* $y \leq x$ *imply* $y \in F$.

(3). $x, y \in F$ *imply* $x \vee y \in F$.

Definition 2.11 ([6])**.** *Let A and B be two lattice pseudo-Wajsberg algebras. A function $h : A \longrightarrow B$ is a homomorphism if and only if it satisfies the following axioms for all $x, y \in A$,*

(1). $h(0) = 0$.

(2). $h(x \longrightarrow y) = h(x) \longrightarrow h(y)$.

(3). $h(x \rightsquigarrow y) = h(x) \rightsquigarrow h(y)$.

(4). $h\left(x^{-}\right) = (h(x))^{-}$.

(5). $h(x^{\sim}) = (h(x))^{\sim}$.

Definition 2.12 ([6])**.** *Let A and B be two lattice pseudo-Wajsberg algebras. The kernel of a homomorphism $h : A \rightarrow B$ is a set and denoted as $Ker(h) = \{x \in A / h(x) = 0\}$.*

Remark 2.13 ([6])**.** *If $h : A \rightarrow B$ is one to one then h is an injective homomorphism. If the homomorphism $h : A \rightarrow B$ is onto then h is surjective. If h is one to one and onto then h is called an isomorphism.*

Definition 2.14 ([10])**.** *Let A be a set. A function $\mu : A \rightarrow [0, 1]$ is called a fuzzy subset on A, for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of x in μ.*

Definition 2.15 ([10])**.** *Let μ be a fuzzy set in a set A. Then for $t \in [0, 1]$, the set $\mu_t = \{x \in A / \mu(x) \geq t\}$ is called level subset of μ.*

Definition 2.16 ([10])**.** *Let μ be a fuzzy set in a set A. Then for $t \in [0, 1]$, the set $\mu^t = \{x \in A / \mu(x) \leq t\}$ is called the lower t-level cut of μ.*

3. Main Results

3.1. Fuzzy Pseudo-Wajsberg Implicative Ideal (Fuzzy PWI-ideal)

In this section, we define fuzzy PWI-ideal of lattice pseudo-Wajsberg algebra and obtain some results with illustrations.

Definition 3.1. *Let A be a lattice pseudo-Wajsberg algebra. A fuzzy subset μ of A is called a fuzzy PWI-ideal of A if it satisfies the following for all $x, y \in A$,*

(1). $\mu(0) \geq \mu(x)$.

(2). $\mu(x) \geq \min\{\mu((x \longrightarrow y)^{-}), \mu(y)\}$.

(3). $\mu(x) \geq \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\}$.

Example 3.2. *Consider a set $A = \{0, a, b, c, 1\}$. Define a partial ordering "$<$" on A, such that $0 < a < b < c < 1$ and the binary operations "$\longrightarrow$", "$\rightsquigarrow$" and the quasi complements "$-$", "$\sim$" given by the following Tables 1, 2, 3 and 4.*

x	x^{-}
0	1
a	b
b	a
c	a
1	0

Table 1

$\longrightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Table 2

x	$x^{\sim}$
0	1
a	c
b	a
c	0
1	0

Table 3

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	c

Table 4

Then, $A = (A, \wedge, \vee, \longrightarrow, \rightsquigarrow, 0, 1)$ is a lattice pseudo-Wajsberg algebra and consider the fuzzy subsets μ_1 and μ_2 on A as,

$$\mu_1(x) = \begin{cases} 0.8 & \text{if } x \in \{0, a\} \\ 0.2 & \text{if } x \in \{b, c, 1\} \end{cases}$$

Then $\mu_1(x)$ is a fuzzy PWI-ideal of A. But,

$$\mu_2(x) = \begin{cases} 0.6 & \text{if } x \in \{0, 1\} \\ 0.2 & \text{if } x \in \{a, b, c\} \end{cases}$$

Then $\mu_2(x)$ is not a fuzzy PWI-ideal of A, since $\mu(a) \geq \min\{\mu((a \longrightarrow 1)^{-}), \mu(1)\}$ and $\mu(a) \geq \min\{\mu((a \rightsquigarrow 1)^{\sim}), \mu(1)\} \Rightarrow \mu(a) \geq \min\{\mu(0), \mu(1)\}$ implies that $0.4 \ngeq 0.6$. Thus $\mu(a) \ngeq \min\{\mu((a \longrightarrow 1)^{-}), \mu(1)\}$.

