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Common Fixed Point Theorems in Menger Space

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Abstract: In this paper, the concept of occasionally weakly compatible maps in Menger space has been introduced to prove common fixed point theorems. Which generalize the well known results.

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1. Introduction and Preliminaries

In 1942 Menger [9] introduced the notion of a probabilistic metric space (PM-space) which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pair, say (p, q), denoted by $F(p, q, t)$ where $t > 0$ and in-terpret this function as the probability that distance between p and q is less than t, whereas in the metric space the distance function is a single positive number. Sehgal [21] initiated the study of fixed points in probabilistic metric spaces. The important development of fixed point theory in Menger spaces were due to Sehgal and Bharucha-Reid [21]. A probabilistic metric space shortly PM-Space, is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X \to L$, where L is the collection of all distribution functions (a distribution function F is non decreasing and left continuous mapping of reals in to $[0, 1]$ with properties, $\inf F(x) = 0$ and $\sup F(x) = 1$).

(1). The value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The function $F_{x,y}$ are assumed satisfy the following conditions;

(2). $(FM-0)$ $F_{x,y}(t) = 1$, for all $t > 0$, iff $x = y$;

(3). $(FM-1)$ $F_{x,y}(0) = 0$, if $t = 0$;

(4). $(FM-2)$ $F_{x,y}(t) = F_{y,x}(t)$;

(5). $(FM-3)$ $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$.

(6). A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm, if it satisfies the following conditions;

(7). $(FM-4)$ $T(a, 1) = a$ for every $a \in [0, 1]$;

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(8). $(FM-5)$ $T(0,0)=0$,

(9). $(FM-6)$ $T(a,b)=T(b,a)$ for every $a,b\in[0,1]$;

(10). $(FM-7)$ $T(c,d)\geq T(a,b)$ for $c\geq a$ and $d\geq b$;

(11). $(FM-8)$ $T(T(a,b),c)=T(a,T(b,c))$ where $a,b,c,d\in[0,1]$;

(12). A Menger space is a triplet (X,F,T), where (X,F) is a PM-Space, X is a non-empty set and a t-norm satisfying instead of (FM-8) a stronger requirement.

(13). $(FM-9)$ $F_{x,z}(t+s)\geq T(F_{x,y}(t),F_{y,z}(s))$ for all $x\geq 0$, $y\geq 0$.

(14). For a given metric space (X,d) with usual metric d, one can put $F_{x,y}(t)=H(t-d(x,y))$ for all $x,y\in X$ and $t>0$, where H is defined as:

$$(t)=\begin{cases}1 & \text{if } s>0,\\ 0 & \text{if } s\leq 0.\end{cases}$$

and t-norm T is defined as $T(a,b)=\min\{a,b\}$.

For the proof of our result we required the following definitions.

Definition 1.1. *Let $(X,F,*)$ be a Menger space and be a continuous t-norm.*

(a). A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n\to x$) iff for every $\epsilon>0$ and $\lambda\in(0,1)$, there exists an integer $n_0=n_0(\epsilon,\lambda)$ such that $F_{x_n,x}(\epsilon)>1-\lambda$ for all $n\geq n_0$.

(b). A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon>0$ and $\lambda\in(0,1)$, there exists an integer $n_0=n_0(\epsilon,\lambda)$ such that $F_{x_n,x_{n+p}}(\epsilon)>1-\lambda$ for all $n\geq n_0$ and $p>0$.

(c). A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 1.2. *If is a continuous t-norm, it follows from $(FM-4)$ that the limit of sequence in Menger space is uniquely determined.*

Definition 1.3. *Self maps A and B of a Menger space $(X,F,*)$, are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax=Bx$ for some $x\in X$ then $ABx=BAx$.*

Definition 1.4. *Self maps A and B of a Menger space $(X,F,*)$ are said to be compatible if $F_{ABx_m,BAx_n}(t)\to 1$ for all $t>0$, whenever x_n is a sequence in X such that $Ax_n\to x$, $Bx_n\to x$ for some x in X as $n\to\infty$.*

Definition 1.5. *Let S and T be weakly compatible of a Menger space $(X,M,*)$ and $Su=Tu$ for some u in X then $STu=TSu=SSu=TTu$.*

Example 1.6. *Let $X=[0,3]$ be equipped with the usual metric $d(x,y)=|x-y|$. Define $f,g:[0,3]\to[0,3]$ by*

$$f(x)=\begin{cases}x & \textit{if } x\in[0,1),\\ 3 & \textit{if } x\in[1,3].\end{cases}$$

And

$$g(x)=\begin{cases}3-x & \textit{if } x\in[0,1),\\ 3 & \textit{if } x\in[1,3].\end{cases}$$

Then for any $x\in[1,3]$, x is a coincidence point and $fgx=gfx$, showing that f,g are weakly compatible maps on $[0,3]$.

