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General Solution and Generalized Hyers-Ulam Stability of n-dimensional Cubic Functional Equation in Various Normed Space: Direct and Fixed Point Methods

Research Article

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Abstract: In this paper, the authors investigate the general solution in vectors space and generalized Hyers-Ulam Stability of n-dimensional Cubic functional Equation

$$f\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} f\left(-x_j+\sum_{i=1; i\neq j} x_i\right)=(n-5)\sum_{1\leq i<j\leq k\leq n} f\left(x_i+x_j+x_k\right)+\left(-n^2+8n-11\right)\sum_{i=1;i\neq j} f\left(x_i+x_j\right) -\sum_{j=1}^{n} f\left(2x_j\right)+\frac{1}{2}\left(n^3-10n^2+23n+2\right)\sum_{i=1}^{n} f\left(x_i\right)$$

with $n>5$, and n is a positive integer using FNS,RNS,IFNS and FELBIN'S type spaces, using direct and fixed point methods.

MSC: 39B52, 32B72, 32B82.

Keywords: Cubic functional equations, FNS, RNS, IFNS and FELBIN'S type spaces, fixed point, generalized Hyers-Ulam Stability.

1. Introduction

In 2003, V. Radu [49] observed that many theorems concerning the stability problem of various functional equation follows from direct and fixed point alternative. Indeed, he applied the fixed point method to prove the Stability of Cauchy functional Equations, Jenson's functional Equations, quadratic functional equations, and Cubic Functional Equations (see[2,3,4,5]). After his work, many authors used the fixed point method to prove the stability of various functional Equations[13,14,15,16,17,18,19,20,23,24,25].

In this paper, we consider the fuzzy version stability problem in the fuzzy normed linear space setting. In 2008, A.K. Mirmostafaee and M.S. Moslehion[36,37,38] used the definition of a fuzzy norm in [3] to obtain a fuzzy version of Stability for the cauchy functional equation:

$$f(x+y)-f(x)-f(y)=0 \tag{1}$$

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and the quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{2}$$

we call a solution of (1) an additive mapping and a Solution of (2) is called a quadratic mapping. In particular, every Solution of the quadratic equation (2) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x)=B(x,x)$ for all x (see[1,16]). The bi-additive function B is given by

$$B(x,y)=\frac{1}{4}\left[f(x+y)-f(x-y)\right] \tag{3}$$

Hyers-Ulam-Rassias Stability Problem for the quadratic functional equation(2) was proved by skof for the functions $f: A\to B$, where A is normed space and B is Banach space (see[2]). Cholewa noticed that the theorem of skof is still true if relevant domain A is replaced by an abelian group. In the paper[10], Czerwik proved the Hyers-Ulam-Rassias Stability of the functional equation(2). Grabice has generalized these result mentioned above. We only mention here the papers [20],[28],[43],[48] concerning the stability of the quadratic functional equations. The following cubic functional equation, which is the oldest cubic functional, and was introduced by J.M. Rassios[45](in 2001):

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y) \tag{4}$$

Jun and Kim [21] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{5}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias Stability for the functional equation (5) (in this case we have a much better possible upper bound for (3) than the Hyers-Ulam-Rassions Stability). The function $f(x)=x^3$ satisfies the functional equation(5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be cubic function. There are many works in the very active area of the stability of functional equations. We only mention here the papers [39] and [8] concerning the stability of the cubic functional equation. The generalized Hyers-Ulam-Rassias Stability of different functional equation in random normed and fuzzy normed spaces has been recently studied in [3-13],[15],[23]. The solution and stability of the succeeding cubic functional equation,

$$f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)=2\left[f(x+y)+2f(x+z)+2f(y+z)+2f(x-z)-12f(y-z)\right] \tag{6}$$

$$g(2x-y)+g(x-2y)=6g(x-y)+3g(x)-3g(y) \tag{7}$$

$$\begin{aligned} f(2x\pm y\pm z)+f(\pm y\pm z)+2f(\pm y)+2f(\pm y)&=2f(x\pm y\pm z)\\ &+f(x\pm y)+f(x\pm z)+f(-x\pm y)+f(-x\pm z)+6f(x) \end{aligned} \tag{8}$$

$$f\left(ax_i+b\sum_{i=2}^{n}x_i\right)+f\left(ax_i-b\sum_{i=2}^{n}x_i\right)+2a\left(b^2-a^2\right)f(x_i)=ab^2\left[f\left(\sum_{i=1}^{n}x_i\right)+f\left(x_i-\sum_{i=2}^{n}x_i\right)\right] \tag{9}$$

$$f\left(\sum_{j=1}^{n-1}x_j+2x_n\right)+f\left(\sum_{j=1}^{n-1}x_j-2x_n\right)+\sum_{j=1}^{n-1}(2x_j)=2f\left(\sum_{j=1}^{n-1}2x_j\right)+4\sum_{j=1}^{n-1}\left(f(x_j+x_n)+f(x_j-x_n)\right) \tag{10}$$

was dealt by Y.S. Jung, I.S. chong[23], M. Arunkumar [3,4,47], H.y.Chu, D.S.Kang[8]. In this paper, the authors investigate the general solution and generalized Hyers-Ulam Stability of a new form of n-dimentional cubic functional equation,

$$f\left(\sum_{i=1}^{n} x_i\right) + f\left(-x_j + \sum_{i=1;i\neq j} \right) = (n-5)\sum_{1\leq i\leq j<k\leq n} (x_i + x_j + x_k) + \left(-n^2 + 8n - 11\right)\sum_{i=1;i\neq j} (x_i + x_j)$$
$$- \sum_{j=1}^{n} (2x_j) + \frac{1}{2}\left(n^3 - 10n^2 + 23n + 2\right)\sum_{i=1}^{n} f(x_i) \tag{11}$$

with $n > 5$, and n is a positive integer using FNS,RNS,IFNS and FELBIN'S type spaces by direct and fixed point methods.

2. General Solution of the n-dimensional Cubic Functional Equation (11)

In this section, the authors discuss the general solution of the functional equation (11) by considering X and Y are real vector space.

Theorem 2.1. *If $f : X \to Y$ satisfies the functional equation (11) for all $x_1, x_2, x_3, ..., x_n \in X$ and $n > 5$ then there exists a function $B : X^3 \to Y$ such that $f(x) = B(x, x, x)$ for all $x \in X$, where B is symmetric for each fixed one variable and additive for each fixed two variables.*

Proof. Assume that $f : X \to Y$ satisfies the functional equation(11). Letting $x_1, x_2, x_3, ..., x_n$ by $(x, 0, 0, ..., 0)$ in (11), we get

$$(2n-2) f(x) = (2n+14) f(x) - 2f(2x) \tag{12}$$

for all $x \in X$. It follows from (12), that

$$f(2x) = (2)^3 f(x) \tag{13}$$

for all $x \in X$. Replacing x by $2x$ in (13) we obtain $f(4x) = 4^3 f(x)$. for all $x \in X$. In general for any positive integers a, we arrive

$$f(2ax) = (2a)^3 f(x) \tag{14}$$

for all $x \in X$. Setting the above equation (12), we get $f(0) = 0$.
Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x_1, x_2, x_3, x_4, x_5, x_6, 0..., 0)$ in (11), we have

$$\begin{aligned}
&f(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + f(-x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + f(x_1 - x_2 + x_3 + x_4 + x_5 + x_6)\\
&+ f(x_1 + x_2 - x_3 + x_4 + x_5 + x_6) + f(x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + f(x_1 + x_2 + x_3 + x_4 - x_5 + x_6)\\
&+ f(x_1 + x_2 + x_3 + x_4 + x_5 - x_6) = (n-5)\left[f(x_1 + x_2 + x_3) + f(x_1 + x_2 + x_4) + f(x_1 + x_2 + x_5)\right]\\
&+ (n-5)\left[f(x_1 + x_2 + x_6) + f(x_1 + x_3 + x_4) + f(x_1 + x_3 + x_5)\right]\\
&+ (n-5)\left[f(x_1 + x_3 + x_6) + f(x_1 + x_4 + x_5) + f(x_1 + x_4 + x_6)\right]\\
&+ (n-5)\left[f(x_1 + x_5 + x_6) + f(x_2 + x_3 + x_4) + f(x_2 + x_3 + x_5)\right]\\
&+ (n-5)\left[f(x_2 + x_3 + x_6) + f(x_2 + x_4 + x_5) + f(x_2 + x_4 + x_6)\right]\\
&+ (n-5)\left[f(x_2 + x_5 + x_6) + f(x_3 + x_4 + x_5) + f(x_3 + x_4 + x_6)\right]\\
&+ (n-5)\left[f(x_3 + x_5 + x_6) + f(x_4 + x_5 + x_6)\right]
\end{aligned}$$

$$+\left(-n^{2}+8n-11\right)(n-5)\left[f\left(x_1+x_2\right)+f\left(x_1+x_3\right)+f\left(x_1+x_4\right)+f\left(x_1+x_5\right)\right]$$
$$+\left(-n^{2}+8n-11\right)(n-5)\left[f\left(x_1+x_6\right)+f\left(x_2+x_3\right)+f\left(x_2+x_4\right)+f\left(x_2+x_5\right)\right]$$
$$+\left(-n^{2}+8n-11\right)(n-5)\left[f\left(x_2+x_6\right)+f\left(x_3+x_4\right)+f\left(x_3+x_5\right)+f\left(x_3+x_6\right)\right]$$
$$+\left(-n^{2}+8n-11\right)(n-5)\left[f\left(x_4+x_5\right)+f\left(x_4+x_6\right)+f\left(x_5+x_6\right)\right]$$
$$-f\left(2x_1\right)-f\left(2x_2\right)-f\left(2x_3\right)-f\left(2x_4\right)-f\left(2x_5\right)-f\left(2x_6\right)$$
$$+\frac{1}{2}\left(n^{3}-10n^{2}+23n+2\right)\left[f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)+f\left(x_6\right)\right] \tag{15}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6 \in X$. Using (13)in (15), we obtain
$f\left(x_1+x_2+x_3+x_4+x_5+x_6\right) + f\left(-x_1+x_2+x_3+x_4+x_5+x_6\right) + f\left(x_1-x_2+x_3+x_4+x_5+x_6\right) + f\left(x_1+x_2-x_3+x_4+x_5+x_6\right) + f\left(x_1+x_2+x_3-x_4+x_5+x_6\right) + f\left(x_1+x_2+x_3+x_4-x_5+x_6\right) + f\left(x_1+x_2+x_3+x_4+x_5-x_6\right) = f\left(x_1+x_2+x_3\right) + f\left(x_1+x_2+x_4\right) + f\left(x_1+x_2+x_5\right) + f\left(x_1+x_2+x_6\right) + f\left(x_1+x_3+x_4\right) + f\left(x_1+x_3+x_5\right) + f\left(x_1+x_3+x_6\right) + f\left(x_1+x_4+x_5\right) + f\left(x_1+x_4+x_6\right) + f\left(x_1+x_5+x_6\right) + f\left(x_2+x_3+x_4\right) + f\left(x_2+x_3+x_5\right) + f\left(x_2+x_3+x_6\right) + f\left(x_2+x_4+x_5\right) + f\left(x_2+x_4+x_6\right) + f\left(x_2+x_5+x_6\right) + f\left(x_3+x_4+x_5\right) + f\left(x_3+x_4+x_6\right) + f\left(x_3+x_5+x_6\right) + f\left(x_4+x_5+x_6\right) + f\left(x_1+x_2\right) + f\left(x_1+x_3\right) + f\left(x_1+x_4\right) + f\left(x_1+x_5\right)+f\left(x_1+x_6\right)+f\left(x_2+x_3\right)+f\left(x_2+x_4\right)+f\left(x_2+x_5\right)+f\left(x_2+x_6\right)+f\left(x_3+x_4\right)+f\left(x_3+x_5\right)+f\left(x_3+x_6\right)+f\left(x_4+x_5\right)+f\left(x_4+x_6\right)+f\left(x_5+x_6\right)-8f\left(x_1\right)-8f\left(x_2\right)-8f\left(x_3\right)-8f\left(x_4\right)-8f\left(x_5\right)-8f\left(x_6\right)$

$$-2\left[f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)+f\left(x_6\right)\right] \tag{16}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6 \in X$. Replacing $(x_1, x_2, x_3, x_4, x_5, x_6)$ by $(x, 0, 0, 0, 0, 0)$ in (16) and using (13), we get

$$f(-x) = -f(x) \tag{17}$$

for all $x \in X$. Hence f is an odd function. Setting $(x_1, x_2, x_3, x_4, x_5, x_6)$ by $(x, x, y, 0, 0, 0)$ in(16), we obtain

$$3f(2x+y)+2f(y)+f(2x-y)=8f(x+y)+24f(x)+6f(y) \tag{18}$$

for all $x, y \in X$. Replacing y by $-y$ in (18) and using (17), we arrive

$$3f(2x-y)-2f(y)+f(2x+y)=8f(x-y)+24f(x)-6f(y) \tag{19}$$

for all $x, y \in X$. Adding (18) and (19), we get

$$f(2x+y)+(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{20}$$

for all $x, y \in X$. By theorem 2.1 of [16], we desired our result. Hence the proof is complete.

