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Semi Prime Ideal in a-Distributive Lattice

Ashok S. Kulkarni[1,*] and A. D. Lokhande[1]

1 Department of Mathematics, D.A.B.N. College, Chikhali, Shirala, Sangli, Maharashtra, India.

2 Department of Mathematics, Y. C. Warana Mahavidyalay Warananagar, Kolhapur, Maharashtra, India.

Abstract: In this paper we define semi prime ideal in a a-distributive lattice L. For a non empty subset B of a a-distributive lattice L, we observe if L is a a-distributive lattice then annihilator of B as $B^a = \{x \in L/x \wedge b \leq a \ \ \forall \ \ b \in B\}$ is a semi prime ideal for any non empty subset B of L and observe some properties.

Keywords: Prime Ideal, Semi Prime Ideal, Distributive Lattice.

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1. Introduction

Varlet [2] defined the concept of 0-distributive lattice (with 0) and pseudo complemented lattice. As a generalization of 0-distributive lattices Varlet introduced a-distributive lattices, a-distributive semilattices. M. V. Patil [6], in her Doctoral thesis define relative annihilators $B^a = \{x \in L : x \wedge b \leq a \ \ \forall \ \ b \in B\}$ for a bounded lattice L and observed some characterizations of a a-distributive lattice L in terms of B^a. Also in [6] observed the necessary and sufficient conditions for a lattice L to be a-distributive by using the properties of relative annihilator B^a of B in 'a'. Yehuda Rav [9] defined and studied semi prime ideals in general lattices. M.Ayub Ali [4] and R.M. Hafizur [8] also studied semi prime ideals in lattices. Momtaz Begum and A.S.A. Noor [5] studied semi prime ideals in meet semilattices.

In this paper we define concept of semi prime ideal in a-distributive lattice and we shows if L is a a-distributive lattice then relative annihilators of B in 'a', $B^a = \{x \in L/x \wedge b \leq a \ \ \forall \ \ b \in B\}$ is a semi prime ideal for any non empty subset B of L.

2. Preliminaries

For basic definitions in lattice theory we refer George Gratzer [1].

By a lattice $\langle L, \wedge, \vee\rangle$ we mean a non empty set L together with the binary operations $\wedge$ and $\vee$ defined on L, satisfying the following conditions, for all a_1, b_1, c_1 in L.

(1). $a_1 \wedge a_1 = a_1$, $a_1 \vee a_1 = a_1$ (Idempotent).

(2). $a_1 \wedge b_1 = b_1 \wedge a_1$, $a_1 \vee b_1 = b_1 \vee a_1$ (Commutative).

(3). $a_1 \wedge (b_1 \wedge c_1) = (a_1 \wedge b_1) \wedge c_1$, $a_1 \vee (b_1 \vee c_1) = (a_1 \vee b_1) \vee c_1$ (Associative).

* *E-mail: ashokkulkarni02@gmail.com*

(4). $a_1 \wedge (a_1 \vee b_1) = a_1 = a_1 \vee (a_1 \wedge b_1)$ (Absorption).