Example 3.3. *Consider a set $A = \{0, a, b, c, 1\}$. Define a partial ordering "$<$" on A, such that $0 < a < b < c < 1$ and the binary operations "$\longrightarrow$", "$\rightsquigarrow$" and the quasi complements "$-$", "$\sim$" given by the following Tables 5, 6, 7 and 8. Then*

x	x^{-}
0	1
a	b
b	c
c	a
1	0

Table 5

$\longrightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	c	c	1	c	1
c	a	b	b	1	1
1	0	a	b	c	1

Table 6

x	$x^{\sim}$
0	1
a	b
b	a
c	b
1	0

Table 7

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

Table 8

$A = (A, \wedge, \vee, \longrightarrow, \rightsquigarrow, 0, 1)$ is a lattice pseudo-Wajsberg algebra and consider the fuzzy subsets μ_1 and μ_2 on A as,

$$\mu_1(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{otherwise} \end{cases}$$

Then $\mu_1(x)$ *is fuzzy PWI-ideal of A. But,*

$$\mu_2(x) = \begin{cases} 0.7 & \text{if } x \in \{0, a\} \\ 0.2 & \text{if } x \in \{b, c, 1\} \end{cases}$$

Then $\mu_2(x)$ *is not a fuzzy PWI-ideal of A. Since*

$$\mu(b) \geq \min\{\mu((b \longrightarrow 0)^{-}),\ \mu(0)\} = \mu(b) \geq \min\{\mu(a),\ \mu(0)\} = 0.3 \ngeq 0.7$$

$$\mu(b) \geq \min\{\mu((b \rightsquigarrow 0)^{\sim}),\ \mu(0)\} = \mu(b) \geq \min\{\mu(b),\ \mu(0)\} = 0.3 = 0.3$$

Thus $\mu(b) \ngeq \min\{\mu((b \longrightarrow 0)^{-}), \mu(0)\}$.

Proposition 3.4. *Every fuzzy PWI-ideal* μ *of a lattice pseudo-Wajsberg algebra A is order reversing.*

Proof. If $y \leq x$, then $(y \longrightarrow x)^{-} = (y \rightsquigarrow x)^{\sim} = 1^{-} = 1^{\sim} = 0 \in F$ and, so

$$\mu(y) \geq \min\{\mu((y \longrightarrow x)^{-}), \mu(x)\} = \min\{\mu((y \rightsquigarrow x)^{\sim}), \mu(x)\} \geq \min\{\mu(0), \mu(x)\} = \mu(x).$$

Thus $\mu(y) \geq \mu(x)$. This shows that μ is order reversing. □

Definition 3.5. *Let A be a lattice pseudo-Wajsberg algebra. A fuzzy subset* μ *of A is called a fuzzy pseudo lattice ideal if it satisfies the following for all* $x, y \in A$,

(1). If $y \leq x$ *then* $\mu(y) \geq \mu(x)$.