Throughout this paper we use the following notation for a given mapping $f : X \to Y$

$$\begin{aligned} Df\left(x_1, x_2, x_3, ..., x_n\right) = f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1; i\neq j} x_i\right) - (n-5) \sum_{;1\leq i<j\leq k\leq n} f\left(x_i+x_j+x_k\right) \\ - \left(-n^{2}+8n-11\right) \sum_{i=1; i\neq j} f\left(x_i+x_j\right) + \sum_{j=1}^{n} f\left(2x_j\right) - \frac{1}{2}\left(n^{3}-10n^{2}+23n+2\right) \sum_{i=1}^{n} f\left(x_i\right) \end{aligned}$$

for all $x_1, x_2, x_3, ..., x_n \in X$. □

3. Preliminaries of Fuzzy Normed spaces

We use the definition of fuzzy normed spaces given in [15] and [32,50,58].

Definition 3.1. *Let X be a real linear space. A function $N : X \times R \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in R$,*

(F_1) $N(x,c) = 0$ *for* $c \leq 0$;

(F_2) $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

(F_3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$

(F_4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

(F_5) $N(x,.)$ *is a non-decreasing function on* $\Re$ *and* $\lim_{t\to\infty} N(x,t) = 1$;

(F_6) *for* $x \neq 0, N(x,.)$ *is (upper semi) continuous on* $\Re$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,t)$ as the truth-value of the statement the norm of x is less than or equal to the real numbers t'.

Example 3.2. *Let $(X, \|.\|)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, \quad x \in X \\ 0 \quad ; & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 3.3. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} (x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.*

Definition 3.4. *A sequence x_n in X is called cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 3.5. *Every convergent sequence in a fuzzy normed space is cauchy. If each cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 3.6. *A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in x, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

4. Fuzzy Stability Results: Direct Method

Throughout this section, assume that $X, \left(Z, N'\right)$ and $\left(Y, N'\right)$ are linear space, fuzzy normed space and fuzzy Banach space,respectively. Now, we investigate the generalized Hyers-Ulam Stability of n-dimensional cubic functional (11).

Theorem 4.1. *Let $\beta \in \{-1, 1\}$ be fixed and let $\alpha : X^n \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2^3}\right)^\beta < 1$*

$$N'\left(\alpha\left(2^\beta x, 0, 0, ..., 0\right), r\right) \geq N'\left(d^\beta \alpha(x, 0, 0, ..., 0), r\right) \tag{21}$$

for all $x \in X$ and all $r > 0, d > 0$, and

$$\lim_{k\to\infty} N'\left(\alpha\left(2^{\beta k}x_1, 2^{\beta k}x_2, 2^{\beta k}x_3, ..., 2^{\beta k}x_n\right), 2^{\beta 3k}r\right) = 1 \tag{22}$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(Df\left(x_1, x_2, x_3, ..., x_n\right), r\right) \geq N'\left(\alpha\left(x_1, x_2, x_3, ..., x_n\right), r\right) \tag{23}$$

for all $r > 0$ and $x_1, x_2, x_3, ..., x_n \in X$. Then the limit

$$C\left(x\right) = N - \lim_{k\to\infty} \frac{f\left(2^{\beta k}x\right)}{2^{\beta 3k}} \tag{24}$$

for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping such that

$$N\left(f\left(x\right) - C\left(x\right), r\right) \geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), r\left|2^3 - d\right|\right) \tag{25}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\beta = 1$. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in (23),we get

$$\begin{aligned} N\left(2f\left(2x\right) - 2^4 f\left(x\right), r\right) &\geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), r\right) \\ N\left(f\left(2x\right) - 8f\left(x\right), r\right) &\geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), r\right) \end{aligned} \tag{26}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^k x$ in (26), we obtain

$$N\left(\frac{f\left(2^{k+1}x\right)}{2^3} - f\left(2^k x\right), \frac{r}{2^3}\right) \geq N'\left(\alpha\left(2^k x, 0, 0, ..., 0\right), r\right) \tag{27}$$

for all $x \in X$ and all $r > 0$. Using (21), F_3 in (27), we arrive

$$N\left(\frac{f\left(2^{k+1}x\right)}{2^3} - f\left(2^k x\right), \frac{r}{2^3}\right) \geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), \frac{r}{d^k}\right) \tag{28}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (28), that

$$N\left(\frac{f\left(2^{k+1}x\right)}{2^{3(K+1)}} - \frac{f\left(2^k x\right)}{2^{3k}}, \frac{r}{2^3.2^{3k}}\right) \geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), \frac{r}{d^k}\right) \tag{29}$$

holds for all $x \in X$ and all $r > 0$. Replacing r by $d^k r$ in (29), we get

$$N\left(\frac{f\left(2^{k+1}x\right)}{2^{3(K+1)}} - \frac{f\left(2^k x\right)}{2^{3k}}, \frac{d^k r}{2^3.2^{3k}}\right) \geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), r\right) \tag{30}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f\left(2^k x\right)}{2^{3k}} - f\left(x\right) = \sum_{i=0}^{k-1}\left[\frac{f\left(2^{i+1}x\right)}{2^{3(i+1)}} - \frac{f\left(2^i x\right)}{2^{3i}}\right] \tag{31}$$

for all $x \in x$. From the equations (30) and (31), we have

$$\begin{aligned} N\left(\frac{f\left(2^k x\right)}{2^{3k}} - f\left(x\right), \sum_{i=0}^{k-1}\frac{d^i r}{2^3.2^{3k}}\right) &\geq min\bigcup_{i=0}^{k-1}\left\{\frac{f\left(2^{i+1}x\right)}{2^{3(i+1)}} - \frac{f\left(2^i x\right)}{2^{3i}}, \frac{d^i r}{2^3.2^{3k}}\right\} \\ &\geq min\bigcup_{i=0}^{k-1}\left\{N'\left(\alpha\left(x, 0, 0, ..., 0\right), r\right)\right\} \\ &\geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), r\right) \end{aligned} \tag{32}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (32) and using (21), (F_3), we obtain

$$N\left(\frac{f\left(2^{k+m}x\right)}{2^{3(k+m)}} - \frac{f\left(2^m x\right)}{2^{3m}}, \sum_{i=0}^{k-1} \frac{d^i r}{2^3.2^{3(i+m)}}\right) \geq N^{'}\left(\alpha\left(x,0,0,...,0\right), \frac{r}{d^m}\right) \tag{33}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Replacing r by $d^m r$ in (33), we get

$$N\left(\frac{f\left(2^{k+m}x\right)}{2^{3(k+m)}} - \frac{f\left(2^m x\right)}{2^{3m}}, \sum_{i=m}^{m+k-1} \frac{d^i r}{2^3.2^{3i}}\right) \geq N^{'}\left(\alpha\left(x,0,0,...,0\right), r\right) \tag{34}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Using (F_3) in (33), we obtain

$$N\left(\frac{f\left(2^{k+m}x\right)}{2^{3(k+m)}} - \frac{f\left(2^m x\right)}{2^{3m}}, r\right) \geq N^{'}\left(\alpha\left(x,0,0,...,0\right), \frac{r}{\sum_{i=m}^{m+k-1} \frac{d^i}{2^3.2^{3i}}}\right) \tag{35}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Since $0 < d < 2^3$ and $\sum_{i=0}^{k}\left(\frac{d}{2^3}\right)^i < \infty$, the cauchy criterion for convergence and (F_5) implies that $\left\{\frac{f\left(2^k x\right)}{2^{3k}}\right\}$ is a cauchy sequence in (Y,N),since (Y,N) is a fuzzy Banach space, this sequence convergence to some point $C(x) \in Y$. So one can define the mapping $C : X \to Y$ by

$$C(x) = N - \lim_{k\to\infty}\{\frac{f\left(2^k x\right)}{2^{3k}}$$

for all $x \in X$. Letting $m = 0$ in (35), we get

$$N\left(\frac{f\left(2^k x\right)}{2^{3(k)}} - f(x), r\right) \geq N^{'}\left(\alpha\left(x,0,0,...,0\right), \frac{r}{\sum_{i=0}^{k-1} \frac{d^i}{2^3.2^{3i}}}\right) \tag{36}$$

For all $x \in x$ and all $r > 0$. Letting $K \to \infty$ in (36) and using (F_6), we arrive

$$\begin{aligned} N\left(f(x) - C(x), r\right) &\geq N^{'}\left(\alpha\left(x,0,0,...,0\right), \frac{r}{\frac{1}{8}\left[\frac{d^0}{2^0} + \frac{d^1}{2^3} + \frac{d^1}{2^3} + \frac{d^2}{\left(2^3\right)^2} + ...\right]}\right) \\ &\geq N^{'}\left(\alpha\left(x,0,0,...,0\right), r\left(2^3 - d\right)\right) \end{aligned}$$

for all $x \in x$ and all $r > 0$. To prove C satisfies the (11), replacing $(x_1, x_2, x_3, ..., x_n)$ by $\left(2^k x_1, 2^k x_2, 2^k x_3, ..., 2^k x_n\right)$ in (23),respectively, we obtain

$$N\left(\frac{1}{2^{3k}} Df\left(2^k x_1, 2^k x_2, 2^k x_3, ..., 2^k x_n\right), r\right) \geq N^{'}\left(\alpha\left(2^k x_1, 2^k x_2, 2^k x_3, ..., 2^k x_n\right), 2^{3k} r\right) \tag{37}$$

for all $r > 0$ and all $x_1, x_2, x_3, ..., x_n \in X$. Now

$$\begin{aligned} N&\left(C\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} C\left(-x_j + \sum_{i=1; i\neq j}^{n} x_i\right) - (n-5) \sum_{1\leq i<j\leq k\leq n} C\left(x_i + x_j + x_k\right)\right) \\ &- N\left(-\left(-n^2 + 8n - 11\right) \sum_{i=1; i\neq j}^{n} C\left(x_i + x_j\right) - \sum_{j=1}^{n} C\left(2x_j\right) - \frac{1}{2}\left(n^3 - 10n^2 + 23n + 2\right) \sum_{i=1}^{n} C\left(x_i\right)\right) \\ &\geq min\left\{N\left(C\left(\sum_{i=1}^{n} x_i\right) - \frac{1}{2^{3k}} f\left(\sum_{i=1}^{n} 2^k x_i\right), \frac{r}{7}\right)\right\}, \\ &\min\left\{N\left(\sum_{j=1}^{n} C\left(\left(-x_j + \sum_{i=1; i\neq j}^{n} x_i\right)\right) - \frac{1}{2^{3k}} \sum_{j=1}^{n} f\left(2^k\left(-x_j + \sum_{i=1; i\neq j}^{n} x_i\right)\right), \frac{r}{7}\right), \right\}, \end{aligned}$$