Let $L = \langle L, \wedge, \vee \rangle$ be a lattice. Define a relation '$\leq$' on L by $a \leq b \Rightarrow a \wedge b = a$ (Dually $a \vee b = b$). Then '$\leq$ ' is a partial ordering relation on L and hence $\langle L, \leq \rangle$ is a poset. If $0 \in L$ such that $0 \leq x$ for all x in L, then 0 is called the zero element in L and if $1 \in L$ such that $1 \geq x$ for all x in L, then 1 is called the unit element in L. A lattice L with zero element and the unit element is called a bounded lattice. A bounded lattice L is complemented if for every x in L there exists y in L such that $x \wedge y = 0$ and $x \vee y = 1$. A lattice L is said to be distributive if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for all $x, y, z \in L$.
A lattice L with 0 is 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for all $a, b, c \in L$. A lattice L with 1 is called 1-distributive if $a \vee b = 1$ and $a \vee c = 1$ imply $a \vee (b \wedge c) = 1$ for all $a, b, c \in L$. A 0-1 distributive lattice is a bounded lattice which is both 0-distributive and 1-distributive.
A non empty subset I_1 of a lattice L is called an ideal if (1) $x' \vee y' \in I_1$, for all $x' \in I_1$, $y' \in I_1$ and (2) for $x', y' \in L$, $x' \leq y'$, $y' \in I_1$ imply $x' \in I_1$. A non empty subset F_1 of a lattice L is called a filter if (1) $x' \wedge y' \in F_1$, for all $x' \in F_1$, $y' \in F_1$ and (2) for x', $y' \in L$, $x' \leq y'$, $x' \in F_1$ imply $y' \in F_1$. A filter F_1 of L is called a maximal filter if $F_1 \neq L$ and it is not contained in any other proper filter of L. P. Balasubramani and P.V. Venkatanarasimhan [7] has given characterizations of 0-distributive lattices. As a generalization of 0-distributive lattices J. C. Varlet [3] introduced a-distributive lattices, a-distributive semilattices.

Definition 2.1. *A lattice L is a-distributive, if for $x, y, z \in L$, $x \wedge y \leq a$, $x \wedge z \leq a$ implies $x \wedge (y \vee z) \leq a$, for 'a' fixed element of L.*

Definition 2.2. *A non empty subset I of a-distributive lattice L is called an ideal of L if*

(1). for $x_1, y_1 \in L$, $x_1 \leq y_1$, $y_1 \in I$ imply $x_1 \in I$.

(2). $x_1 \vee y_1 \in I$, for all $x_1, y_1 \in I$.

A proper ideal I of a a-distributive lattice L is called prime if $x_1 \wedge y_1 \in I$ implies $x_1 \in I$ or $y_1 \in I$.

Definition 2.3. *A non empty subset F_1 of a-distributive lattice L is called a filter of L if*

(1). for $x_1, y_1 \in L$, $x_1 \leq y_1$, $x_1 \in F_1$ imply $y_1 \in F_1$.

(2). $x_1 \wedge y_1 \in F_1$, for all $x_1, y_1 \in F_1$.

A proper filter F_1 of a a-distributive lattice L is called prime if $x_1 \vee y_1 \in F_1$ implies $x_1 \in F_1$ or $y_1 \in F_1$.

3. Semi Prime Ideal in a-Distributive Lattice

Yehuda Rav [9] studied semi prime ideals in general lattices. Similarly in this section we define semi prime ideal for a a-distributive lattice L. In this section we consider L as a a-distributive lattice.

Definition 3.1. *An ideal I of a a-distributive lattice L is called a semi prime ideal if for all $x_1, y_1, z_1 \in L$, $x_1 \wedge y_1 \in I$ and $x_1 \wedge z_1 \in I$ imply $x_1 \wedge (y_1 \vee z_1) \in I$.*

Now we define down set and semi down set in a a-distributive lattice L.

Definition 3.2. *A non empty subset D of a-distributive lattice L is called down set of L if for $x, y \in L$, $x \leq y$, $y \in D$ imply $x \in D$.*

Definition 3.3. *Down set D of a-distributive lattice L is called a semi prime down set if for all $x, y, z \in D$, $x \wedge y \in D$ and $x \wedge z \in D$ imply $x \wedge (y \vee z) \in D$.*

Example 3.4. *Let $L = [\{0, a, b, c, 1\}, \wedge, \vee]$ be a-distributive lattice whose diagrammatic representation is as shown in figure 1 below*

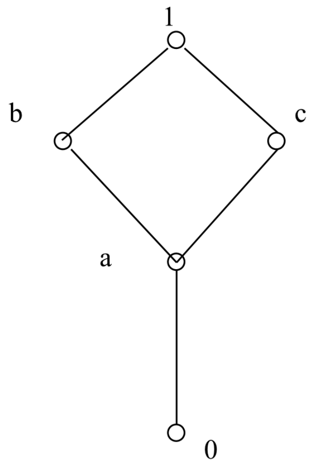

Figure 1.

