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Entire Double Sequence Space of Interval Numbers

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Abstract: In the past decades, modal analysis has become a major technology in the quest for determining, improving and optimizing dynamic characteristics of engineering structures. Not only has it been recognized in mechanical and aeronautical engineering, but modal analysis has also been discovered in profound applications for civil and building structures, space structures, transportation and nuclear problems [4]. In this paper we introduced the new concept of double interval sequence spaces $\Gamma(gI)$ and $\Lambda(gI)$. We present the different properties like completeness, solidness etc. Also, we have given some new definitions and theorems about the sequence space of double interval numbers.

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1. Introduction

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [9] in 1959 and Moore and Yang [10] 1962. Furthermore, Moore and others [11] have developed applications to differential equations. Chiao in [5] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengönül and Eryilmax [12] in 2010 studied bounded and convergent sequence space of interval numbers and showed that these spaces are complete metric space. A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $I\Re$. Any elements of $I\Re$ is called closed interval and denoted by $\bar{x}$. That is $\bar{x} = \{x \in \Re : a \leq x \leq b\}$. An interval number $\bar{x}$is a closed subset of real numbers. Let x_l and x_r be be respectively first and last points of the interval number $\bar{x}$. For $\bar{x}_1, \bar{x}_2 \in I\Re$, we define $\bar{x}_1 = \bar{x}_2$ if and only if $x_{1l} = x_{2l}$ and $x_{1r} = x_{2r}$

$$\bar{x}_1 + \bar{x}_2 = \{x \in \Re : x_{1l} + x_{2l} \leq x \leq x_{1r} + x_{2r})\}$$

$$\bar{x}_1 \times \bar{x}_2 = \{x \in \Re : \min(x_{1l}x_{2l}, x_{1l}x_{2r}, x_{1r}x_{2l}, x_{1r}x_{2r}) \leq x \leq \max(x_{1l}x_{2l}, x_{1l}x_{2r}, x_{1r}x_{2l}, x_{1r}x_{2r})\}$$

The set of all interval numbers $I\Re$ is a complete metric space defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|\bar{x}_{1l} - \bar{x}_{2l}|, |\bar{x}_{1r} - \bar{x}_{2r}|\}$$

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In the special case $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of $\Re$. Let us define transformation $f : N \times N \to \Re$, $k, l \to f(k, l) = \bar{x}_{k,l}$, then $\bar{x} = (\bar{x}_{k,l})$ is called double sequence of interval numbers. $\bar{x}_{k,l}$ is called k,l$^{\text{th}}$ term of sequence $\bar{x} = (\bar{x}_{k,l})$. We denote by $\omega^2(IR)$ the set of all double sequence of interval numbers.

A sequence $\bar{x} = (\bar{x}_{k,l})$ of double sequence interval numbers is said to be convergent in the Pringsheim's sense or P-convergent to the interval number $\bar{x}_0$if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $d(\bar{x}_{k,l}, \bar{x}_0) < \varepsilon$ for all $k, l \geq k_0$. A sequence $\bar{x} = (\bar{x}_{k,l})$ of double sequence of interval numbers is said to be double interval fundamental sequence if for every $\varepsilon > 0$ there exists $k_0 \in$ N such that $d(\bar{x}_{k,l}, \bar{x}_{m,n}) < \varepsilon$ whenever $m, n, k, l > k_0$. Let $p = (p_{k,l})$ be a double sequence of positive real numbers. We define convergent series, bounded series and p-absolute convergent series of sequences spaces of double interval numbers which are denoted $cs^2(IR)$, $bs^2(IR)$, $l_p^2(IR)$ respectively, that is

$$cs^2(IR) = \left\{ \bar{x} = (\bar{x}_{k,l}) \in \omega^2(IR) : \lim_m \lim_n \left(d(\sum_{k=1}^{m} \sum_{l=1}^{n} \bar{x}_{k,l}, \bar{x}_0) \right) = \bar{0} \right\},$$