$$\min\left\{N\left(-(n-5)\sum_{i=1;1\leq i<j\leq k\leq n}C(x_i+x_j+x_k)+\frac{1}{2^{3k}}(n-5)\sum_{i=1;1\leq i<j\leq k\leq n}f\left((x_i+x_j+x_k)2^k\right),\frac{r}{7}\right),\right\},$$
$$\min\left\{N\left(-(-n^2+8n-11)\sum_{i=1;i\neq j}^{n}C(x_i+x_j)+\frac{1}{2^{3k}}(-n^2+8n-11)\sum_{i=1;i\neq j}^{n}f\left(2^k(x_i+x_j)\right),\frac{r}{7}\right),\right\},$$
$$min\left\{N\left(-\sum_{j=1}^{n}C(2x_j)+\frac{1}{2^{3k}}\sum_{j=1}^{n}f\left(2^k2x_j\right),\frac{r}{7}\right),\right\},$$
$$\min\left\{N\left(-\frac{1}{2}(n^3-10n^2+23n+2)\sum_{i=1}^{n}C(x_i)+\frac{1}{2^{3k}}\left(\frac{n^3-10n^2+23n+2}{2}\right)\sum_{i=1}^{n}f\left(2^kx_i\right),\frac{r}{7}\right)\right\},$$
$$\min\left\{N\left(\frac{1}{2^{3k}}f\left(2^k\sum_{i=1}^{n}x_i\right)\right)+\frac{1}{2^{3k}}\sum_{j=1}^{n}f\left(2^k\left(-x_j+\sum_{i=1;i\neq j}x_i\right)\right)-\frac{(n-5)}{2^{3k}}\sum_{i=1;1\leq i<k\leq j\leq n}f\left(2^k(x_i+x_j+x_k)\right)\right\}$$
$$min\left\{N\frac{(-n^2+8n-11)}{2^{3k}}\sum_{i=1;i\neq j}f\left(2^k(x_i+x_j)\right)+\frac{1}{2^{3k}}\sum_{j=1}^{n}f\left(2^k.2x_j\right)-\frac{1}{2}\frac{(n^3-10n^2+23n+2)}{2^{3k}}\sum_{i=1}^{n}f\left(2^kx_i\right),\frac{r}{7}\right\}$$

for all $x_1,x_2,x_3,...,x_n\in X$ and all $r>0$. Using (37) and (F_5) in (38), we arrive

$$N\left(C\left(\sum_{i=1}^{n}x_i\right)+\sum_{j=1}^{n}C\left(-x_j+\sum_{i=1;i\neq j}x_i\right)-(n-5)\sum_{1\leq i<k\leq j\leq n}C((x_i+x_j+x_k)\right)$$

$$\begin{aligned}&-N\left((-n^2+8n-11)\right)\sum_{i=1;i\neq j}C(x_i+x_j)+N\left(\sum_{j=1}^{n}C(2x_j)-\frac{1}{2}(n^3-10n^2+23n+2)\sum_{i=1}^{n}C(x_i),r\right)\\&\geq min\left\{1,1,1,1,1,1,N'\left(\alpha\left(2^kx_1,2^kx_2,2^kx_3,...,2^kx_n\right),2^{3k}r\right)\right\}\\&\geq N'\left(\alpha\left(2^kx_1,2^kx_2,2^kx_3,...,2^kx_n\right),2^{3k}r\right)\end{aligned}\tag{38}$$

for all $x_1,x_2,x_3,...,x_n\in X$ and all $r>0$. Letting $K\to\infty$ in (39) and using (22), we see that

$$N\left(C\left(\sum_{i=1}^{n}x_i\right)+\sum_{i=1;i\neq j}^{n}C\left(-x_j+\sum_{i=1;i\neq j}x_i\right)-(n-5)\sum_{i=1;1\leq i<k\leq j\leq n}C((x_i+x_j+x_k)\right)$$

$$-N\left((-n^2+8n-11)\sum_{i=1;i\neq j}C(x_i+x_j)+\sum_{j=1}^{n}C(2x_j)-\frac{1}{2}(n^3-10n^2+23n+2)\sum_{i=1}^{n}C(x_i)\right)=1\tag{39}$$

for all $x_1,x_2,x_3,...,x_n\in X$ and all $r>0$. Using (F_2) in the above inequality gives

$$\begin{aligned}\sum_{i=1}^{n}C(x_i)+\sum_{j=1}^{n}C\left(-x_j+\sum_{i=1;i\neq j}x_i\right)&=(n-5)\sum_{1\leq i<k\leq j\leq n}C((x_i+x_j+x_k)\\&+(-n^2+8n-11)\sum_{i=1;i\neq j}C(x_i+x_j)-\sum_{j=1}^{n}C(2x_j)+\frac{1}{2}(n^3-10n^2+23n+2)\sum_{i=1}^{n}C(x_i)\end{aligned}$$

for all $x_1,x_2,x_3,...,x_n\in X$. Hence C satisfies the cubic functional equation (11). In order to prove $C(x)$ is unique, let $C'(x)$ be another cubic functional equation satisfies (11) and (25). Hence,

$$\begin{aligned}N\left(\left(C(x)-C'(x)\right),r\right)=N\left(\frac{C(2^kx)}{2^{3k}}-\frac{C'(2^kx)}{2^{3k}},r\right)&\geq\min\left\{N\left(\frac{C(2^kx)}{2^{3k}}-\frac{f(2^kx)}{2^{3k}},\frac{r}{2}\right),N\left(\frac{f(2^kx)}{2^{3k}}-\frac{C'(2^kx)}{2^{3k}},\frac{r}{2}\right)\right\}\\&\geq N'\left(\alpha\left(2^kx,0,0,...,0\right),\frac{r2^{3k}(2^3-d)}{2}\right)\\&\geq N'\left(\alpha(x,0,0,...,0),\frac{r2^{3k}(2^3-d)}{2^{dk}}\right)\end{aligned}$$

for all $x \in X$ and all $r > 0$. Since

$$\lim_{k\to\infty} \frac{r2^{3k}\left(2^3-d\right)}{2^{dk}} = \infty$$

we obtain

$$\lim_{k\to\infty} N^{'}\left(\alpha\left(x,0,0,...,0\right), \frac{r2^{3k}\left(2^3-d\right)}{2^{dk}}\right) = 1$$

Thus

$$N\left(C\left(x\right)-C^{'}\left(x\right),r\right) = 1$$

for all $x \in X$ and all $r > 0$, hence $C\left(x\right) = C^{'}\left(x\right)$. Therefore $C\left(x\right)$ is unique. For $\beta = -1$, we can prove the result by a similar methods. This completes the proof of the theorem. From Theorem 4.1, we obtain the following corollary concerning the generalized Hyers-Ulam Stability for the functional equation (11). □

Corollary 4.2. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df\left(x_1,x_2,x_3,...,x_n\right),r\right) \geq \begin{cases} N^{'}\left(\epsilon,r\right) \\ N^{'}\left(\epsilon\sum_{i=1}^{n}\|x_i\|^s,r\right); \quad s\neq 3 \\ N^{'}\left(\epsilon\left(\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right),r\right); \quad s\neq \frac{3}{n} \end{cases} \tag{40}$$

for all $x_1,x_2,x_3,...,x_n \in X$ and all $r > 0$, where ϵ, s are constants, there exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f\left(x\right)-C\left(x\right),r\right) \geq \begin{cases} N^{'}\left(\epsilon,r\left|2^3-1\right|\right); \\ N^{'}\left(\epsilon\|x_i\|^s,r\left|2^3-2^s\right|\right); \quad s\neq 3 \\ N^{'}\left(\epsilon\left(\|x_i\|^{ns}\right),r\left|2^3-2^{ns}\right|\right); \quad s\neq \frac{3}{n} \end{cases} \tag{41}$$

for all $x \in X$ and all $r > 0$.

5. Fuzzy Stability Results: Fixed Point Method

In this section, the authors presented the generalized Hyers-Ulam Stability of the functional equation(11) in Fuzzy normed spaces using fixed point method,[12]. Now we recall the fundamental results in fixed point theory.

Theorem 5.1 ([23] The alternative of fixed point)**.** *Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

(B_1) $d\left(T^n x, T^{n+1}x\right) = \infty, \forall n \geq 0$

(or)

(B_2) *there exists a natural number n_0 such that:*

(i) $d\left(T^n x, T^{n+1}x\right) < \infty$ for all $n \geq n_0$

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii)y^* is the unique fixed point of T in the set $Y = \{y \in X : d\left(T^{n_0}x, y\right) < \infty\}$;

(iv)$d\left(y^*, y\right) \leq \frac{1}{1-L} d\left(y, Ty\right)$ for all $y \in Y$.

For to prove the stability result we define the following :

δ_i is a constant such that

$$\delta_i = \begin{cases} 2 & if \quad i = 0 \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g/g : X \to Y, g(0) = 0\}.$$

Theorem 5.2. *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^n \to Z$ with the condition,*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, ..., \delta_i^k x_n\right), \delta_i^{3k} r\right) = 1 \quad \forall x_1, x_2, x_3, ..., x_n \in X, r > 0 \tag{42}$$

and satisfying the functional in equality

$$N(Df(x_1, x_2, x_3, ..., x_n), r) \geq N'(\alpha(x_1, x_2, x_3, ..., x_n), r) \quad \forall x_1, x_2, x_3, ..., x_n \in X, r > 0 \tag{43}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \alpha\left(\frac{x}{2}, 0, 0, ..., 0\right), \tag{44}$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'(\beta(x), r), \forall x \in X, r > 0 \tag{45}$$

Then there exists unique cubic functions $C : X \to Y$ satisfies the functional equation (11) and

$$N(f(x) - c(x), r) \geq N'\left(\frac{L^1 - i}{1 - L}\beta(x), r\right), \forall x \in X, r > 0 \tag{46}$$

Proof. Let d be a general metric on Ω, such that $d(g, h) = inf\left\{K \in (0, \infty)/N(g(x) - h(x), r) \geq N'(k\beta(x), r), x \in X, r > 0\right\}$ It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3}g(\delta_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have

$$\begin{aligned} d(g, h) \leq k &\Rightarrow N(g(x) - h(x), r) \geq N'(k\beta(x), r) \\ &\Rightarrow N\left(\frac{g(\delta_i x)}{\delta_i^3} - \frac{h(\delta_i x)}{\delta_i^3}, r\right) \geq N'\left(\frac{k}{\delta_i^3}\beta(\delta_i x), r\right) \\ &\Rightarrow N(Tg(x) - Th(x), r) \geq N'(kL\beta(x), r) \\ &\Rightarrow d(Tg(x), Th(x)) \leq KL \\ &\Rightarrow d(Tg, Th) \leq Ld(g, h) \end{aligned} \tag{47}$$

for all $g, h \in \Omega$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in (44) we get

$$N(f(2x) - 8f(x), r) \geq N'(\alpha(x, 0, 0, ..., 0), r) \tag{48}$$

for all $x \in X, r > 0$. Using (F_3) in (48,) we arrive

$$N\left(\frac{f(2x)}{2^3} - f(x), r\right) > N'\left(\frac{1}{2^3}\alpha(x, 0, 0, ..., 0), r\right) \tag{49}$$

for all $x \in X, r > 0.$, with the help of (45) when $i = 0$,it follows from(49), we get

$$\begin{aligned} \Rightarrow \quad & N\left(\frac{f(2x)}{2^3} - f(x), r\right) \geq N'(L\beta(x), r) \\ \Rightarrow \quad & d(Tf, f) \leq L = L^1 = L^{1-i} \end{aligned} \tag{50}$$