(2). $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

Example 3.6. *Consider a set* $A = \{0, a, b, 1\}$. *Define a partial ordering "<" on A, such that* $0 < a < 1$; $0 < b < 1$ *and the binary operations "*$\longrightarrow$*", "*$\rightsquigarrow$*" and the quasi complements "*$-$*", "*$\sim$*" given by the following Tables 9, 10, 11 and 12.*

x	x^{-}
0	1
a	b
b	a
1	0

Table 9

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Table 10

x	$x^{\sim}$
0	1
a	b
b	a
1	0

Table 11

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Table 12

Then, $A = (A, \wedge, \vee, \longrightarrow, \rightsquigarrow, 0, 1)$ *is a lattice pseudo-Wajsberg algebra and consider the fuzzy subsets* μ_1 *and* μ_2 *on A as*

$$\mu_1(x) = \begin{cases} 0.5 & \text{if } x \in \{0, b\} \\ 0.3 & \text{if } x \in \{a, 1\} \end{cases}$$

Then $\mu_1(x)$ *is a fuzzy pseudo lattice ideal of A. But,*

$$\mu_2(x) = \begin{cases} 0.8 & \text{if } x \in \{0, a, b\} \\ 0.2 & \text{if } x \in \{1\} \end{cases}$$

Then $\mu_2(x)$ *is not a fuzzy pseudo lattice ideal of A, [from (2) of Definition 2.3]. Since* $\mu(a \vee b) = \mu((b \rightsquigarrow a) \longrightarrow a) = \mu((a \longrightarrow a)) = \mu(1) \geq \min\{\mu(a), \mu(b)\} = 0.2 \ngeq 0.8$. *Thus* $\mu(a \vee b) \ngeq \min\{\mu(a), \mu(b)\}$.

Proposition 3.7. *Let A be a lattice pseudo-Wajsberg algebra. Every fuzzy PWI-ideal of A is a fuzzy pseudo lattice ideal.*

Proof. Let μ be a fuzzy PWI-ideal of A, Proposition 3.4 shows that F satisfies (1) of Definition 3.5. Let $x, y \in A$ then,

$$((x \vee y) \longrightarrow y)^{-} = (x \longrightarrow y)^{-} \leq x \quad \text{[from (4) (a) of Proposition 2.6]}$$
$$((x \vee y) \rightsquigarrow y)^{\sim} = (x \rightsquigarrow y)^{\sim} \leq x \quad \text{[from (4) (b) of Proposition 2.6]}$$

It is enough to prove

$$\begin{aligned} \mu(x \vee y) &\geq \min\{\mu(x), \mu(y)\} \\ \mu(x \vee y) &\geq \min\left\{\mu\left(((x \vee y) \longrightarrow y)^{-}\right), \mu(y)\right\} \\ &\geq \min\{\mu(((x \vee y) \rightsquigarrow y)^{\sim}), \mu(y)\} \\ &\geq \min\{\mu(x), \mu(y)\}. \end{aligned}$$

□

Remark 3.8. *The converse of Proposition 3.7 may not be true. In Example 3.6, μ is a fuzzy subset of A defined by*

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in \{0, b\} \\ 0.4 & \text{if } x \in \{a, 1\} \end{cases}$$

Then μ is a fuzzy pseudo lattice ideal of A, but not a fuzzy PWI-ideal, since$\mu(a) \geq \min\left\{\mu\left((a \longrightarrow b)^{-}\right), \mu(b)\right\} = 0.4 \geq \min\{\mu(0), \mu(b)\} = 0.4 \ngeq 0.7$. *Thus* $\mu(a) \ngeq \min\left\{\mu\left((a \longrightarrow b)^{-}\right), \mu(b)\right\}$.

Proposition 3.9. *Let μ be a fuzzy subset of lattice pseudo-Wajsberg algebra A. Then μ is a fuzzy PWI-ideal if and only if μ_t is a PWI-ideal when $\mu_t \neq \phi$, $t \in [0,1]$.*

Proof. Let μ be a fuzzy PWI-ideal of A, when $\mu_t \neq 0$ such that $t \in [0,1]$. Clearly $0 \in \mu_t$. Suppose $x, y \in A$, we have

(1). If $(x \longrightarrow y)^{-} \in \mu_t$, $y \in \mu_t$ then $\mu\left((x \longrightarrow y)^{-}\right) \geq t$ and $mu(y) \geq t$.