$$bs^2(IR) = \left\{ \bar{x} = (\bar{x}_{k,l}) \in \omega^2(IR) : \sup_{m,n} d(\sum_{k=1}^{m} \sum_{l=1}^{n} \bar{x}_{k,l}, \bar{0}) < \infty \right\},$$

$$l_p^2(IR) = \left\{ \bar{x} = (\bar{x}_{k,l}) \in \omega^2(IR) : \left(\sum_{k=1}^{m} \sum_{l=1}^{n} \left(d(\sum_{k=1}^{m} \sum_{l=1}^{n} \bar{x}_{k,l}, \bar{0}) \right)^p \right)^{1/p} < \infty, p \geq 1 \right\}$$

Clearly we see that the spaces $cs^2(IR)$, $bs^2(IR)$, $l_p^2(IR)$ are sub vector spaces in accordance with scalar product and addition on $\omega^2(IR)$ which are metric spaces.

2. Main Results

We define the entire sequence spaces of symmetric modals which are denoted by $\Gamma^2(IR)$ and $\Lambda^2(IR)$ respectively.

$$\Gamma^2(IR) = \left\{ \bar{x} = (\bar{x}_{k,l}) \in \omega^2(IR) : \lim_{k,l} (D(\bar{x}_{k,l}, \bar{0})) = 0 \right\} \Lambda^2(IR) = \left\{ \bar{x} = (\bar{x}_{k,l}) \in \omega^2(IR) : \sup_{k,l} (D(\bar{x}_{k,l}, \bar{0})) < \infty \right\}$$

where $D(\bar{x}_{k,l}, \bar{y}_{k,l}) = \max \left\{ \left| \underline{x}_{k,l} - \underline{y}_{k,l} \right|^{1/p_{k,l}}, |\bar{x}_{k,l} - \bar{y}_{k,l}|^{1/p_{k,l}} \right\}$ the metric defined by

$$\bar{d}(\bar{x}_{k,l}, \bar{y}_{k,l}) = \sup_{k,l} \max \left\{ \left| \underline{x}_{k,l} - \underline{y}_{k,l} \right|^{1/p_{k,l}}, |\bar{x}_{k,l} - \bar{y}_{k,l}|^{1/p_{k,l}} \right\} = \sup_{k,l} D(\bar{x}_{k,l}, \bar{y}_{k,l}) \tag{1}$$

which satisfies the metric space axioms.

Theorem 2.1. *The sequence space $\Gamma^2(IR)$ is a complete metric space with respect to the metric defined by* (1).

Proof. Let $(\bar{x}^{(n)})$ be a fundamental double sequence of interval numbers in $\Gamma^2(IR)$. Then for a given $\varepsilon > 0$ there exists a positive integer n_0 such that $\bar{d}(\bar{x}_{k,l}^{(n)}, \bar{x}_{k,l}^{(m)}) = \sup_{k,l} D(\bar{x}_{k,l}^{(n)}, \bar{x}_{k,l}^{(m)}) < \varepsilon$ for all $n, m \geq n_0$. This is true for all k, l, we have $D(\bar{x}_{k,l}^{(n)}, \bar{x}_{k,l}^{(m)}) < \varepsilon$ for all $n, m \geq n_0$

$$\max \left\{ \left| \underline{x}_{k,l}^{(n)} - \underline{x}_{k,l}^{(m)} \right|^{1/p_{k,l}}, \left| \bar{x}_{k,l}^{(n)} - \bar{x}_{k,l}^{(m)} \right|^{1/p_{k,l}} \right\} < \varepsilon \quad \text{for all } n, m \geq n_0$$

$$\left| \underline{x}_{k,l}^{(n)} - \underline{x}_{k,l}^{(m)} \right|^{p_{k,l}} < \varepsilon^{p_{k,l}} \quad \textit{and} \quad \left| \bar{x}_{k,l}^{(n)} - \bar{x}_{k,l}^{(m)} \right|^{1/p_{k,l}} < \varepsilon^{p_{k,l}} \quad \text{for all } n, m \geq n_0$$