Here $I = \{0, a, b\}$ is semi prime ideal.

M. V. Patil [6] define relative annihilator of B in 'a' denoted by B^a, for any non empty subset of bounded lattice L as $B^a = \{x \in L / x \wedge b \leq a \;\; \forall \;\; b \in B\}$ and proved some properties of B^a. Also it is proved in a a-distributive lattice L, B^a is a-ideal for any non empty subset B of L. If $b \in L$, clearly $\{b\}^a = \{x \in L / x \wedge b \leq a\}$. We write $(b)^a$ for $\{b\}^a$. In the next theorem we prove relative annihilator of B in 'a', B^a is a semi prime ideal in a a-distributive lattice L.

Theorem 3.5. *If L is a-distributive lattice then B^a is a semi prime ideal for any non empty subset B of L.*

Proof. Let L be a-distributive lattice. We show B^a is an ideal of a-distributive lattice L. Let $x, y \in B^a$ implies $x \wedge b \leq a$ for all $b \in B$ and $y \wedge b \leq a$ for all $b \in B$ implies $x \wedge b \leq a$ and $y \wedge b \leq a$ for all $b \in B$ implies $b \wedge x \leq a$ and $b \wedge y \leq a$ for all $b \in B$. Since L is a-distributive lattice we get $b \wedge (x \vee y) \leq a$ for all $b \in B$ implies $(x \vee y) \wedge b \leq a$ for all $b \in B$ implies $x \vee y \in B^a$. For all $x, y \in L$ let $x \leq y$ and $y \in B^a$ then $y \wedge b \leq a$ for all $b \in B$. As $x \leq y$ implies $x \wedge b \leq a$ for all $b \in B$. Therefore $x \in B^a$. Therefore B^a is an ideal of L. Now to show B^a is a semi prime ideal. Let for all $x, y, z \in L$, $x \wedge y \in B^a$ and $x \wedge z \in B^a$. Implies $(x \wedge y) \wedge b \leq a$ and $(x \wedge z) \wedge b \leq a$ for all $b \in B$ implies $(x \wedge b) \wedge y \leq a$ and $(x \wedge b) \wedge z \leq a$ for all $b \in B$. As L is a-distributive, $(x \wedge b) \wedge (y \vee z) \leq a$ implies $x \wedge (y \vee z) \wedge b \leq a$ implies $[x \wedge (y \vee z)] \wedge b \leq a$ for all $b \in B$ implies $x \wedge (y \vee z) \in B^a$ implies B^a is semi prime ideal of L. □

Necessary and sufficient condition for a lattice L to be a-distributive lattice is B^a is a semi prime ideal under some condition is given in the next theorem.

Theorem 3.6. *Lattice L is a a-distributive lattice if and only if B^a is a semi prime ideal for any non empty subset B of L satisfying condition if $x \wedge y \leq a$, $x \wedge z \leq a$ then $x \in B$ for all $x, y, z \in L$.*

Proof. By Theorem 3.5, If L be a a-distributive lattice then B^a is a semi prime ideal for any non empty subset B of L. Now let B^a is semi prime ideal to prove that lattice L is a a-distributive. As B^a is semi prime ideal of L clearly B^a is an ideal of L. Let for $x, y, z \in L$, $x \wedge y \leq a$, $x \wedge z \leq a$. As B be a set satisfying above condition therefore $x \in B$ implies $y, z \in B^a$

for $x \in B \subseteq L$. But B^a is an ideal of L implies $y \vee z \in B^a$ implies $(y \vee z) \wedge x \leq a$ implies $x \wedge (y \vee z) \leq a$ implies L is a a-distributive lattice. □

As in Theorem 3.5, similarly we can prove in a a-distributive lattice $(a \neq 1)$, $\{b\}^a$ is semi prime ideal for $b \in L$.