(2). If $(x \rightsquigarrow y)^{\sim} \in \mu_t$, $y \in \mu_t$ then $\mu((x \rightsquigarrow y)^{\sim}) \geq t$ and $\mu(y) \geq t$.

It follows that, $\mu(x) \geq \min\left\{\mu\left((x \longrightarrow y)^{-}\right), \mu(y)\right\} \geq t$ and $\mu(x) \geq \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\} \geq t$ so that $x \in \mu_t$. Hence μ_t is a PWI-ideal of A.

Conversely, suppose that μ_t; $[t \in [0,1]]$ is a PWI-ideal of A. When $\mu_t \neq \phi$ for any $x \in A$, $x \in \mu_{\mu(x)}$ it follows that $\mu_{\mu(x)}$ is a PWI-ideal of A. Hence $0 \in \mu_{\mu(x)}$ that is $\mu(0) \geq \mu(x)$ for any $x, y \in A$. Let $t = \min\left\{\mu\left((x \longrightarrow y)^{-}\right), \mu(y)\right\}$ (or) $t = \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\}$ it follows that μ_t is a PWI-ideal. Therefore $(x \longrightarrow y)^{-}$, $(x \rightsquigarrow y)^{\sim} \in \mu_t$ and $y \in \mu_t$ this implies that $x \in \mu_t$, and also

$$\mu(x) \geq t = \min\left\{\mu\left((x \longrightarrow y)^{-}\right), \mu(y)\right\} \quad \textit{and} \quad \mu(x) \geq t = \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\}.$$

Let $t = \min\left\{\mu\left((x \longrightarrow y)^{-}\right), \mu(y)\right\}$, and also μ_t is a PWI-ideal. Therefore $(x \longrightarrow y)^{-} \in \mu_t$ and $y \in \mu_t$ implies $x \in \mu_t$. Similarly, let $t = \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\}$, and also μ_t is a PWI-ideal. Therefore $(x \rightsquigarrow y)^{\sim} \in \mu_t$ and $y \in \mu_t$ implies $x \in \mu_t$. Therefore $\mu(x) \geq t = \min\left\{\mu\left((x \longrightarrow y)^{-}\right), \mu(y)\right\}$ and $\mu(x) \geq t = \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\}$. Thus μ is a fuzzy PWI-ideal of A. □

Proposition 3.10. *A fuzzy subset μ in A if and only if satisfies the following for all $x, y, z \in A$.*

(1). $\mu(0) \geq \mu(x)$.

(2). If $\left((z \longrightarrow y)^{-} \longrightarrow x\right)^{-} = 0$, *then* $\mu(z) \geq \min\{\mu(x), \mu(y)\}$.

(3). If $((z \rightsquigarrow y)^{\sim} \rightsquigarrow x)^{\sim} = 0$, *then* $\mu(z) \geq \min\{\mu(x), \mu(y)\}$.

Proof. Let μ be a fuzzy PWI-ideal and let $x, y, z \in A$. Suppose that $\left((z \longrightarrow y)^{-} \longrightarrow x\right)^{-} = 0$, since μ is a fuzzy PWI-ideal, we have

$$\mu\left((z \longrightarrow y)^{-}\right) \geq \min\left\{\mu\left(\left((z \longrightarrow y)^{-} \longrightarrow x\right)^{-}\right), \mu(x)\right\} = \min\{\mu(0), \mu(x)\} = \mu(x),$$

$$\mu\left((z \rightsquigarrow y)^{\sim}\right) \geq \min\{\mu(((z \rightsquigarrow y)^{\sim} \rightsquigarrow x)^{\sim}), \mu(x)\} = \min\{\mu(0), \mu(x)\} = \mu(x)$$

$$\text{and} \quad \mu(z) \geq \min\left\{\mu\left((z \longrightarrow y)^{-}\right), \mu(y)\right\}, \mu(z) \geq \min\{\mu((z \rightsquigarrow y)^{\sim}), \mu(y)\}.$$

Therefore, we get $\mu(z) \geq \min\{\mu(x), \mu(y)\}$.