This leads to the fact $\bar{x}_{k,l}^{(n)}$ is a fundamental sequence in IR. Since IR is a complete metric space, $\bar{x}_{k,l}^{(n)}$ is convergent. $\lim_n \bar{x}_{k,l}^{(n)} = \bar{x}_{k,l}$ for each $k, l \in$ N. This is true for all k, l, $\sup_{k,l} D(\bar{x}_{k,l}^{(n)}, \bar{x}_{k,l}) < \varepsilon$. So $\bar{x}_{k,l}^{(n)} \to \bar{x}_{k,l}$ as $n \to \infty$ in $\Gamma^2(IR)$, we have to show that $\bar{x} = (\bar{x}_{k,l}) \in \Gamma^2(IR)$. Since $\bar{x}_{k,l}^{(n)} \in \Gamma^2(IR)$, we have $\bar{d}(\bar{x}_{k,l}^{(n)}, \bar{0}) < \varepsilon$. Consider $\bar{d}(\bar{x}_{k,l}, \bar{0}) = \sup_{k,l} D(\bar{x}_{k,l}, \bar{0}) \leq \sup_{k,l} D(\bar{x}_{k,l}^{(n)}, \bar{x}_{k,l}) + \sup_{k,l} D(\bar{x}_{k,l}^{(n)}, \bar{0}) < \varepsilon + \varepsilon = 2\varepsilon$. Hence $(\bar{x}_{k,l}) \in \Gamma^2(IR)$. This completes the proof. □

Theorem 2.2. *A necessary and sufficient condition that $D(\sum \bar{x}_{k,l}\bar{y}_{k,l},\bar{0})$ should be convergent for every $(\bar{x}_{k,l})$ for which $\lim_{k,l}(D(\bar{x}_{k,l},\bar{0}))=0$ is that $D(\bar{y}_{k,l},\bar{0})$ should be bounded.*

Proof. Suppose $D(\bar{y}_{k,l},\bar{0})$ is bounded. then we can find M so that $D(\bar{y}_{k,l},\bar{0})\le M$ for $k,l\ge 1$, since $D(\sum \bar{x}_{k,l},\bar{0})\to\tilde{0}$ as $k,l\to\infty$, we can find k_0 so that $D(\sum \bar{x}_{k,l},\bar{0})\le\frac{1}{2M}, k,l\ge k_0$

$$\begin{aligned}[D(\bar{x}_{k,l}\bar{y}_{k,l},\bar{0})]^{p_{k,l}} &\le [D(\sum \bar{x}_{k,l},\bar{0})]^{p_{k,l}}[D(\bar{y}_{k,l},\bar{0})]^{p_{k,l}}\\ &< \left(\frac{1}{2M}\right)^{p_{k,l}} M^{p_{k,l}}=\frac{1}{2^{p_{k,l}}}\end{aligned}$$

So $\sum[D(\bar{x}_{k,l}\bar{y}_{k,l},\bar{0})]^{p_{k,l}}$ converges.

Conversely, suppose $D(\bar{y}_{k,l},\bar{0})$ is not bounded .Then we can find an increasing sequence {k$_q$,{l$_q$} of integers such that $D(\bar{y}_{k_q,l_q},\bar{0})\ge q,\ q=1,2,....$ That is, $[D(\bar{y}_{k_q,l_q},\bar{0})]^{p_{k_q,l_q}}\ge q^{p_{k_q,l_q}},\ q=1,2,....$ Take $\bar{x}_{k,l}=\begin{cases}\left[\frac{1}{q^{p_{k_q,l_q}}},0\right] & \text{if } k=k_q, l=l_q\\ [0,0] & \text{if } k\ne k_q, l\ne l_q\end{cases}$.
Then $\lim_{k,l}(D(\bar{x}_{k,l},\bar{0}))=0$. But