Theorem 3.7. *Let $a \in L(a \neq 1)$, If L is a-distributive lattice then $\{b\}^a$ is semi prime ideal for $b \in L$.*

Now we define set B'^a in a-distributive lattice L and show B'^a is a down set of L.

Definition 3.8. *For non empty subset B of a-distributive lattice L we define $B'^a = \{x \in L/x \wedge b \leq a \ \forall \ \ b \notin B\}$. Clearly for all $x, y \in L$ such that if $x \leq y$ and $y \in B'^a$ then $y \wedge b \leq a$ for all $b \notin B$. As $x \leq y$ implies $x \wedge b \leq a$ for all $b \notin B$. Therefore $x \in B'^a$. Hence B'^a is a down set of L.*

Now we give some equivalent statements for a-distributive lattice L.

Theorem 3.9. *Following statements are equivalent*

(1). L be a-distributive lattice.

(2). B^a is a semi prime ideal for every subset B of L.

(3). $(\{x\})^a$ for each $x \in L$ is a semi prime ideal.

Proof. $(1) \Rightarrow (2)$: Has already proved in Theorem 3.5.
$(2) \Rightarrow (3)$: As B^a is a semi prime ideal for every subset B of L. Considering $B = (\{x\})$. We get $(\{x\})^a$ for each $x \in L$ is a semi prime ideal.
$(3) \Rightarrow (1)$: Suppose $(\{x\})^a$ for each $x \in L$ is a semi prime ideal. Now to show L is a-distributive, Let $x, y, z \in L$ with $x \wedge y \leq a$ and $x \wedge z \leq a$. This implies $y, z \in (\{x\})^a$ and as $(\{x\})^a$ is an ideal therefore $y \vee z \in (\{x\})^a$ implies $x \wedge (y \wedge z) \leq a$ therefore L is a-distributive lattice. □

M.V. Patil [6] stated some properties of relative annihilators of B in 'a' that is of B^a in a bounded lattice with bounds 0 and 1. Here we give some properties of relative annihilators of B in 'a' that is of B^a in a a-distributive lattice L in the following theorem.

Theorem 3.10. *If B and C are non empty subsets of a-distributive lattice L, then*

(1). If $B \subseteq C$ then $C^a \subseteq B^a$.

(2). If $B \subseteq C$ then $(B^a)^a \subseteq (C^a)^a$.

(3). $(B \cap C)^a = B^a \cap C^a$.

(4). $B \subseteq (B^a)^a$.

Proof.

(1). If $x' \in C^a$ then $x' \wedge c \leq a$ for all $c \in C$ implies $x' \wedge b \leq a$ for all $b \in B$ since $B \subseteq C$ implies $x' \in B^a$. Therefore $C^a \subseteq B^a$.

(2). Let $x' \in (B^a)^a$ implies $x' \wedge b \leq a$ for all $b \in B^a$ implies $x' \wedge b \leq a$ for all $b \in B^a$ since $B \subseteq C$ therefore by (1) $C^a \subseteq B^a$ implies $x' \wedge c \leq a$ for all $c \in C^a$ implies $x' \in (C^a)^a$. Therefore $(B^a)^a \subseteq (C^a)^a$.

(3). To prove $(B\cap C)^a = B^a \cap C^a$. Let $x' \in (B\cap C)^a$ implies $x'\wedge d \le a$ for all $d\in (B\cap C)$ implies $x'\wedge d\le a$ for all $d\in B$ and $d\in C$ implies $x'\wedge d\le a$ for all $d\in B$ and $x'\wedge d\le a$ for all $d\in C$ implies $x'\in B^a$ and $x'\in C^a$ implies $x'\in B^a\cap C^a$. Therefore