Replacing x by $\left(\frac{x}{2}\right)$ in,(48) we obtain

$$N\left(f(x) - 8f\left(\frac{x}{2}\right), r\right) \geq N'\left(\frac{1}{2^3}\alpha\left(\frac{x}{2}, 0, 0, ..., 0\right), r\right) \tag{51}$$

for all $x \in X, r > 0$, with the help of (45) when $i = 1$, it follows from (51), we get

$$\Rightarrow N\left(f(x) - 8f\left(\frac{x}{2}\right), r\right) \geq N'(\beta(x), r)$$

$$\Rightarrow d(Tf, f) \leq 1 = L^0 = L^{1-i} \tag{52}$$

Then from (50) and (52), we can conclude,

$$d(Tf, f) \leq L^{1-i} < \infty$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x) = N - \lim_{k\to\infty} \frac{f(2^k x)}{2^{3k}} \quad \forall x \in X, r > 0. \tag{53}$$

Replacing $(x_1, x_2, x_3, ..., x_n)$ by $\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, ..., \delta_i^k x_n\right)$ in (44),we arrive

$$N\left(\frac{1}{\delta_i^{3k}} Df\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, ..., \delta_i^k x_n\right), r\right) \geq N'\left(\alpha\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, ..., \delta_i^k x_n\right), \delta_i^{3k} r\right) \tag{54}$$

for all $r > 0$ and all $x_1, x_2, x_3, ..., x_n \in X$. By proceeding the same procedure as in the Theorem 4.1, we can prove the function , $C : X \to Y$ satisfies the functional equation(11). By fixed point alternative, since C is unique fixed point of T in the set $\Delta = \{f \in \Omega / d(f, c) < \infty\}$, therefore C is a unique function such that

$$N(f(x) - C(x), r) \geq N'(k\beta(x), r) \tag{55}$$

for all $x \in X$ and all $r > 0$ and $k > 0$. Again using the fixed point alternative, we obtain

$$\begin{aligned} d(f, C) &\leq \frac{1}{1-L} d(T, Tf) \\ \Rightarrow d(f, C) &\leq \frac{L^{1-i}}{1-L} \\ N(f(x) - C(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \end{aligned} \tag{56}$$

for all $x \in X$ and $r > 0$. This completes the proof of the theorem. From Theorem 5.2, we obtain the following corollary concerning the stability for the functional equation(11). □

Corollary 5.3. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df\left(x_1,x_2,x_3,...,x_n\right),r\right) \geq \begin{cases} N'\left(\epsilon,r\right) \\ N'\left(\epsilon\sum_{i=1}^{n}\|x_i\|^s,r\right); \quad s\neq 3 \\ N'\left(\epsilon\left(\prod_{i=1}^{n}\|x_i\|^s\right),r+\sum_{i=1}^{n}\|x_i\|^{ns},r\right); \quad s\neq \frac{3}{n} \end{cases} \tag{57}$$

for all $x_1,x_2,x_3,...,x_n \in X$ and $r>0$, where ϵ, S are constants with $\epsilon>0$. There exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f\left(x\right)-C\left(x\right),r\right) \geq \begin{cases} N'\left(\epsilon,r\left|2^3-1\right|\right), \\ N'\left(\epsilon\|x\|^s,r\left|2^3-2^s\right|\right), \\ N'\left(\epsilon\|x\|^{ns},r\left|2^3-2^{ns}\right|\right), \end{cases} \tag{58}$$

for all $x \in X$ and all $r>0$.

Proof. Setting

$$\alpha\left(x_1,x_2,x_3,...,x_n\right) = \begin{cases} \epsilon, \\ \epsilon\sum_{i=1}^{n}\|x_i\|^s, \\ \epsilon\left(\prod_{i=1}^{n}\|x_i\|^s+\sum_{i=1}^{n}\|x_i\|^{ns}\right), \end{cases}$$

for all $x_1,x_2,x_3,...,x_n \in X$. Then

$$N'\left(\alpha\left(\delta_i^k x_1,\delta_i^k x_2,\delta_i^k x_3,...,\delta_i^k x_n\right),\delta_i^{3k}r\right) = \begin{cases} N'\left(\epsilon,\delta_i^{3k}r\right) \\ N'\left(\epsilon\sum_{i=1}^{n}\|x_i\|^s,\delta_i^{(3-s)k}r\right), \\ N'\left(\epsilon\left(\prod_{i=1}^{n}\|x_i\|^s,r\right)+\sum_{i=1}^{n}\|x_i\|^{ns},\delta_i^{(3-ns)k}r\right), \end{cases}$$

$$= \begin{cases} \to 1 \quad as \quad k\to\infty \\ \to 1 \quad as \quad k\to\infty \\ \to 1 \quad as \quad k\to\infty \end{cases}$$

Thus, (33) is holds. But we have $\beta(x) = \alpha\left(\frac{x}{2},0,0,...,0\right)$ has the property $N'\left(L\frac{1}{\delta_i^3}\beta\left(\delta_i x\right),r\right) \geq N'\left(\beta\left(x\right),r\right) \quad \forall x\in X, r>0$. Hence

$$N'\left(\beta\left(x\right),r\right) = N'\left(\alpha\left(\frac{x}{2},0,0,...,0\right),\right) = \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\frac{\epsilon}{2^s}\|x\|^s,r\right), \\ N'\left(\frac{\epsilon}{2^{ns}}\|x\|^{ns},r\right), \end{cases}$$

Now

$$N'\left(\frac{1}{\delta_i^3}\beta\left(\delta_i x\right),r\right) = \begin{cases} N'\left(\frac{\epsilon}{\delta_i^3},r\right), \\ N'\left(\frac{\epsilon}{\delta_i^3}\left(\frac{1}{2^s}\right)\|\delta_i x\|^s,r\right), \\ N'\left(\frac{\epsilon}{\delta_i^3}\left(\frac{1}{2^{ns}}\right)\|\delta_i x\|^{ns},r\right), \end{cases}$$

$$= \begin{cases} N'\left(\delta_i^{-3}\beta\left(x\right),r\right), \\ N'\left(\delta_i^{s-3}\beta\left(x\right),r\right), \\ N'\left(\delta_i^{ns-3}\beta\left(x\right),r\right), \end{cases}$$

Now from (36), we prove the following cases for conditions (i) and (ii).

case:1 $L = 2^{-3}$ for $s = 0$ if $i = 0$

$$\begin{aligned} N\left(f\left(x\right)-C\left(x\right),r\right) &\geq N^{'}\left(\frac{\left(2^{-3}\right)^{1-0}}{1-2^{-3}}\beta\left(x\right),r\right) \\ &= N^{'}\left(\epsilon 2^{-3}\times\frac{2^3}{2^3-1}\left\|x\right\|^s,r\right) \\ &= N^{'}\left(\epsilon\left\|x\right\|^s,r\left[2^3-1\right]\right) \end{aligned}$$

case:2 $L = 2^{3}$ for $s = 0$ if $i = 1$

$$\begin{aligned} N\left(f\left(x\right)-C\left(x\right),r\right) &\geq N^{'}\left(\frac{\left(2^{3}\right)^{1-1}}{1-2^{3}}\beta\left(x\right),r\right) \\ &\geq N^{'}\left(\epsilon\frac{1}{1-2^3}\left\|x\right\|^s,r\right) \\ &\geq N^{'}\left(\epsilon\left\|x\right\|^s,r\left[1-2^3\right]\right) \end{aligned}$$

case:3 $L = 2^{s-3}$ for $s > 3$ if $i = 0$

$$\begin{aligned} N\left(f\left(x\right)-c\left(x\right),r\right) &\geq N^{'}\left(\epsilon\frac{\left(2^{s-3}\right)^{1-0}}{1-2^{s-3}}\beta\left(x\right),r\right) \\ &\geq N^{'}\left(\frac{\epsilon}{2^3-2^s}\left\|x\right\|^s,r\right) \\ &\geq N^{'}\left(\epsilon\left\|x\right\|^s,r\left[2^3-2^s\right]\right) \end{aligned}$$

case:4 $L = 2^{3-s}$ for $s < 3$ if $i = 1$

$$\begin{aligned} N\left(f\left(x\right)-C\left(x\right),r\right) &\geq N^{'}\left(\frac{\epsilon\left(2^{3-s}\right)^{1-1}}{1-2^{3-s}}\beta\left(x\right),r\right) \\ &\geq N^{'}\left(\epsilon\frac{1}{2^s-2^3}\left\|x\right\|^s,r\right) \\ &\geq N^{'}\left(\epsilon\left\|x\right\|^s,r\left[2^s-2^3\right]\right) \end{aligned}$$

case:5 $L = 2^{3-ns}$ for $s < \frac{3}{n}$ if $i = 1$

$$\begin{aligned} N\left(f\left(x\right)-C\left(x\right),r\right) &\geq N^{'}\left(\epsilon\frac{\left(2^{3-ns}\right)^{1-1}}{1-2^{3-ns}}\beta\left(x\right),r\right) \\ &\geq N^{'}\left(\epsilon\frac{1}{2^{ns}-2^3}\left\|x\right\|^s,r\right) \\ &\geq N^{'}\left(\epsilon\left\|x\right\|^{ns},r\left[2^{ns}-2^3\right]\right) \\ N\left(f\left(x\right)-C\left(x\right),r\right) &\geq N^{'}\left(\epsilon\left\|x\right\|^{ns},r\left[2^{ns}-2^3\right]\right) \end{aligned}$$

case:6 $L = 2^{ns-3}$ for $s > \frac{3}{n}$ if $i = 0$

$$\begin{aligned} N\left(f\left(x\right)-C\left(x\right),r\right) &\geq N^{'}\left(\epsilon\frac{\left(2^{ns-3}\right)^{1-0}}{1-2^{ns-3}}\beta\left(x\right),r\right) \\ &\geq N^{'}\left(\epsilon\frac{1}{2^3-2^{ns}}\left\|x\right\|^{ns},r\right) \\ &\geq N^{'}\left(\epsilon\left\|x\right\|^{ns},r\left[2^3-2^{ns}\right]\right) \end{aligned}$$

Hence the Proof is complete. □

6. Preliminaries of Random Normed Space

In the sequel, we adopt the usual terminology,notations and conventions of the theory of random normed space as in [32,36,38]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : R \cup \{-\infty, \infty\} \to [0,1]$, such that F is left continuous and non decreasing on R, $F(0) = 0$ and $F(+\infty) = 1$, D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x,that,$l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \Re$. The maximal element for Δ^+ in this order the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, if t \le 0 \\ 1, if t \ge 0 \end{cases} \tag{59}$$

Definition 6.1 ([53]). *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm, if T satisfies the following condition:*

(a) T is commutative and associative;

(b) T is continuous

(c) $T(a,1) = a$ for all $a \in [0,1]$

(d) $T(a,b) \le T(c,d)$ when $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$

Typical examples of continuous t-norms are $T_p(a,b) = ab$, $T_m(a,b) = min(a,b)$ and $T_L(a,b) = max(a+b-1,0)$ (The Lukasiewicz t-norm). Recall (see[15,17]) that if T is a t-norm and x_n is a given sequence of numbers in [0,1], then $T_{i=1}^n x_{n+i}$ is defined recurrently by $T_{i=1}^{'} x_i = x_i$ and $T_{i=1}^n x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right)$ for $n \ge 2$, $T_{i=1}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known [16] that, for the Lukasiewicz t-norm, the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty \tag{60}$$

Definition 6.2. *Shers A random normed space (briefly,RN-Space) is a triple (X, μ, T), where X is a vector space. T is a continuous t-norm and μ is a mapping from X into D^+ satisfies the following conditions:*

(RN_1) $\mu_x(t) = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$

(RN_2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$, and $\alpha \in \Re$ with $\alpha \ne 0$

(RN_3) $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $T, s \ge 0$

Example 6.3. *Every normed space $(X, \|.\|)$ defines a random normed space (X, μ, T),where*

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

and T is the minimum T-norm. This space is called the induced random normed spaces.