$$\begin{aligned}[D(\bar{x}_{k,l}\bar{y}_{k,l},\bar{0})]^{p_{k_q,l_q}}[D(\tilde{x}_k\tilde{y}_k,\tilde{0})]^{k_p} &\ge [D(\bar{x}_{k,l},\bar{0})]^{p_{k_q,l_q}}[D(\bar{y}_{k,l},\bar{0})]^{p_{k_q,l_q}}\\ &=\left[\left(\frac{1}{q^{p_{k_q,l_q}}}\right)^{\frac{1}{p_{k_q,l_q}}}\right]^{p_{k_q,l_q}}\left[(q^{p_{k_q,l_q}})^{\frac{1}{p_{k_q,l_q}}}\right]^{p_{k_q,l_q}}\\ &=1\end{aligned}$$

so that $\sum[D(\bar{x}_{k,l}\bar{y}_{k,l},\bar{0})]$ does not converges. Hence $D(\bar{y}_{k,l},\bar{0})$ is bounded. ☐

Theorem 2.3. *The double sequence spaces of interval numbers $\Gamma^2(IR)$ and $\Lambda^2(IR)$ are solid.*

Proof. We consider $\Gamma^2(IR)$ Now let $\bar{d}(\bar{y}_{k,l},\bar{0})\le\bar{d}(\bar{x}_{k,l},\bar{0})$ for all $k,l\in N$ and for some $\bar{x}\in\Gamma^2(IR)$. Then, we have

$$\sup_{k,l}\max\left\{\left|\underline{y}_{k,l}\right|^{1/p_{k,l}},|\bar{y}_{k,l}|^{1/p_{k,l}}\right\}\le\sup_{k,l}\max\left\{\left|\underline{x}_{k,l}\right|^{1/p_{k,l}},|\bar{x}_{k,l}|^{1/p_{k,l}}\right\}$$

$\underline{y}_{k,l}\le\underline{x}_{k,l}$ and $\bar{y}_{k,l}\le\bar{x}_{k,l}$. That is $\bar{y}\le\bar{x}$. It is clear that $\bar{y}\in\Gamma^2(IR)$. Therefore $\Gamma^2(IR)$ is solid. ☐

Theorem 2.4. *The sequence $(\bar{e}_{1,l},\bar{e}_{2,l}....\bar{e}_{k,l},..)$ is schauder interval base for $\Gamma^2(IR)$, where $\bar{e}_{k,l}=\{\bar{0},\bar{0},...[1,1],\bar{0},...\}$.*

Proof. Let $\bar{x}=(\bar{x}_{k,l})\in\Gamma^2(IR)$. Therefore for every $\varepsilon>0$ there exists a positive integer $n\in N$ such that $k,l\ge n$, $\bar{d}(\bar{x}_{k,l},\bar{0})=\sup_{k,l}D(\bar{x}_{k,l},\bar{0})<\varepsilon$. Now we should show the following statement. $\lim_{k,l\to\infty}\bar{d}((\bar{x}_{k,l}-\sum\bar{e}_{k,l}\bar{x}_{k,l}),\bar{0})=0$. From here we can write next steps

$$\begin{aligned}\bar{d}((\bar{x}_{k,l}-\sum\bar{e}_{k,l}\bar{x}_{k,l}),\bar{0}) &= \bar{d}(([\underline{x}_{1,l},\bar{x}_{1,l}],[\underline{x}_{2,l},\bar{x}_{2,l}],......[\underline{x}_{k,l},\bar{x}_{k,l}],...)-([\underline{x}_{1,l},\bar{x}_{1,l}],[\underline{x}_{2,l},\bar{x}_2],......[\underline{x}_{n,l},\bar{x}_{n,l}]),\bar{0})\\ &=\bar{d}(([\bar{0},\bar{0},......[\underline{x}_{n+1,l},\bar{x}_{n+1,l}],[\underline{x}_{n+2,l},\bar{x}_{n+2,l}]),\bar{0})\\ &=\sup_{k,l\ge n+1}\max\left\{\left|\underline{x}_{k,l}\right|^{1/p_{k,l}}|\bar{x}_{k,l}|^{1/p_{k,l}}\right\}\to 0\quad as\ n\to\infty\end{aligned}$$