$$(B\cap C)^a \subseteq B^a\cap C^a. \tag{1}$$

Now let $x'\in B^a\cap C^a$ implies $x'\in B^a$ and $x'\in C^a$ implies $x'\wedge b\le a$ for all $b\in B$ and $x'\wedge c\le a$ for all $c\in C$ implies $x'\wedge d\le a$ for all $d\in B$ and $d\in C$ implies $x'\wedge d\le a$ for all $d\in (B\cap C)$ implies $x'\in (B\cap C)^a$. Therefore

$$(B^a\cap C^a)\subseteq (B\cap C)^a. \tag{2}$$

From (1) and (2) we get, $(B\cap C)^a = B^a\cap C^a$.

(4). To prove that $B\subseteq (B^a)^a$. Let $p\in B$, $y'\in B^a$, $x'\in (B^a)^a$. As $y'\in B^a$ implies $y'\wedge p\le a$ for all $p\in B$. Also as $x'\in (B^a)^a$ implies $x'\wedge y'\le a$ for all $y'\in B^a$. Implies $p\wedge y'\le a$ for all $y'\in B^a$ implies $p\in (B^a)^a$. Therefore $B\subseteq (B^a)^a$. □

Momtaz Begum and A.S.A. Noor [5] proved a theorem for directed above meet semilattice with 0 and in [4] M.Ayub Ali, et al stated similar theorem for lattice similarly we show result is hold for a-distributive lattice which is in the following theorem.

Theorem 3.11. *Let L be a-distributive lattice for all $a\in A$ where A be a non empty subset of L and J be an ideal of L. Consider the following statements*

(1). $J=\{x\in L/x\wedge b\le a\}$, is semi prime ideal.

(2). $\{a\}^{-J}=\{x\in L/x\wedge a\in J\}$ is a semi prime ideal for all $a\in A$.

(3). $A^{-J}=\{x\in L/x\wedge a\in J\ \ \forall\ \ a\in A\}$ is a semi prime ideal.

Then (1) implies (2) implies (3).

Proof. (1) ⇒ (2) : First we show $\{a\}^{-J}$ is an ideal of L. Let $x,y\in\{a\}^{-J}$ implies $x\wedge a\in J$, $y\wedge a\in J$. As J is semi prime ideal $a\wedge(x\vee y)=(x\vee y)\wedge a\in J$ implies $x\vee y\in\{a\}^{-J}$. Now for $x,y\in L$. Let $x\le y$, $y\in\{a\}^{-J}$. As $y\in\{a\}^{-J}$ implies $y\wedge a\in J$. As $x\le y$ therefore $x\wedge a\le y\wedge a$ and $y\wedge a\in J$, J is semi prime ideal implies $x\wedge a\in J$ implies $x\in\{a\}^{-J}$. Hence $\{a\}^{-J}$ is an ideal in L. Now to show $\{a\}^{-J}$ is semi prime ideal. Let $x\wedge y\in\{a\}^{-J}$ and $x\wedge z\in\{a\}^{-J}$ implies $(x\wedge y)\wedge a\in J$ and $(x\wedge z)\wedge a\in J$ implies $(x\wedge a)\wedge y\in J$ and $(x\wedge a)\wedge z\in J$. As J is semi prime, $(x\wedge a)\wedge(y\vee z)\in J$ implies $x\wedge(y\vee z)\wedge a\in J$ implies $x\wedge(y\vee z)\in\{a\}^{-J}$. Therefore $\{a\}^{-J}$ is semi prime ideal.
(2) ⇒ (3) : First we show A^{-J} is an ideal. Let $x,y\in A^{-J}$ implies $x\wedge a\in J$, $y\wedge a\in J$ for all $a\in A$. As $x\wedge a\in J$, $y\wedge a\in J$ implies $x,y\in\{a\}^{-J}$ for all $a\in A$ implies $x\vee y\in\{a\}^{-J}$ for all $a\in A$ implies $(x\vee y)\wedge a\in J$ for all $a\in A$ implies $x\vee y\in A^{-J}$. Now for $x,y\in L$, let $x\le y$, $y\in A^{-J}$. As $y\in A^{-J}$ implies $y\wedge a\in J$ for all $a\in A$, implies $y\in\{a\}^{-J}$ for all $a\in A$. As $x\le y$ and $\{a\}^{-J}$ is ideal therefore $x\in\{a\}^{-J}$ for all $a\in A$ implies $x\wedge a\in J$ for all $a\in A$ implies $x\in A^{-J}$. Therefore A^{-J} is an ideal. Now to show A^{-J} is semi prime ideal. Let $x\wedge y\in A^{-J}$ and $x\wedge z\in A^{-J}$ implies $(x\wedge y)\wedge a\in J$ and $(x\wedge z)\wedge a\in J$ for all $a\in A$, implies $x\wedge y\in\{a\}^{-J}$ and $x\wedge z\in\{a\}^{-J}$ for all $a\in A$. And as $\{a\}^{-J}$ is semi prime $x\wedge(y\vee z)\in\{a\}^{-J}$ for all $a\in A$ Implies $x\wedge(y\vee z)\wedge a\in J$ for all $a\in A$ implies $x\wedge(y\vee z)\in A^{-J}$ for all $a\in A$ implies A^{-J} is semi prime ideal. Hence (1) implies (2) implies (3). □