Definition 6.4. *Let (X, μ, T) be a RN-space*

(1) A Sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if , for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integers N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ for all $n > N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ for all $n \ge m \ge N$.

(3) A RN-space (X, μ, T) is said to be complete , if every cauchy sequence in X is convergent to a point in X.

Theorem 6.5. *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

7. Random Stability Results: Direct Method

In this section, the generalized Ulam-Hyers Stability of the cubic functional equation (11) in RN-space is provided. Through out this section, let us consider X be a linear space (Y, μ, T) is a complete RN-space. The proof of the following theorem and corollary is similar to that of results of section 4 and 5. Hence the detail of the proof are ommited.

Theorem 7.1. *Let $j = \pm 1$ $f : X \to Y$ be a mapping for which there exists a function $\eta : X^n \to D^+$ with the condition.*

$$\lim_{k\to\infty} T_{i=0}^{\infty} = \left(\eta_{2^{(k+i)}x_1, 2^{(k+i)}x_2, 2^{(k+i)}x_3, \dots, 2^{(k+i)}x_n}\left(2^{3(k+i+1)j}t\right)\right) = 1 \tag{61}$$

$$= \lim_{k\to\infty} \eta_{2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, \dots, 2^{kj}x_n}\left(2^{3kj}t\right) \tag{62}$$

such that the functional inequality with $f(0) = 0$ such that

$$\mu_{Df(x_1,x_2,x_1,\dots,x_n)}(t) \ge \eta_{(x_1,x_2,x_1,\dots,x_n)}(t) \tag{63}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and all $t > 0$. Then there exsits a unique cubic mapping $C : X \to Y$ satisfies the functional equation (11) and

$$\mu_{c(x)-f(x)}(t) \ge T_{i=0}^{\infty}\left(\eta_{2^{(i+1)j}x,0,0,\dots,0}\left(2^{3(i+1)j}t\right)\right) \tag{64}$$

for all $s \in X$ and all $t > 0$. The mapping $C(x)$ is defined by

$$\mu_{c(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f\left(2^{kj}x\right)}{2^{3kj}}}(t) \tag{65}$$

for all $x \in X$ and all $t > 0$.

Proof. Assume $j = 1$. Setting $(x_1, x_2, x_3, \dots, x_n) = (x, 0, 0, \dots, 0)$ in (61), we get

$$\mu_{f(2x)-8(x)}(t) \ge \eta_{x,0,0,\dots,0}(t) \tag{66}$$

for all $x \in X$ and all $t > 0$. It follows from (65) and (RN_2), we have

$$\mu_{f\frac{2x}{2^3}-f(x)}(t) \ge \eta_{x,0,0,\dots,0}\left(2^3 t\right) \tag{67}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^k x$ in (66), we arrive

$$\frac{\mu_{f\left(2^{k+1}x\right)}}{2^{3(k+1)}} - \frac{f\left(2^k x\right)}{2^3 k}(t) \ge \eta_{2^k x,0,0,\dots,0}\left(2^{3(k+1)}t\right) \tag{68}$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of Theorem 4.1 $\square$

The following corollary is an immediate consequence of Theorem 7.1, concerning the stability of (11).

Corollary 7.2. *Let ϵ and s be non-negative real numbers. Let a cubic functionf $: X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,x_3,\ldots,x_n)}(t) \geq \begin{cases} \eta_\epsilon(t), \\ \eta_\epsilon \sum_{i=1}^n \|x_i\|^s(t), s \neq 3 \\ \eta_\epsilon \left(\prod_{i=1}^n \|x_i\|^s + \sum_{i=1}^n \|x_i\|^{ns}\right)(t), s \neq \frac{3}{n} \end{cases} \tag{69}$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique cubic function $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{|2^3-1|}}(t), \\ \eta_{\frac{\epsilon\|x\|^s}{|2^3-2^s|}}(t), \\ \eta_{\frac{\epsilon\|x\|^{ns}}{|2^3-2^{ns}|}}(t) \end{cases} \tag{70}$$

for all $x \in X$ and all $t > 0$.

8. Random stability Results: Fixed Point Method.

In this section, the authors present the generalized Ulam-Hyers Stability of the functional equation (11),in Random normed spaces using fixed point method.

Theorem 8.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^n \to D^+$ with the condition*

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \ldots, \delta_i^k x_n}\left(\delta_i^{3k} t\right) = 1, \quad \forall x_1, x_2, \ldots, x_n \in X, t > 0 \tag{71}$$

and satisfying the functional inequality

$$\mu_{Df(x_1,x_2,x_3,\ldots,x_n)}(t) \geq \eta_{x_1,x_2,x_3,\ldots,x_n}(t), \forall x_1, x_2, x_3, \ldots, x_n \in X, t > 0 \tag{72}$$

If there exists $L = L(i)$ such that the function

$$x \to \beta(x,t) = \eta_{\frac{x}{2},0,0,\ldots,0}(t),$$

has the property, that

$$\beta(x,t) \leq L\frac{1}{\delta_i^3}\beta(\delta_{ix}, t), \forall x \in X, t > 0 \tag{73}$$

Then there exists a unique cubic function $C : X \to Y$ satisfying the functional equation (11) and

$$\mu_{c(x)-f(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t), \forall x \in X, t > 0 \tag{74}$$

Proof. Let d be a general metric on Ω, such that $d(g,h) = inf\left\{k \in (0,\infty) / \mu_{(g(x)-h(x)}(kt) \geq \beta(x,t), x \in X, t > 0\right\}$. It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i^3}g(\delta_i x), \forall x \in X$. Now for $g, h \in \Omega$, we have $d(g,h) \leq K$

$$\begin{aligned} &\Rightarrow \mu_{(g(x)-h(x)}(Kt) \geq \beta(x,t) \\ &\Rightarrow \mu_{(Tg(x)-Th(x))}\frac{Kt}{\delta_i^3} \geq \beta(x,t) \\ &\Rightarrow d(Tg(x), Th(x)) \leq KL \\ &\Rightarrow d(Ts, Th) \leq Ld(g,h) \end{aligned} \tag{75}$$

for all $g, h \in \Omega$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant Constant L. The rest of the proof is similar to that of Theorem 5.2. From the Theorem 8.1, we obtain the following corollary concerning the stability for the functional equation(11). $\square$

Corollary 8.2. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,x_3,..,x_n)}(t) \geq \begin{cases} \eta_\epsilon(t), \\ \eta_\epsilon \sum_{i=1}^{n} \|x_i\|^s (t), \qquad s \neq 3 \\ \eta_\epsilon \left(\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns}\right), s \neq \frac{3}{n} \end{cases} \tag{76}$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and $t > 0$, where ϵ, S are constants with $\epsilon > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{|2^3-1|}}(t), \\ \eta_{\frac{\epsilon\|x\|^s}{|2^3-2^s|}}(t), \\ \eta_{\frac{\epsilon}{|2^3-2^{ns}|}}(t), \end{cases}$$

for all $x \in X$ and all $t > 0$.

Proof. Setting

$$\eta_{Df(x_1,x_2,x_3,..,x_n)}(t) \geq \begin{cases} \eta_\epsilon(t), \\ \eta_\epsilon \sum_{i=1}^{n} \|x_i\|^s (t), \\ \eta_\epsilon \left(\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns}\right), \end{cases}$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $t > 0$. The rest of the proof is similar that of Corollary 5.3. $\square$

9. Stability Result: Intuitionistic Fuzzy Normed Space

In this section, we give some basic definition and notations about intuitionistic fuzzy metric space introduced by J. H. Park[40] and R. Saadati and J.H. Park[51,52].

Definition 9.1. *Let μ and v be membership and non membership degree of an intutionistic fuzzy set from $X \times (0, +\infty)$ to [0,1] such that $\mu_x(t) + v_x(t) \leq 1$ for all $x \in X$ and all $t > 0$. The triple $(X, P_{\mu,v}, M)$ is said to be an intutionistic fuzzy normed space (briefly IFN-space) if X is a vector space, M is a continuous t-representable and $P_{\mu,v}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,*

(IFN_1) $P_{\mu,v}(x, 0) = 0_{L^*}$;

(IFN_2) $P_{\mu,v}(x, t) = 1_{L^*}$; *if and only if* $x = 0$

(IFN_3) $P_{\mu,v}(\alpha x, t) = P_{u,v}\left(x, \frac{t}{|\alpha|}\right)$ *for all* $\alpha \neq 0$;

(IFN_4) $P_{\mu,v}(x + y, t + s) \geq_{L^*} M(P_{\mu,v}(x, t), P_{\mu,v}(y, s))$;

In this case $P_{\mu,v}$ is called intutionistic fuzzy norm. Here $P_{\mu,v}(x, t) = (\mu_x(t), v_x(t))$.

Example 9.2. *Let $(X, \|.\|)$ be a normed space. Let $T(a, b) = (a, b\min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, v be membership degree of an intutionistic fuzzy set defined by*

$$P_{\mu,v}(x, t) = (\mu_x(t), v_x(t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \forall t \in R^+.$$

Then $(X, P_{\mu,v}, T)$ is an IFN-space.

Definition 9.3. *A sequence $\{x_n\}$ in an IFN-space $(x, P_{u,v}, T)$ is called a Cauchy sequence if,for any $\epsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that $P_{u,v}(x_n - x_m, t) >_{L^*} (N_s(\epsilon), \epsilon), \forall n, m \geq n_0$, where N_s is the standard negator.*

Definition 9.4. *The sequence $\{x_n\}$ is said to be convergent to point $x \in X$ (denoted by $x_n \stackrel{P_{u,v}}{\rightarrow} x$) if $P_{u,v}(x_n - x, t) \rightarrow 1_{L^*}$ as $n \rightarrow \infty$ for every $t > 0$.*

Definition 9.5. *An IFN-space $(X, P_{\mu,v}, T)$ is said be complete if every Cauchy sequence in X is convergent to a point $x \in X$. Further details about IFN-space one can see ([4,7,8,12,24,29-33,35-36]). Throughout this section, let us consider $X, (z, p_{\mu,v}, M)$ and $\left(y, P'_{\mu,v}, M\right)$ are linear space, intuitionistic fuzzy normed space and complete intuitionistic fuzzy normed space.*