Conversely, let μ satisfies (1) (a) (b) of Proposition 2.4, we have $(x \longrightarrow y)^{-} \longrightarrow (x \longrightarrow y)^{-} = 1$ and $(x \rightsquigarrow y)^{\sim} \rightsquigarrow (x \rightsquigarrow y)^{\sim} = 1$. From (2) and (3) of Definition 3.1, $\mu(x) \geq \min\left\{\mu\left((x \longrightarrow y)^{-}\right), \mu(y)\right\}, \mu(x) \geq \min\{\mu((x \rightsquigarrow y)^{\sim}), \mu(y)\}$. Then μ is fuzzy PWI-ideal of A. □

3.2. Properties of homomorphism and kernel

In this section, we investigate some of the properties of homomorphism and kernel in fuzzy PWI-ideal of lattice pseudo-Wajsberg algebra.

Definition 3.11. *Let A and B be two lattice pseudo-Wajsberg algebras, μ be any fuzzy subset in A, and $f : A \to B$ be any function. Set $\overline{f}(y) = \{x \in A : f(x) = y\}$ for $y \in B$. The fuzzy subset ν in B defined by,*

$$\nu(y) = \begin{cases} \sup\left\{\mu(x) : x \in \overline{f}(y)\right\} & \textit{if } \overline{f}(y) \neq \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

for all $y \in B$, is called the image of μ under f and is denoted by $f(\mu)$.

Definition 3.12. *Let A and B be lattice pseudo-Wajsberg algebras, $f : A \to B$ be any function and let ν be any fuzzy subset in $f(A)$. The fuzzy subset μ in A defined by $\mu(x) = \nu(f(x))$, for all $x \in A$ is called the preimage of ν under f and is denoted by $\overline{f}(\nu)$.*

Proposition 3.13. *Let A and B be two lattice pseudo-Wajsberg algebras and let $f : A \to B$ be a homomorphism and ν is fuzzy PWI-ideal of B then $\overline{f}(\nu)$ is a fuzzy PWI-ideal of A.*

Proof. Let $x \in A$. Since $f(x) \in B$ and ν is fuzzy PWI-ideal of B. We have $\nu(0) \geq \nu(f(x)) = \left(\overline{f}(\nu)\right)(x)$ but $\nu(0) \geq \nu(f(0)) = \left(\overline{f}(\nu)\right)(0)$. Thus, we get $\left(\overline{f}(\nu)\right)(0) \geq \left(\overline{f}(\nu)\right)(x)$ for any $x \in A$ that is $\overline{f}(\nu)$ satisfies (1) of Definition 3.1. Now let $x, y \in A$ since ν is a fuzzy PWI-ideal of B, we obtain

$$\nu(f(x)) \geq \min\left\{\nu\left((f(x) \longrightarrow f(y))^{-}\right), \nu(f(y))\right\}$$

$$\nu(f(x)) \geq \min\left\{\nu\left(f(x \longrightarrow y)^{-}\right), \nu(f(y))\right\} \quad \text{[from (2) of Definition 2.11]}$$

Hence $\left(\overline{f}\left(\nu\right)\right)(x) \geq \min\left\{\left(\overline{f}\left(\nu\right)\right)\left((x \longrightarrow y)^{-}\right), \left(\overline{f}\left(\nu\right)\right)(y)\right\}$, and also

$$\nu\left(f\left(x\right)\right) \geq \min\left\{\nu\left((f(x) \rightsquigarrow f(y))^{\sim}\right), \nu\left(f\left(y\right)\right)\right\}$$

$$\nu\left(f\left(x\right)\right) \geq \min\left\{\nu\left((f(x \rightsquigarrow y)^{\sim}\right), \nu\left(f\left(y\right)\right)\right\} \quad \text{[from (3) of Definition 2.11]}$$