We have

$$\bar{x}_{k,l}=\sum_{k,l=1}^{\infty}\bar{e}_{k,l}\bar{x}_{k,l}\tag{2}$$

Let us show uniqueness of the representation given by (2) for $\bar{x} = (\bar{x}_{k,l}) \in \Gamma^2(IR)$. Suppose that there exists a representation $\bar{x}_{k,l} = \sum_{k,l=1}^{\infty} \bar{e}_{k,l}\bar{y}_{k,l}$, then for $n \to \infty$, we have

$$\bar{d}(\sum_{k,l=1}^{n} (\bar{x}_{k,l} - \bar{y}_{k,l})\bar{e}_{k,l}, \bar{0}) = \sum_{k,l=1}^{n} \bar{d}((\bar{x}_{k,l} - \bar{y}_{k,l})\bar{e}_{k,l}, \bar{0})$$
$$= \sup_{k,l \geq n+1} \max\left\{ \left|\underline{y}_{k,l} - \underline{x}_{k,l}\right|^{1/p_{k,l}}, |\bar{y}_{k,l} - \bar{x}_{k,l}|^{1/p_{k,l}} \right\} \to 0 \quad as\ n \to \infty$$

$\left|\underline{y}_{k,l} - \underline{x}_{k,l}\right|^{1/p_{k,l}} \to \tilde{0}$ and $|\bar{y}_{k,l} - \bar{x}_{k,l}|^{1/p_{k,l}} \to \tilde{0}$ as $n \to \infty$. Therefore $\underline{y}_{k,l} = \underline{x}_{k,l}$ and $\bar{y}_{k,l} = \bar{x}_{k,l}$. That is $\bar{y} = \bar{x}$. □

3. α, β, γ Duals of Sequence Space the Modals

For the sequence spaces $\lambda^2(IR)$ and $\mu^2(IR)$, we define the set $S(\lambda^2(IR), \mu^2(IR))$ by

$$S(\lambda^2(IR), \mu^2(IR)) = \left\{(\bar{y}_{k,l}) \in \omega^2(IR) : (\bar{x}_{k,l}, \bar{y}_{k,l}) \in \mu^2(IR)\right\} \ for\ all\ \bar{x}_{k,l} \in \lambda^2(IR) \tag{3}$$

with the notation of (3), the α, β, γ duals of double sequence space $\lambda^2(IR)$ which are denoted by $\lambda^{2,\alpha}(IR)$, $\lambda^{2,\beta}(IR)$ and $\lambda^{2,\gamma}(IR)$ are defined by

$$\lambda^{2,\alpha}(IR) = S(\lambda^2(IR), l_1^2(IR)),$$
$$\lambda^{2,\beta}(IR) = S(\lambda^2(IR), cs^2(IR))$$
$$\lambda^{2,\gamma}(IR) = S(\lambda^2(IR), bs^2(IR))$$

Theorem 3.1. *The β dual of sequence space $\Gamma^2(IR)$ is $\Lambda^2(IR)$.*

Proof. Let us suppose that and $\bar{y} = (\bar{y}_{k,l}) \in \Lambda^2(IR)$ for every $\bar{x} = (\bar{x}_{k,l}) \in \Gamma^2(IR)$, then $\sup D(\bar{y}_{k,l}, \bar{0}) < \infty$, we can write