Theorem 9.6. *Let $K \in \{-1, 1\}$ be fixed and let $\epsilon : X^n \rightarrow Z$ be mapping such that for some b with $0 < \left(\frac{b}{2}\right)^k < 1$,*

$$P'_{u,v}\left(\epsilon\left(2^k x, 0, 0, .., 0\right), r\right) \geq_{L^*} P'_{u,v}\left(b^k \epsilon(x, 0, 0, .., 0), r\right) \tag{77}$$

for all $x \in X$ and all $r > 0, b > 0$ and

$$\lim_{k\rightarrow\infty} P'_{u,v}\left(\epsilon\left(2^k x_1, 2^k x_2, 2^k x_3, .., 2^k x_n\right), 2^{3k} r\right) = 1_{L^*} \tag{78}$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $r > 0$. Suppose that a function $f : X \rightarrow Y$ satisfying the inequality

$$P_{u,v}(Df(x_1, x_2, x_3, ..., x_n), r) \geq_{L^*} P'_{u,v}(\epsilon(x_1, x_2, x_3, ..., x_n), r) \tag{79}$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $r > 0$. Then the limit

$$P_{u,v}\left(C(x) - \frac{f(2^k x)}{2^{3k}}, r\right) \rightarrow 1_{L^*} as K \rightarrow \infty, r > 0 \tag{80}$$

exists for all $x \in X$ and the mapping $C : X \rightarrow Y$ is a unique cubic mapping satisfying (11) and

$$P_{u,v}(f(x) - C(x), r) \geq_{L^*} P'_{u,v}\left(\epsilon(x, 0, 0, ..., 0), r\left(2^3 - b\right)\right) \tag{81}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $k = 1$. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in(9.3),we arrive

$$P_{u,v}(f(2x) - 8f(x), r) \geq_{L^*} P'_{u,v}(\epsilon(x, 0, 0, ..., 0), r)$$

for all $x \in X$ and all $r > 0$. Using (IFN_3) in the above equation, we get

$$P_{u,v}\left(\frac{f(2x)}{2^3} - f(x), r\right) \geq_{L^*} P'_{u,v}(\epsilon(x, 0, 0, ..., 0), r) \tag{82}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^k x$ in (81), we obtain

$$P_{u,v}\left(\frac{f(2^{k+1}x)}{2^{3(k+1)}} - f\left(2^k x\right), r\right) \geq_{L^*} P'_{u,v}\left(\epsilon\left(2^k x, 0, 0, ..., 0\right), r\right) \tag{83}$$

for all $x \in X$ and all $r > 0$. Using (76),(IFN_3) in (82), we arrive

$$P_{u,v}\left(\frac{f(2^{k+1}x)}{2^{3(k+1)}} - f\left(2^k x\right), r\right) \geq_{L^*} P'_{u,v}\left(\epsilon(x, o, o, ..., 0), \frac{r}{b^k}\right) \tag{84}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 4.1. □

Corollary 9.7. *Let ϵ and S be an nonnegative real numbers. Let a cubic function $f : X \to Y$ satisfies the inequality*

$$P'_{u,v}\left(f\left(x_1, x_2, x_3, ..., x_n\right), r\right) \geq_{L^*} \begin{cases} P'_{u,v}\left(\epsilon, r\right) \\ P'_{u,v}\left(\epsilon \sum_{k=1}^{n} x_k^s, r\right), \qquad s \neq 3 \\ P'_{u,v}\left(\epsilon\left(\sum_{k=1}^{n} x_k^{ns} + \prod_{i=1}^{n} x_k^s\right), r\right), s \neq \frac{3}{n} \end{cases}$$

for all $x_1, x_2, x_3, ..., x_n \in X$. Where ϵ, S are constant with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$P_{u,v}\left(f\left(x\right) - c\left(x\right), r\right) \geq \begin{cases} P'_{u,v}\left(\epsilon, \left|2^3 - 1\right|\right), \\ P'_{u,v}\left(\epsilon x_k^s, r\left|2^3 - 2^s\right|\right), \\ P'_{u,v}\left(\epsilon x_k^{ns}, r\left|2^3 - 2^{ns}\right|\right), \end{cases} \tag{85}$$

for all $x \in X$ and all $r > 0$.

10. Stability Results: Fixed Point Method

In this section, the authors investigate the generalized Ulam-Hyers Stability of the functional equation (11) in IFN-space using fixed point method.

Theorem 10.1. *Let $f : X \to Z$ be a mapping for which there exists a function $\epsilon : X^n \to Y$ with the condition,*

$$\lim_{k\to\infty} P'_{u,v}\left(\epsilon\left(a_i^k x_1, a_i^k x_2, a_i^k x_3, ..., a_i^k x_n\right), a_i^{3k} r\right) = 1_{L*} \tag{86}$$

for all $x_1, x_2, x_3, ..., x_n \in X$,and all $r > 0$,and satisfies the functional inequality

$$P_{u,v}\left(f\left(x_1, x_2, x_3, ..., x_n\right), r\right) \geq_{L^*} P'_{u,v}\left(\epsilon\left(x_1, x_2, x_3, ..., x_n\right), r\right) \tag{87}$$

for all $x_1, x_2, x_3, ..., x_n \in X$,and all $r > 0$. If there exists $L = L\left(i\right)$ such that the function $x \to \psi\left(x\right) = \epsilon\left(\frac{x}{2}, 0, 0, ..., 0\right)$,has the property,

$$P'_{u,v}\left(L\frac{\psi\left(a_i x\right)}{a_i^3}, r\right) = P'_{u,v}\left(\psi\left(x\right), r\right), \tag{88}$$

for all $x \in X$ and all $r > 0$. Then there exists a unique cubic function $C : X \to Y$ satisfying the functional equation(11),and

$$P_{u,v}\left(f\left(x\right) - c\left(x\right), r\right) \geq_{L^*} P'_{u,v}\left(\psi\left(x\right), \left(\frac{L^{1-i}}{1-L} r\right)\right) \tag{89}$$

for all $x \in X$ and all $r > 0$.

Proof. Let d be the general metric on A, such that

$$d\left(g, h\right) = inf\left\{K \in (0, \infty) \left| P_{u,v}\left(g\left(x\right) - h\left(x\right)\right) \geq_{L^*} P'_{u,v}\left(\psi\left(x\right), kr\right), x \in X, r > 0\right.\right\}$$

It is easy to see that (A, d) is complete. Define $T : A \to A$ by $Tg\left(x\right) = \frac{1}{a_i^3} g\left(a_i x\right)$. for all $x \in X$. By [21], we see that T is strictly contractive mapping A with Lipschitz constant L. It is follows from (38) that

$$P_{u,v}\left(\frac{f\left(2x\right)}{2^3} - f\left(x\right), r\right) \geq_{L_*} P'_{u,v}\left(\epsilon\left(x, 0, 0, ..., 0\right), 2^3 r\right), \tag{90}$$

for all $x \in X$ and all $r > 0$. With the help of (87) when $i = 1$, it follows from (89), that

$$P_{u,v}\left(\frac{f(2x)}{2^3} - f(x), r\right) \geq_{L_*} P'_{u,v}\left(\psi(x), 2^3 r\right),$$

$$\Rightarrow d(Th, h) \leq 1 = L^0 = L^{1-i} \tag{91}$$

Replacing x by $\frac{x}{2}$ in (89), we obtain

$$P_{u,v}\left(f(x) - 2^3 f\left(\frac{x}{2}\right), r\right) \geq_{L^*} P'_{u,v}\left(\epsilon\left(\frac{x}{2}, 0, 0, ..., 0\right), r\right) \tag{92}$$

for all $x \in X$ and all $r > 0$, with the help of (87) when $i = 0$, it follows from (91), that

$$P_{u,v}\left(f(x) - 2^3 f\left(\frac{x}{2}\right), r\right) \geq_{L^*} P'_{u,v}(\psi(x), Lr), \forall x \in X, r > 0$$

$$\Rightarrow d(f, Th) \leq L = L^1 = L^{1-i} \tag{93}$$

Then from (90) and (92), we conclude $d(f, Th) \leq L^{1-i} < \infty$. Now from the fixed point alternative in both cases,it follows that there exists a fixed point C of T in A such that

$$\lim_{k\to\infty} P_{u,v}\left(\frac{f\left(a_i^k x\right)}{a_i^{3k}} - C(x), r\right) \to 1_{L^*}, \forall x \in X, r > 0. \tag{94}$$

Replacing $(x_1, x_2, x_3, ..., x_n)$ by $\left(a_i^k x_1, a_i^k x_2, a_i^k x_3, ..., a_i^k x_n\right)$ in (86), we arrive

$$P_{u,v}\left(\frac{1}{a_i^{3k}} Df\left(a_i^k x_1, a_i^k x_2, a_i^k x_3, ..., a_i^k x_n\right), r\right) \geq_{L^*} P'_{u,v}\left(\epsilon\left(a_i^k x_1, a_i^k x_2, a_i^k x_3, ..., a_i^k x_n\right), a_r^{3k}\right)$$

for all $x_1, x_2, x_3, ..., x_n \in X$ and all $r > 0$. By proceeding the same procedure in the Theorem 9.1, we can prove the function , $C : X \to Y$ is cubic and it satisfies the functional equation (11), since C is Unique fixed point of T in the set $B = \{f \in A | d(f, c) < \infty\}$, such that

$$P_{u,v}(f(x) - c(x), r) \geq_{L^*} P'_{u,v}(\psi(x), kr), \forall x \in X, r > 0 \tag{95}$$

Again using the fixed point alternative,we obtain

$$d(f, c) \leq \frac{1}{1-L} d(f, Tf) \to d(f, c) \leq \frac{L^{1-i}}{1-L}$$

Hence we have

$$P_{u,v}(f(x) - c(x), r) \geq_{L^*} P'_{u,v}\left(\psi(x), \left(\frac{L^{1-i}}{1-L} r\right)\right), \forall x \in X, r > 0 \tag{96}$$

This complete the proof of the theorem. □

From Theorem 10.1, we obtain the following corollary concerning the stability for the functional equation(11).

Proof. Suppose that a function $f : X \to Y$ satisfies the inequality

$$P_{u,v}\left(Df\left(x_1,x_2,x_3,...,x_n\right),r\right) \geq_{L^*} \begin{cases} P'_{u,v}\left(\epsilon,r\right), \\ P'_{u,v}\left(\epsilon\sum_{i=1}^{n} x_i^s, r\right), s \neq 3; \\ P'_{u,v}\left(\epsilon\left(\prod_{i=1}^{n} x_i^s + \sum_{i=1}^{n} x_i^{ns}\right), r\right), s \neq \frac{3}{n} \end{cases} \tag{97}$$

for all $x_1,x_2,x_3,...,x_n \in X$ and all $r > 0$, where ϵ, S are constants with $\epsilon > 0$. Then there exists a Unique cubic mapping $C : X \to Y$ such that

$$P_{u,v}\left(f\left(x\right)-c\left(x\right),r\right) \geq_{L^*} \begin{cases} P'_{u,v}\left(\epsilon,\left(\frac{8}{|7|}r\right)\right); \\ P'_{u,v}\left(\epsilon\left\|x\right\|^s, \frac{2^{3+s}}{|2^s-2^3|}\right); \\ P'_{u,v}\left(\epsilon\left\|x\right\|^{ns}, \frac{2^{3+s}}{|2^{ns}-2^3|}\right); \end{cases} \tag{98}$$

for all $x \in X$ and all $r > 0$. □

Proof. The Proof follows by replacing $L = 2^3$ for $i = 0$ and $L = 2^{-3}$ for $i = 1$; $L = 2^{3-s}$ for $s > 3, i = 0$ and $L = 2^{s-3}$ for $s < 3, i = 1$; $L = 2^{3-ns}$ for $s > \frac{3}{n}, i = 0$ and $L = 2^{ns-3}$ for $s < \frac{3}{n}, i = 1$. Hence the proof is complete. □

11. Stability Results: Felbin's type Spaces

In this, we give some basic definition and notations about Felbin's type spaces using direct Method.

Definition 11.1 ([8])**.** *A fuzzy subset η on R is called a fuzzy real number, whose α-level set is denoted by $[\eta]_\alpha$*

$$ie., [\eta]_\alpha = \{t : \eta(t) \geq \alpha\},$$

if it is satisfies two axioms:

- *There exists $t_0 \in R$ such that $\eta(t_0) = 1$*
- *For each $\alpha \in (0,1], [\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$ where $-\infty < \eta_\alpha^- \geq \eta_\alpha^+ < +\infty$.*

The set of all fuzzy real numbers denoted by

$F(R)$. If $\eta \in F(R)$ and $\eta(t) = 0$

Whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*[R]$ denotes the set of all non-negative fuzzy real numbers. The numbers $\overline{0}$ stands for the fuzzy real number as:

$$\overline{0} = \begin{cases} t, t = 0 \\ 0, t \neq 0 \end{cases}$$

Clearly,$\overline{0} \in F^*[R]$. Also the set of real numbers can be embedded in $F(R)$ because if $r \in (-\infty,\infty)$, then $r^- \in F(R)$ satisfies $r^-(t) = \overline{O}(t-r)$.