Hence $\left(\overline{f}\left(\nu\right)\right)(x) \geq \min\left\{\left(\overline{f}\left(\nu\right)\right)\left((x \rightsquigarrow y)^{\sim}\right), \left(\overline{f}\left(\nu\right)\right)(y)\right\}$. Then $\overline{f}\left(\nu\right)$ is a fuzzy PWI-ideal of A. □

Proposition 3.14. *Let A and B be two lattice pseudo-Wajsberg algebras and let $f : A \to B$ be a homomorphism and μ be a fuzzy PWI-ideal of A. If μ is a constant on $Ker f = \overline{f}\left(0\right)$, then $\overline{f}\left(f\left(\mu\right)\right) = \mu$.*

Proof. Let $x \in A$ and $f\left(x\right) = y$. Hence $\left(\overline{f}\left(f\left(\mu\right)\right)\right)(x) = \left(f\left(\mu\right)\right)\left(f\left(x\right)\right) = \left(f\left(\mu\right)\right)(y) = \min\left\{\mu\left(a\right) : a \in \overline{f}\left(y\right)\right\}$ for all $a \in \overline{f}\left(y\right)$, we have $f\left(x\right) = f\left(a\right)$. Hence

(1). $f\left((a \longrightarrow x)^{-}\right) = 0$, that $(a \longrightarrow x)^{-} \in Ker\ f$.

(2). $f\left((a \rightsquigarrow x)^{\sim}\right) = 0$, that $(a \rightsquigarrow x)^{\sim} \in Ker\ f$.

Thus $\mu\left((a \longrightarrow x)^{-}\right) = \mu\left(0\right)$ and $\mu\left((a \rightsquigarrow x)^{\sim}\right) = \mu\left(0\right)$. Therefore, $\mu\left(a\right) \geq \min\left\{\mu\left((a \longrightarrow x)^{-}\right), \mu\left(x\right)\right\} \geq \min\left\{\mu\left(0\right), \mu\left(x\right)\right\} \geq \mu\left(x\right)$, and also $\mu\left(a\right) \geq \min\left\{\mu\left((a \rightsquigarrow x)^{\sim}\right), \mu\left(x\right)\right\} \geq \min\left\{\mu\left(0\right), \mu\left(x\right)\right\} \geq \mu\left(x\right)$ that is $\mu\left(x\right) \geq \mu\left(a\right)$. Similarly, $\mu\left(a\right) \geq \mu\left(x\right)$. Hence, we have $\mu\left(a\right) = \mu\left(x\right)$. Thus, we have $\left(\overline{f}\left(f\left(\mu\right)\right)\right)(x) = \min\left\{\mu\left(a\right) : a \in \overline{f}\left(y\right)\right\} = \mu\left(x\right)$ that is, $\overline{f}\left(f\left(\mu\right)\right) = \mu$. □

Remark 3.15. *If μ is a fuzzy PWI-ideal of lattice pseudo-Wajsberg algebra A, then set $A_{\mu} = \{x \in A : \mu\left(x\right) = \mu\left(0\right)\}$ is a PWI-ideal of A. But the converse may not be true. In Example 3.3, μ is a fuzzy subset of A defined by*

$$\mu\left(x\right) = \begin{cases} 0.4 & \text{if } x \in \{0\} \\ 0.7 & \text{if } x \in \{a, b, c, 1\} \end{cases}$$

then $A_{\mu} = \{0\}$ is a PWI-ideal of A, but μ is not a fuzzy PWI-ideal of A, since $\mu\left(0\right) \ngeq \mu\left(x\right)$.