$$\lim_{m,n} D\left(\sum_{k,l=1}^{m,n} \bar{x}_{k,l}\bar{y}_{k,l}, \tilde{0}\right) = \lim_{n} D\left(\sum_{k,l=1}^{m,n} [\underline{y}_{k,l}, \bar{y}_{k,l}], [\underline{x}_{k,l}, \bar{x}_{k,l}], \bar{0}\right)$$
$$= \lim_{m,n} D\left(\sum_{k,l=1}^{m,n} [\underline{y}_{k,l}\underline{x}_{k,l}, \bar{y}_{k,l}\bar{x}_{k,l}], \bar{0}\right)$$
$$= \lim_{m,n} \max\left\{ \left|\sum_{k,l=1}^{m,n} \underline{y}_{k,l}\underline{x}_{k,l}\right|^{1/p_{k,l}}, \left|\sum_{k,l=1}^{m,n} \bar{y}_{k,l}\bar{x}_{k,l}\right|^{1/p_{k,l}} \right\}$$
$$\leq \lim_{m,n} \max\left\{ \sum_{k,l=1}^{m,n} \left|\underline{y}_{k,l}\underline{x}_{k,l}\right|^{1/p_{k,l}}, \sum_{k,l=1}^{m,n} |\bar{y}_{k,l}\bar{x}_{k,l}|^{1/p_{k,l}} \right\}$$
$$= \lim_{m,n} M \max\left\{ \sum_{k,l=1}^{m,n} |\underline{x}_{k,l}|^{1/p_{k,l}}, \sum_{k,l=1}^{m,n} |\bar{x}_{k,l}|^{1/p_{k,l}} \right\}$$

Where $M = \max\{M_1, M_2\}$; $M_1 = \sup_{k,l} \left|\underline{y}_{k,l}\right|^{1/p_{k,l}}$, $M_2 = \sup_{k,l} |\bar{y}_{k,l}|^{1/p_{k,l}}$.

$$\lim_{m,n} D\left(\sum_{k,l=1}^{m,n} \bar{x}_{k,l}\bar{y}_{k,l}, \bar{0}\right) \leq \lim_{m,n} MD\left(\sum_{k,l=1}^{m,n} \bar{x}_{k,l}, \bar{0}\right)$$
$$= MD\left(\sum_{k,l=1}^{\infty} \bar{x}_{k,l}, \bar{0}\right)$$

$$= M \sum_{k,l=1}^{\infty} D(\bar{x}_{k,l}, \bar{0}) < \infty$$

Therefore, we get $\bar{x}_{k,l}y_{k,l} \in cs^2(IR)$. Hence $(\bar{y}_k) \in \Gamma^{2,\beta}(IR)$.

$$\Lambda^2(IR) \subset \Gamma^{2,\beta}(IR) \tag{4}$$

Let $\bar{y} = (\bar{y}_{k,l}) \in \Gamma^{2,\beta}(IR)$, then $\sum D(\bar{x}_{k,l}\bar{y}_{k,l}, \bar{0})$ converges for every $\bar{x} = (\bar{x}_{k,l}) \in \Gamma^2(IR)$. By Theorem 2.2, $D(\bar{y}_{k,l}, \bar{0})$ is bounded. $\sup_{k,l} D(\bar{y}_{k,l}, \bar{0})$ is bounded, then $\bar{y} = (\bar{y}_{k,l}) \in \Lambda^2(IR)$

$$\Gamma^{2,\beta}(IR) \subset \Lambda^2(IR) \tag{5}$$

From (4) and (5), $\Gamma^{2,\beta}(IR) = \Lambda^2(IR)$. □

Theorem 3.2. $\Gamma^{2,\alpha}(IR) = \Gamma^{2,\beta}(IR) = \Gamma^{2,\gamma}(IR) = \Lambda^2(IR)$.

Proof. From Theorem 3.1, $\Gamma^{2,\beta}(IR) = \Lambda^2(IR)$. From Theorem 2.3 and Theorem 2.4, $\Gamma^{2,\alpha}(IR) = \Gamma^{2,\beta}(IR) = \Gamma^{2,\gamma}(IR)$. Hence $\Gamma^{2,\alpha}(IR) = \Gamma^{2,\beta}(IR) = \Gamma^{2,\gamma}(IR) = \Lambda^2(IR)$. □

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