Definition 11.2 ([3])**.** *Fuzzy arithmetic operations $\oplus, !, \otimes, \%$ on $F(R) \times F(R)$ can be defined as:*

- $(\eta \oplus \delta)(t) = Sup_{s\in R}\{\eta(s) \wedge \delta(t-s)\}; t \in R$
- $(\eta!\delta)(t) = Sup_{s\in R}\{\eta(s) \wedge \delta(s-t)\}; t \in R$
- $(\eta \otimes \delta)(t) = Sup_{s\in R}\{\eta(s) \wedge \delta(t/s)\}; t \in R$
- $(\eta\%\delta)(t) = Sup_{s\in R}\{\eta(s) \wedge \delta(s)\}; t \in R$

The additive and multiplicative identities in $F(R)$ are $\overline{0}$ and T, respectively. Let $!\ \eta$ be defined as $\overline{0}!\eta$. It is clear that $\eta!\delta = \eta \oplus (!\delta)$.

Definition 11.3 ([3]). *For $k \in P$, fuzzy scalar multiplication Keeta is defined as $(ke\eta)(t) = \eta(t/k)$ and $oe\eta$ is denoted to be 0.*

Lemma 11.4. *Let η, δ be fuzzy real numbers. Then $\forall t \in R$. $\eta(t) = \delta(t) \Leftrightarrow \forall a \in (0,1], [\eta]_\alpha = [\delta]_\alpha$.*

Lemma 11.5. *Let $\eta, \delta \in F(R)$ and $[\eta]_\alpha = \left[\eta_a^-, \eta_a^+\right], [\delta]_\alpha = \left[\delta_a^-, \delta_a^+\right]$. Then*

- $[\eta \oplus \delta]_\alpha = \left[\eta_\alpha^- + \delta_a^+, \eta_\alpha^+ + \delta_a^+\right]$,
- $[\eta ! \delta]_\alpha = \left[\eta_\alpha^- - \delta_a^+, \eta_\alpha^+ - \delta_a^+\right]$,
- $[\eta \otimes \delta]_\alpha = \left[\eta_\alpha^- \delta_a^+, \eta_\alpha^+ \delta_a^+\right], \eta, \delta \in F^*[R]$
- $[\eta \% \delta]_\alpha = \left[1/\delta_a^+, 1/\delta_a^-\right], \delta_\alpha^- > 0.$

Definition 11.6 ([3]). *Define a partial ordering $\leq$ in $F(R)$ by $n \leq s$ if and only if $\eta_\alpha^- \leq \delta_a^-$ and $\eta_\alpha^+ \leq \delta_a^+$ for all $\alpha \in (0,1]$. The Strict inequality in $F(R)$ is defined by $n < s$ if and only if $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^+ < \delta_a^+$ for all $\alpha \in (0,1]$.*

Definition 11.7 ([58]). *Let X be a real linear space, L and R (respectively, left norm, right norm) be symmetric and non-decreasing mapping from $[0,1] \times [0,1] \to [0,1]$ satisfying $L(0,0) = 0, R(1,1) = 1$. Then $\|.\|$ is called a fuzzy norm and $(X, \|.\|, L, R)$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\|.\| : X \to F^*(R)$ satisfies the following axioms, where $[\|x\|]_\alpha = \left[\|x\|_a^-, \|x\|_a^+\right]$ for $x \in X$ and $\alpha \in (0,1]$:*

- $\|x\| = 0$ *if and only if* $x = 0$,
- $\|rx\| = |r| e \|x\|$ *for all* $x \in X$ *and* $r \in (-\infty, \infty)$,
- *For all $x, y \in X$, if $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$,then, $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$,if $s \geq \|x\|_1^-, t \geq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$, then $\|x+y\|(s+t) \leq L(\|x\|(s), \|y\|(t))$.*

Theorem 11.8 ([51]). . *Let $(X, \|.\|, L, R)$ be an FNLS and $\lim_{\alpha \to 0^+} R(a,a) = 0$. Then $(X, \|.\|, L, R)$ is a Hausdorff topological vector space, whose neighbourhood base of origin is $\{N(\epsilon, a); \epsilon > 0, \alpha \in (0,1]\}$, where $N(\epsilon, \alpha) = \left\{x : \|x\|_\alpha^+ \leq \epsilon\right\}$.*

Definition 11.9. *Let $(X, \|.\|, L, R)$ be an FNLS. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges to $x \in X$, if $\lim_{n\to\infty} \|x_n - x\|_\alpha^-$, for every $\alpha \in (0,1]$ denoted by $\lim_{n\to\infty} x_n = x$.*

Definition 11.10. *Let $(X, \|.\|, L, R)$ be an FNLS. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$.*

Definition 11.11. *Let $(X, \|.\|, L, R)$ be an FNLS. A sequence $A \subseteq X$ is said to be complete if every Cauchy sequence in A, converges in A. The fuzzy normed space $(X, \|.\|, L, R)$ is said to be a fuzzy Banach space if it is complete.*

Theorem 11.12. *Let $j = \pm 1$. Let $\phi : U^n \to F^*(R)$ be a function such that $\sum_{l=0}^\infty \frac{1}{2^{3lj}} \phi\left(2^{lj} x\right)_\alpha^+$ converges and*

$$\lim_{l\to\infty} \frac{1}{2^{3lj}} \phi\left(2^{lj} x\right)_\alpha^+ = 0 \tag{99}$$

for all $x \in U^n$,and let $f : U \to V$ be a function satisfying the inequality

$$\left\|D^+(x)\right\|_\alpha^+ \leq (x)_\alpha^+ \tag{100}$$

for all $x \in U^n$. Then there exists a Unique cubic function $C : U \to V$ such that

$$\|f(x) - c(x)\|_\alpha^+ \leq \frac{1}{2^3} e \sum_{k=\frac{1-j}{2}} \frac{\left(2^{kj} x\right)_\alpha^+}{2^{3kj}} \tag{101}$$

for all $x \in U$. The function $C(x)$ is defined by

$$C(x) = \lim_{l\to\infty} \frac{f\left(2^{lj} x\right)}{2^{3lj}}, \forall x \in U. \tag{102}$$

Proof. Assume j=1. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in (99), we get

$$\|f(2x) - 8f(x)\|_{\alpha}^{+} \leq e\phi(x, 0, 0, ..., 0) \tag{103}$$

for all $x \in U$. Dividing 2^3, we obtain

$$\left\| \frac{f(2x)}{8} - f(x) \right) |_{\alpha}^{+} \leq \frac{1}{2^3} e\phi(x, 0, 0, ..., 0) \tag{104}$$

for all $x \in U$. Let $\phi(x)_{\alpha}^{+} = \phi(x, 0, 0, ..., 0)$ in (10.6), we arrive

$$\left\| \frac{f(2x)}{2^3} - f(x) \right) |_{\alpha}^{+} \leq \frac{1}{2^3} e\phi(x)_{\alpha}^{+} \tag{105}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 2^3 in (104) and adding the resultant inequality with (104), we obtain

$$\left\| \frac{f(2x)}{2^3} - f(x) \right) |_{\alpha}^{+} \leq \frac{1}{2^3} e\phi \left[\phi(x)_{\alpha}^{+} + \frac{\phi(2x)_{\alpha}^{+}}{2^3} \right] \tag{106}$$

for all $x \in U$. In general for any positive integer l, we have

$$\begin{aligned} \left\| \frac{f(2^l x)}{2^{3l}} - f(x) \right) |_{\alpha}^{+} &\leq \frac{1}{2^3} e \sum_{k=0}^{l-1} \frac{\phi(2^k x)}{2^{3k}} \\ &\leq \frac{1}{2^3} e \sum_{k=0}^{\infty} \frac{\phi(2^k x)}{2^{3k}} \end{aligned} \tag{107}$$

for all $x \in U$ In order to prove the convergence of the sequence $\left\{ \frac{f(2^l x)}{2^{3l}} \right\}$, replace x by $2^m x$ and dividing by 2^{3m} in (106), for any $m, l > 0$, we arrive

$$\left\| \frac{f(2^{l+m})}{2^{3(l+m)}} - \frac{f(2^m x)}{2^{3m}} \right\|_{\alpha}^{+} = \frac{1}{2^{3m}} e \left\| \frac{f(2^l . 2^m x)}{2^{3l}} - f(2^m x) \right\|_{\alpha}^{+} nonumber \tag{108}$$

$$\begin{aligned} &\leq \frac{1}{2^3} e \sum_{k=0}^{l-1} \frac{\phi(2^{k+m} x)_{\alpha}^{+}}{2^{3(k+m)}} \\ &\leq \frac{1}{2^3} e \sum_{k=0}^{\infty} \frac{\phi(2^{k+m} x)_{\alpha}^{+}}{2^{3(k+m)}} \end{aligned} \tag{109}$$

$\to 0$ as $m \to \infty$ for all $x \in U$. Hence the sequence $\left\{ \frac{f(2^l x)}{2^{3l}} \right\}$ is a Cauchy sequence. Since V is complete,there exists a mapping $C : V \to V$ such that

$$C(x) = \lim_{l \to \infty} \frac{f(2^l x)}{2^{3l}}, \forall x \in U$$

Letting $l \to \infty$ in (107), we see that (100) holds for $x \in U$. To prove C satisfies (11), replacing x by $2^l x$ and divided by2^{3l} in (99), we arrive, $\frac{1}{2^{3l}} e \left\| Df(2^l x) \right\|_{\alpha}^{+} \leq \frac{1}{2^{3l}} e\phi(2^l x)_{\alpha}^{+}$ forall $x \in U^n$. Letting $l \to \infty$ and in the above in equality, we see that $\|DC(x)\|_{\alpha}^{+} = 0$. Hence C satisfies (11) for all $x_1, x_2, x_3, ..., x_n \in U$. To prove C is unique, let $B(x)$ be the another Cubic mapping satisfying (11) and (100). Then

$$\begin{aligned} \|C(x) - B(x)\|_{\alpha}^{+} &= \frac{1}{2^{3l}} \left\{ \left\| C\left(2^l x\right) - B\left(2^l x\right) \right\|_{\alpha}^{+} \right\} \\ &\leq \frac{1}{2^{3l}} e \left\{ \left\| C\left(2^l x\right) - f\left(2^l x\right) \right\|_{\alpha}^{+} \oplus \left\| f\left(2^l\right) - B\left(2^l x\right) \right\|_{\alpha}^{+} \right\} \\ &\leq \frac{1}{2^{3l}} e \sum_{k=0}^{\infty} \frac{\phi(2^{k+l} x)_{\alpha}^{+}}{2^{3(k+l)}} \end{aligned}$$

$\to 0$ as $l \to \infty$ for all $x \in U$. Hence C is Unique. For $j = -1$, we can prove of the similar type of stability result. This completes the proof of the theorem. □

The following corollary is a immediate consequence of Theorem 10.1 concerning the stability of (11).

Corollary 11.13. *Let λ and S be a non-negative real numbers. If a function $f : U \to V$ satisfying the inequality*

$$\|Df(x)\|_\alpha^+ \leq \begin{cases} \lambda, \\ \lambda \sum_{l=1}^n \left(\|x_l\|_\alpha^+\right)^s, \quad s \neq 3 \\ \lambda \left(\sum_{l=1}^n \left(\|x_l\|_\alpha^+\right)^{ns} \oplus \prod_{l=1}^n \left(\left(\|x_l\|_\alpha^+\right)^s\right)\right), \quad s \neq \frac{3}{n} \end{cases} \tag{110}$$

for all $x_1, x_2, x_3, ..., x_n \in U^n$, then there exists a Unique Cubic function $C : U \to V$ such that

$$\|f(x) - c(x)\|_\alpha^+ \leq \begin{cases} \frac{\lambda}{|7|}, \\ \frac{\lambda}{|8-2^s|} e\|x\|^s, \\ \frac{\lambda}{|8-2^{ns}|} e\|x\|^{ns}, \end{cases} \tag{111}$$

for all $x \in U$.