Definition 3.12. *For an element b of a-distributive lattice L, we define $(b]=\{x\in L/x\le b\}$.*

Necessary and sufficient condition for a lattice L to be a-distributive lattice is $(a]$ is a semi prime down set is given in the next theorem.

Theorem 3.13. *Lattice L is a a-distributive lattice if and only if $(a]$ is semi prime down set.*

Proof. **Only if part:** Let L be a-distributive lattice $(a \in L,\ a \neq 1)$. First we show $(a]$ is down set. Let $x \leq y$ such that $y \in (a]$ implies $y \leq a$ therefore $x \leq a$ implies $x \in (a]$. Therefore $(a]$ is down set of L. Let $x \wedge y \in (a]$ and $x \wedge z \in (a]$ for $x, y, z \in L$. Hence $x \wedge y \leq a$, $x \wedge z \leq a$. As L is a-distributive $x \wedge (y \vee z) \leq a$ implies $x \wedge (y \vee z) \in (a]$. Therefore $(a]$ is semi prime down set of L.

If part: For $a \in L$, let $(a]$ be semi prime down set. To show L is a a-distributive lattice. For $x, y, z \in L$, let $x \wedge y \leq a$, $x \wedge z \leq a$ implies $x \wedge y \in (a]$, $x \wedge z \in (a]$. As $(a]$ is semi prime down set. Therefore $x \wedge (y \vee z) \in (a]$ implies $x \wedge (y \vee z) \leq a$. Therefore L is a-distributive lattice. □

Theorem 3.14. *Intersection of two semi prime ideals of a a-distributive lattice L is a semi prime ideal.*

Proof. Let I_1, I_2 are semi prime ideals of L. To show that $I = I_1 \cap I_2$ be semi prime. Let $a \wedge b \in I$ and $a \wedge c \in I$ implies $a \wedge b \in I_1 \cap I_2$ and $a \wedge c \in I_1 \cap I_2$ implies $a \wedge b \in I_1$ and $a \wedge b \in I_2$ and $a \wedge c \in I_1$ and $a \wedge c \in I_2$ implies $a \wedge b \in I_1$, $a \wedge c \in I_1$ and $a \wedge b \in I_2$, $a \wedge c \in I_2$. As I_1 is semi prime implies $a \wedge (b \vee c) \in I_1$. Similarly as I_2 is semi prime ideal implies $a \wedge (b \vee c) \in I_2$. Hence $a \wedge (b \vee c) \in (I_1 \cap I_2)$. $a \wedge (b \vee c) \in I$. Therefore $I_1 \cap I_2$ is semi prime. □

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