Remark 1.7. *If self maps A and B of a Menger space $(X, F, *)$ are compatible then they are weakly compatible.*

Lemma 1.8. *Let $(X, M, *)$ be a Menger space. Then for all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.*

Lemma 1.9. *Let $(X, M, *)$ be a Menger space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M_{x,y}(t) \geq M_{x,y}(t)$ $\forall$ $t > 0$, then $x = y$.*

Lemma 1.10. *Let xn be a sequence in a Menger space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that $M_{x_{n+2},x_{n+1}}(kt) \geq M_{x_{n+1},x_n}(t)$ $\forall$ $t > 0$ and $n \in N$. Then xn is a Cauchy sequence in X.*

Lemma 1.11. *The only t-norm $*$ satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.*

Example 1.12. *Let (X, d) be a metric space. Define $a * b = \min a, b$ and $M_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all $x, y \in X$ and all $t > 0$. Then $(X, M,)$ is a Menger space. It is called the Menger space induced by d.*

2. Weakly Compatible Maps

In 1982, Sessa [17], weakened the concept of commutativity to weakly commuting mappings. Afterwards, Jungck [4] enlarged the concept of weakly commuting mappings by adding the notion of compatible mappings. In 1991, Mishra [16] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Theorem 2.1. *Let $(X, M, *)$ be a Complete Menger Space with $t*t \geq t$ for all $t \in [0, 1]$. Let A, B, S, T, P and Q be mappings from X into itself satisfying the following conditions:*

(2.1) $P(X) \subset AB(X), Q(X) \subset ST(X)$;

(2.2) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$;

(2.3) Pairs (P, AB) and (Q, ST) are compatible of type (α) (or compatible of type (A)),

(2.4) A, B, S and T are continuous,

(2.5) There exists a number $k \in (0, 1)$ such that

$$M_{Px,Qy}(kt) \geq M_{ABx,Px}(t) * M_{STy,Qy}(t) * M_{STy,Px}(\beta t) * M_{ABx,Qy}(2 - \beta)t * M_{ABx,STy}(t),$$

for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

3. Main Results

Now we prove the following results:

Theorem 3.1. *Let $(X, M, *)$ be a Complete Menger Space with $t*t \geq t$ for all $t \in [0, 1]$. Let A, B, S, T, P and Q be mappings from X into itself satisfying the following conditions:*

(3.1) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$;

(3.2) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$;

(3.3) Pairs (P, AB) and (Q, ST) are occasionally weakly compatible;

(3.4) There exists a number $k \in (0, 1)$ such that

$$M_{Px,Qy}(kt) \geq M_{ABx,Px}(t) * M_{STy,Qy}(t) * M_{STy,Px}(\beta t) * M_{ABx,Qy}(2-\beta)t * M_{ABx,STy}(t),$$

for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$.

If the range of the subspaces $P(X)$ or $AB(X)$ or $Q(X)$ or $ST(X)$ is complete, then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, so $\{y_n\}$ converges to a point $z \in X$. Since $\{Px_{2n}\}, \{Qx_{2n+1}\}, \{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, they also converge to the same point z. Since $P(X) \in AB(X)$, there exists a point $u \in X$ such that $ABu = z$. Then, using (3.4)

$$\begin{aligned} M_{Pu,z}(kt) &\geq M_{Pu,Qx2n+1}(kt) \\ &\geq M_{ABu,Pu}(t) * M_{STx2n+1,Qx2n+1}(t) * M_{STx2n+1,Pu}(\beta t) * M_{ABu,Qx2n+1}(2-\beta)t * M_{ABu,STx2n+1}(t). \end{aligned}$$

Proceeding limit as $n \to \infty$ and setting $\beta = 1$,

$$\begin{aligned} M_{Pu,z}(kt) &\geq M_{Pu,z}(t) * M_{z,z}(t) * M_{z,Pu}(\beta t) * M_{z,z}(t) * M_{z,z}(t) \\ &= M_{Pu,z}(t) * 1 * M_{Pu,z}(t) * 1 * 1 \\ &\geq M_{Pu,z}(t). \end{aligned}$$

By Lemma 1.8, $Pu = z$. Therefore, $ABu = Pu = z$. Since $Q(X) \subset ST(X)$, there exists a point $v \in X$ such that $z = STv$. Then, again using (3.4)

$$M_{Pu,Qv}(kt) \geq M_{ABu,Pu}(t) * M_{STv,Qv}(t) * M_{STv,Pu}(\beta t) * M_{ABu,Qv}(2-\beta)t * M_{ABu,STv}(t)$$

Proceeding limit as $n \to \infty$, we have for $\beta = 1$, $Qv = z$. Therefore, $ABu = Pu = STv = Qv = z$. Since pair (P, AB) is occasionally weakly compatible, therefore, $Pu = ABu$ implies that $PABu = ABPu$ i.e., $Pz = ABz$. Now we show that z is a fixed point of P. For $\beta = 1$, we have

$$\begin{aligned} M_{Pz,Qv}(kt) &\geq M_{ABz,Pz}(t) * M_{STv,Qv}(t) * M_{STv,Pz}(\beta t) * M_{ABz,Qv}(2-\beta)t * M_{ABz,STv}(t) \\ &= 1 * 1 * M_{z,Pz}(t) * M_{Pz,z}(t) * M_{Pz,z}(t). \end{aligned}$$