12. Fixed Point Stability Results

In this section, the authors proved the generalized Ulam-Hyers Stability of the n-dimensional cubic functional equation (11) in Felbin's type spaces with the help of the fixed point method.

Theorem 12.1. *Let $f : U \to V$ be a mapping for which there exists a function $\phi : U^n \to F^*(R)$ with the condition*

$$\lim_{k\to\infty} \frac{\phi\left(\epsilon_i^k x\right)_\alpha^+}{\epsilon_i^{3k}} = 0 \tag{112}$$

for all $x \in U$, satisfying the functional inequality

$$\|Df(x)\|_\alpha^+ \leq \phi(x)_\alpha^+ \tag{113}$$

for all $x_1, x_2, x_3, ..., x_n \in U^n$, and $\alpha \in (0, 1]$. If there exists $L = L(i) < 1$ such that the function

$$x \to \phi(x)_\alpha^+ = e\phi\left(\frac{x}{2}, 0, 0, ..., 0\right),$$

has the property

$$\frac{1}{\epsilon_i^3} e\phi(\epsilon_i x)_\alpha^+ \leq Le\phi(x)_\alpha^+ \tag{114}$$

for all $x \in U$. Then there exists a Unique Cubic function $C : U \to V$ satisfying the functional equation (1) and (11)

$$\|f(x) - c(x)\|_\alpha^+ \leq \frac{L^{1-i}}{1-L} e\phi(x)_\alpha^+, \forall x \in U \tag{115}$$

Proof. Let d be the general metric on Ω, such that

$$d(f, s) = inf\left\{k \in (0, \infty); \|f(x) - g(x)\|_\alpha^+ \leq ke\phi(x)_\alpha^+, x \in U\right\}.$$

It is easy to see that (Ω, d) is complete. Define $G : \Omega \to \Omega$ by $Gf(x) = \frac{1}{\epsilon_i^3} f(\epsilon_i x)$, for all $x \in U$. For $f, g \in \Omega$ and $x \in U$, we have

$$\begin{aligned} d(f,g) = k &\Rightarrow \|f(x) - g(x)\|_\alpha^+ \leq ke\phi(x)_\alpha^+ \\ &\Rightarrow \left\| \frac{f(\epsilon_i x)}{\epsilon_i^3} - \frac{g(\epsilon_i x)}{\epsilon_i^3} \right\| \leq \frac{1}{\epsilon_i^3} ke\phi(\epsilon_i x)_\alpha^+ \\ &\Rightarrow \|Gf(x) - Gg(x)\|_\alpha^+ \leq \frac{1}{\epsilon_i^3} ke\phi(x)_\alpha^+ \\ &\Rightarrow d(Gf(x), Gg(x)) \leq KL \\ & d(Gf, Gg) \leq Ld(f,g) \end{aligned}$$

Therefore G is strictly contractive mapping on Ω with Lipschitz constant L. Replacing $(x_1, x_2, x_3, ..., x_n)$ by $(x, 0, 0, ..., 0)$ in (111), we get

$$\|f(2x) - 8f(x)\| \leq e\phi(x, 0, 0, ..., 0) \tag{116}$$

for all $x \in U$. Using the definition $\phi(x)_\alpha^+$ in the above equation and for $i = 0$, we have

$$\left\| \frac{f(2x)}{8} - f(x) \right\|_\alpha^+ \leq \frac{1}{8} e\phi(x)_\alpha^+$$

$$ie., \quad \|Gf(x) - f(x)\| \leq Le\phi(x)_\alpha^+$$

for all $x \in U$. Hence, we arrive

$$d(Gf(x), f(x)) \leq L = L^{1-i} \tag{117}$$

for all $x \in U$. Replacing x by $\frac{x}{2}$ in (114), we obtain

$$\left\| f(x) - 8f\left(\frac{x}{2}\right) \right\|_\alpha^+ \leq e\phi\left(\frac{x}{2}, 0, 0, ..., 0\right) \tag{118}$$

for all $x \in U$. Using definition of $\phi(x)_\alpha^+$ in the above equation and for $i = 1$, we have

$$\left\| f(x) - 8f\left(\frac{x}{2}\right) \right\|_\alpha^+ \leq \phi(x)_\alpha^+$$

ie.,$\|f(x) - Gf(x)\|_\alpha^+ \leq \phi(x)_\alpha^+$ for all $x \in U$. Hence, we arrive

$$d(f(x), Gf(x)) \leq 1 = L^{1-i} \tag{119}$$

for all $x \in U$. From (115) and (117), we can conclude

$$d(f(x), Gf(x)) \leq L^{1-i} < \infty \tag{120}$$

for all $x \in U$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point C of G in Ω such that

$$C(x) = \lim_{k \to \infty} \frac{f(\epsilon_i^k x)}{\epsilon_i^{3k}}, \forall x \in U \tag{121}$$

In order to prove $C : U \to V$ satisfies the functional equation (11), replace x by $\epsilon_i^k x$ and divide by ϵ_i^{3k} in (111), we get

$$\left\| \frac{1}{\epsilon_i^{3k}} Df(\epsilon_i x) \right\|_\alpha^+ \leq \frac{1}{\epsilon_i^{3k}} e\phi(x)_\alpha^+ \tag{122}$$

for all $x \in U$. Letting $K \to \infty$ in the above in equality and using the choice of C and ϕ, we conclude C satisfies the functional equation (11). Since C is Unique fixed point of G in the set $\Delta = \{f \in \Omega / d(f,c) < \infty\}$, therefore C is Unique function such that

$$\|f(x) - C(x)\|_{\alpha}^{+} \leq Ke\phi(x)_{\alpha}^{+} \tag{123}$$

for all $x \in U$. Again using the fixed point alternative, we obtain

$$\begin{aligned} d(f,c) &\leq \frac{1}{1-L} d(f,Gf) \\ \text{ie.,} \quad d(f,c) &\leq \frac{L^{1-i}}{1-L} \\ \text{ie.,} \quad \|f(x) - c(x)\|_{\alpha}^{+} &\leq \frac{L^{1-i}}{1-L} e\phi(x)_{\alpha}^{+} \end{aligned} \tag{124}$$

for all $x \in U$. This completes the proof of the theorem. □

The following corollary is immediate consequence of Theorem 11.1 concerning the stability of (11).

Corollary 12.2. *Suppose that a function $f : U \to V$ satisfies the inequality*

$$\|Df(x)\|_{\alpha}^{+} \leq \begin{cases} \epsilon, \\ \epsilon \sum_{l=1}^{n} \left(\|x_i\|_{\alpha}^{+}\right)^{s}, & s \neq 3 \\ \epsilon \left(\sum_{l=1}^{n} \left(\|x_i\|_{\alpha}^{+}\right)^{ns} \oplus \prod_{l=1}^{n} \left(\|x_i\|_{\alpha}^{+}\right)^{s}\right), & s \neq \frac{3}{n} \end{cases}$$

for all $x_1, x_2, x_3, .., x_n \in U^n$, where $\epsilon > 0$. Then there exists a Unique cubic mapping $C : U \to V$ such that

$$\begin{cases} \frac{\epsilon}{|7|}, \\ \frac{\epsilon}{|8-2^{s}|} e \|x\|^{s}, \\ \frac{\epsilon}{|8-2^{ns}|} e \|x\|^{ns} \end{cases}$$

for all $x \in U$.

Proof. Let $\phi(x)_{\alpha}^{+} = \epsilon \sum_{l=1}^{n} \left(\|x_l\|_{\alpha}^{+}\right)^{s}$ for all $x_1, x_2, x_3, ..., x_n \in U^n$. Then for $s < 3, i = 0$, and for $S > 3, i = 1$, we arrive

$$\begin{aligned} \frac{1}{\epsilon_i^{3k}} e\phi\left(\epsilon_i^k x\right)_{\alpha}^{+} &= \frac{1}{\epsilon_i^{3k}} e\phi\left(\epsilon_i^k x_1, \epsilon_i^k x_2, \epsilon_i^k x_3, .., \epsilon_i^k x_n\right)_{\alpha}^{+} \\ &= \epsilon \epsilon_i^{(s-3)k} e \sum_{l=1}^{n} \left(\|x_l\|_{\alpha}^{+}\right)^{s} \end{aligned}$$

$\to 0$ as $k \to \infty$. Thus, (110) is holds. But we have $\phi(x)_{\alpha}^{+} = e\phi\left(\frac{x}{2}, 0, 0, .., 0\right)$ has the property

$$\frac{1}{\epsilon_i^3} e\phi(\epsilon_i x)_{\alpha}^{+} = Le\phi(x)_{\alpha}^{+}, \forall x \in U.$$

Hence $\phi(x)_{\alpha}^{+} = \epsilon e \|x\|^{s} = \frac{\epsilon}{2^s} e \|x\|^{s}$ for all $x \in U$. Replacing x by $\epsilon_i x$ and divide by ϵ_i^3 in above inequality, we get

$$\begin{aligned} \frac{1}{\epsilon_i^3} e\phi(\epsilon_i x)_{\alpha}^{+} &= \frac{\epsilon}{2^s \epsilon_i^3} e\left(\|\epsilon_i x\|_{\alpha}^{+}\right)^{s} \\ &= \epsilon_i^{s-3} \frac{\epsilon}{2^s} e\left(\|\epsilon_i x\|_{\alpha}^{+}\right)^{s} \\ &= \epsilon_i^{s-3} e\phi(x)_{\alpha}^{+} \end{aligned}$$

for all $x \in U$. Hence the inequality (112) holds when, $L == \epsilon_i^{s-3}$, that is

$$L = \begin{cases} 2^{s-3} & \text{for} \quad i = 0; \quad s < 3 \\ 2^{3-s} & \text{for} \quad i = 1; \quad s > 3 \end{cases}$$

Now from (113), we prove the following cases:

Case:1 2^{s-3} for $s < 3$ if $i = 0$;

$$\begin{aligned} \|f(x) - c(x)\|_\alpha^+ &\leq \frac{2^{s-3}}{1 - 2^{s-3}} \frac{\epsilon}{2^s} e\left(\|x\|_\alpha^+\right)^s \\ &= \frac{\epsilon}{|8 - 2^s|} e\left(\|x\|_\alpha^+\right)^s \end{aligned}$$

Case:2 2^{3-s} for $s > 3$; if $i = 1$

$$\begin{aligned} \|f(x) - c(x)\|_\alpha^+ &\leq \frac{\left(2^{3-s}\right)^{1-1}}{1 - 2^{3-s}} \frac{\epsilon}{2^s} e\left(\|x\|_\alpha^+\right)^s \\ &= \frac{\epsilon}{|2^s - 8|} e\left(\|x\|_\alpha^+\right)^s \end{aligned}$$

From the above two cases we arrive (113) for $\phi(x)_\alpha^+ = \epsilon \sum_{l=1}^{n} \left(\|x_l\|_\alpha^+\right)^s$. Proceding in the similar mannar one can prove the results for

$$\phi(x)_\alpha^+ = \epsilon \quad \text{and} \quad \phi(x)_\alpha^+ = \epsilon \left(\sum_{l=1}^{n} \left(\|x_l\|_\alpha^+\right)^{ns} \oplus \prod_{l=1}^{n} \left(\|x_l\|_\alpha^+\right)^s \right)$$

. Hence the proof is complete. □

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