Proposition 3.16. *Let A and B be two lattice pseudo-Wajsberg algebras and let $f : A \to B$ be a surjective homomorphism and μ is a fuzzy PWI-ideal of A such that $Ker\ f \subseteq A_{\mu}$. Then $f\left(\mu\right)$ is a fuzzy PWI-ideal of B.*

Proof. Since μ is a fuzzy PWI-ideal of A and $0 \in \overline{f}\left(0\right)$, we have $\left(f\left(\mu\right)\right)(0) = \min\left\{\mu\left(a\right) : a \in \overline{f}\left(0\right)\right\} = \mu\left(0\right) \geq \mu\left(x\right)$ for any $x \in A$. Hence, $\left(f\left(\mu\right)\right)(0) \geq \min\left\{\mu\left(x\right) : x \in \overline{f}\left(y\right)\right\} = \left(f\left(\mu\right)\right)(y)$ for any $y \in B$. Thus $f\left(\mu\right)$ satisfies (1) of Definition of 3.1. Suppose that

(1). $\left(f\left(\mu\right)\right)(x_B) < \min\left\{\left(f\left(\mu\right)\right)\left((x_B \longrightarrow y_B)^{-}\right), \left(f\left(\mu\right)\right)(y)\right\}$.

(2). $\left(f\left(\mu\right)\right)(x_B) < \min\left\{\left(f\left(\mu\right)\right)\left((x_B \rightsquigarrow y_B)^{\sim}\right), \left(f\left(\mu\right)\right)(y)\right\}$ for some $x_B, y_B \in B$.

Since f is surjective, there are $x_A, y_A \in A$ such that $f\left(x_A\right) = x_B$ and $f\left(y_A\right) = y_B$. Hence,

(1). $\left(f\left(\mu\right)\right)\left(f(x_A)\right) < \min\left\{\left(f\left(\mu\right)\right)\left(f\left((x_A \longrightarrow y_A)^{-}\right)\right), \left(f\left(\mu\right)\right)\left(f\left(y_A\right)\right)\right\}$.

(2). $\left(f\left(\mu\right)\right)\left(f(x_A)\right) < \min\left\{\left(f\left(\mu\right)\right)\left(f\left((x_A \rightsquigarrow y_A)^{\sim}\right)\right), \left(f\left(\mu\right)\right)\left(f\left(y_A\right)\right)\right\}$.

Therefore,

(1). $\left(\overline{f}\left(f\left(\mu\right)\right)\right)(x_A) < \min\left\{\left(\overline{f}\left(f\left(\mu\right)\right)\right)\left((x_A \longrightarrow y_A)^{-}\right), \left(\overline{f}\left(f\left(\mu\right)\right)\right)(y_A)\right\}$.

(2). $\left(\overline{f}\left(f\left(\mu\right)\right)\right)\left(x_A\right) < \min\left\{\left(\overline{f}\left(f\left(\mu\right)\right)\right)\left(\left(x_A \rightsquigarrow y_A\right)^{\sim}\right), \left(\overline{f}\left(f\left(\mu\right)\right)\right)\left(y_A\right)\right\}$.

Since, $Ker\ f \subseteq A_\mu$, μ is constant on $Ker\ f$, then from the Proposition 3.2, we get

(1). $\mu\left(x_A\right) < \min\left\{\mu\left(\left(x_A \longrightarrow y_A\right)^{-}\right), \mu\left(y_A\right)\right\}$.

(2). $\mu\left(x_A\right) < \min\left\{\mu\left(\left(x_A \rightsquigarrow y_A\right)^{\sim}\right), \mu\left(y_A\right)\right\}$.

Which is a contradiction with the fact that μ is a fuzzy PWI-ideal. Thus, we have $f\left(\mu\right)$ is a fuzzy PWI-ideal of B. □

4. Conclusion

In this paper, we have introduced the notions of fuzzy PWI-ideal and fuzzy pseudo lattice ideal of lattice pseudo-Wajsberg algebra and discussed some of their properties with illustrations. Finally, we have investigated the properties of fuzzy PWI-ideals related to homomorphism and kernel of lattice pseudo-Wajsberg algebra.

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