Therefore, we have by Lemma 1.8, $Pz = z$. Hence $Pz = z = ABz$. Similarly, pair of map Q, ST is occasionally weakly compatible, we have $Qz = STz = z$. Now we show that $Bz = z$, by putting $x = Bz$ and $y = x_{2n+1}$ with $\beta = 1$ in for (3.4), we have

$$M_{PBz,Qx_{2n+1}}(kt) \geq M_{AB(Bz),P(Bz)}(t) * M_{STx_{2n+1},Qx_{2n+1}}(t) * M_{STx_{2n+1},PBz}(t) * M_{AB(Bz),Qx2n}(t) * M_{AB(Bz),STx2n+1}(t).$$

Proceeding limits as $n \to \infty$ and using Lemma 1.8, we have $Bz = z$. Since $ABz = z$, therefore, $Pz = ABz = Bz = z = Qz = STz$. Finally, we show that $Tz = z$, by putting $x = z$ and $y = Tz$ with $\beta = 1$ in (3.4).

$$M_{Pz,Q(Tz)}(kt) \geq M_{ABz,Pz}(t) * M_{ST(Tz),Q(Tz)}(t) * M_{ST(Tz),Pz}(t) * M_{ABz,Q(Tz)}(t) * M_{ABz,ST(Tz)}(t)$$

Therefore, $Tz = z$. Hence, $ABz = Bz = STz = Tz = Pz = Qz = z$. □

Uniqueness follows easily. If we put $B = T = I$, the identity map on X, in Theorem 3.1, we have the following:

Corollary 3.2. *Let $(X, M, *)$ be a complete Menger metric space with $t * t \geq t$ for all $t \in (0, 1)$ and let A, S, P and Q be the mapping from X into itself such that*

(3.5) $P(X) \subset A(X)$, $Q(X) \subset S(X)$.

(3.6) The pairs (A, S) and (Q, S) are occasionally weakly compatible.

(3.7) There exists a number $k \in (0, 1)$ such that

$$M_{Px,Qy}(kt) \geq M_{Ax,Px}(t) * M_{Sy,Qy}(t) * M_{Sy,Px}(\beta t) * M_{Ax,Qy}(2 - \beta)t * M_{Ax,Sy}(t);$$

for all $x, y \in X$, $\beta \in (0, 2)$ with $t > 0$.

If the range of the one subspaces is complete then A, S, P and Q have a unique common fixed point in X.

If we put $A = B = S = T = I$ in Theorem 3.1, we have the following:

Corollary 3.3. *Let $(X, M, *)$ be a complete Menger metric space with $t * t \geq t$ for all $t \in [0, 1]$ and let P and Q be occasionally weakly compatible mapping from X into itself. If there exists a constant $k \in (0, 1)$ such that*

$$M_{Px,Qy}(kt) \geq M_{x,Px}(t) * M_{y,Qy}(t) * M_{y,Px}(\beta t) * M_{x,Qy}(2 - \beta)t * M_{x,y}(t);$$

for all $x, y \in X$, $b \in (0, 2)$ and $t > 0$. If the range of the one subspaces is complete then P and Q have a unique common fixed point in X.

If we put $P = Q, A = S$ and $B = T = I$ in Theorem 3.1, we have the following:

Corollary 3.4. *Let $(X, M, *)$ be a complete Menger metric space with $t * t \geq t$ for all $t \in [0, 1]$ and let P, S be occasionally weakly compatible maps on X such that $P(X) \subset S(X)$ and satisfy the following condition:*

$$M_{Px,Py}(t) \geq M_{Sx,Px}(t) * M_{Sy,Py}(t) * M_{Sy,Px}(\beta t) * M_{Sx,Py}(2 - \beta)t * M_{Sx,Sy}(t),$$

for all $x, y \in X$, $b \in (0, 2)$ and $t > 0$. If the range of the one subspaces is complete then P and S have a unique common fixed point in X.

Example 3.5. *Let $X = [0, 1]$ with usual metric d and for each $t \in [0, 1]$. Define $M_{x,y}(t) = \frac{t}{t+|x-y|}$, $M_{x,y}(0) = 0$ for all $x, y \in X$. Clearly $(X, M, *)$ is a complete fuzzy metric space where $*$ is defined by $a * b = ab$. Let A, B, S, T, P and Q be defined by $Ax = x$, $Bx = x/2$, $Sx = x/5$, $Tx = x/3$, $Px = x/6$ and $Qx = 0$ for all $x, y \in X$. Then $P(X) = [0, 1/6]$ I $[0, 1/2] = AB(X)$ and $Q(X) = 0$ I $[0, 1/5] = STx$. If we take $k = 1/2$, $t = 1$ and $\beta = 1$, we see that all conditions of Theorem 3.1 are satisfied. Moreover, the pair $\{P, AB\}$ and $\{Q, ST\}$ are occasionally weakly compatible.*

4. Conclusion

Theorem 3.1 is a generalization of well known results in the sense that condition of compatibility of type (A) of the pairs of self maps has been restricted to occasionally weakly compatible self maps and continuity of the mappings have been completely removed.

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