International Journal of Mathematics And its Applications
Volume 3, Issue 1 (2015), 1–6.
ISSN: 2347-1557

Some Common Fixed Point Results for Multivalued Mapping in Cone Metric Spaces

A.K.Dubey†,[1], Rohit Verma‡ and R.P.Dubey*

† *Department of Mathematics, Bhilai Institute of Technology, Chhattisgarh, India.*
‡ *Department of Mathematics, Shri Shankracharya Institute of Engineering and Technology, Durg, India.*
* *Department of Mathematics, Dr. C.V.Raman University, Chhattisgarh, India.*

Abstract : Let P be a subset of a Banach space E and P is normal and regular cone on E, we prove the existence of the common fixed point for multivalued maps in cone metric spaces. Our results generalize some well-known recent results in the literature.

Keywords : Cone metric spaces, multivalued mappings, Fixed point.

AMS Subject Classification: 47H10, 54H25.

---

1 Introduction

Huang and Zhang [1] generalized the notion of metric space by replacing the set of real numbers by ordered Banach space, defined a cone metric space, and obtained some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, several other authors [2], [8], [9], [10], [11] studied the existence of common fixed point of mappings satisfying a contractive type condition in cone metric spaces. Seong Hoon Cho and Mi Sun Kim [6] have proved certain fixed point theorems by using multivalued mapping in the setting of contractive constant in metric spaces. In this paper we obtain common fixed points for a pair of multivalued mappings satisfying a generalized contractive type conditions in cone metric spaces. Recall the following definitions which are related to cone metric spaces from [1].

Definition 1.1. *Let E be a real Banach space and a subset P of E is said to be a cone if it satisfies that following conditions,*

(i) P *is closed, non-empty, and* $P \neq \{0\}$,

(ii) $ax + by \in P, a, b \geqslant 0$ *and* $x, y \in P$,

(iii) $x \in P$ *and* $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

The partial ordering $\leqslant$ with respect to the cone P by $x \leqslant y$ if and only if $y - x \in P$. If $y - x \in$interior of P, then it is denoted by $x \ll y$. The cone P is said to be normal if a number $K > 0$ such that,

[1]Corresponding author E-Mail: anilkumardby@rediffmail.com (A.K.Dubey)

for all $x, y \in E, 0 \leqslant x \leqslant y$ implies $\| x \| \leqslant K \| y \|$. The cone P is called regular of every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent. The least positive number K is called the normal constant of P. There are non normal cones also. In the following, we always suppose E is a Banach space, P is a cone in E with $Int\ P \neq \phi\ and \leqslant$ is partial ordering with respect to P.

Definition 1.2. *Let X be a non-empty set of E. Suppose that the map $d : X \times X \to E$ satisfies;*

(i) $0 \leqslant d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.3. *Let $E = R^2, P = \{(x, y) \in E : x, y \geq 0\} \subset R^2, X = R$ and $d : X \times X \to E$ defined by*

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space [1].

Definition 1.4. *Let, (X, d) be a cone metric space,$x \in X$ and $\{x_n\}$ a sequence in X. Then*

(i) $\{x_n\}$ *converges to x whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $d(x_n, x) \ll c$ for all $n \geq N$.*

(ii) $\{x_n\}$ *is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.*

Definition 1.5. *Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.*

Let (X, d) be a metric space. We denote by $CB(X)$ the family of non-empty closed bounded subset of X and let $C(X)$ denote the set of all non-empty compact subsets of X. Let $H(.,.)$ be the Hausdorff distance on $CB(X)$. That is, for $A, B \in CB(X), H(A, B) = max\,\{Sup_{a \in A} d(a, B), Sup_{b \in B} d(A, b)\}$, where $d(a, B) = inf\,\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B.

Theorem 1.6 ([7])**.** *A multivalued mapping $T : X \to CB(X)$ is called a contraction mapping if there exists $k \in (0, 1)$ such that $H(Tx, Ty) \leq kd(x, y)\ \ \forall x, y \in X$ and $x \in X$ is said to be a fixed point of T if $x \in T(X)$.*

2 Fixed Point

In this section, we shall give some results which generalize [7, 10, 11].

Theorem 2.1. *Let (X, d) be a complete cone metric space and let mappings $T_1, T_2 : X \to C(X)$ satisfying the following conditions;*

(i) For each $x \in X, T_1(x), T_2(x) \in CB(X)$,

(ii) $H(T_1(x), T_2(y)) \leq \propto d(x, y) + \beta[d(x, T_2(y)) + d(y, T_1(x))]$

where $\propto, \beta$ *are non negative real numbers and* $\propto +2\beta < 1$. *Then there exists* $p \in X$ *such that* $p \in T_1(x) \cap T_2(x)$.

Proof. Let $x_o \in X, T_1(x_0)$ is a non-empty closed bounded subset of X. Choose $x_1 \in T_1(x_0)$, for this x_1 by the same reason mentioned above $T_2(x_1)$ is non-empty closed bounded subset of X. Since $x_1 \in T_1(x_0)$ and $T_1(x_0)$ and $T_2(x_1)$ are closed bounded subset of $X, \exists x_2 \in T_2(x_1)$ such that $d(x_1, x_2) \leq H(T_1(x_0), T_2(x_1)) + q$, where $q = max\left\{\frac{\propto+\beta}{1-\beta}, \frac{\propto+\beta}{1-\beta}\right\}$

$$\begin{aligned}
d(x_1, x_2) &\leq H(T_1(x_0), T_2(x_1)) + q \\
&\leqslant \propto d(x_0, x_1) + \beta[d(x_0, T_2(x_1)) + d(x_{1,}, T_1(x_0))] + q \\
&\leqslant \propto d(x_0, x_1) + \beta[d(x_0, x_2) + d(x_{1,}, x_1)] + q \\
&\leqslant \propto d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_{1,}, x_2)] + q \\
d(x_1, x_2) &\leq \frac{\propto +\beta}{1-\beta} d(x_0, x_1) + q \\
d(x_1, x_2) &\leq q d(x_0, x_1) + q
\end{aligned}$$

For this $x_2, T_1(x_2)$ is a non-empty closed bounded subset of X. Since $x_2 \in T_2(x_1)$ and $T_2(x_1)$ and $T_1(x_2)$ are closed bounded subset of $X, \exists x_3 \in T_1(x_2)$ such that

$$\begin{aligned}
d(x_2, x_3) &\leq H(T_1(x_2), T_2(x_1)) + q^2 \\
&\leqslant \propto d(x_2, x_1) + \beta[d(x_2, T_2(x_1)) + d(x_1, T_1(x_2))] + q^2 \\
&\leqslant \propto d(x_1, x_2) + \beta[d(x_2, x_2) + d(x_1, x_3)] + q^2 \\
&\leqslant \propto d(x_1, x_2) + \beta[d(x_1, x_2) + d(x_2, x_3)] + q^2 \\
d(x_2, x_3) &\leq \frac{\propto +\beta}{1-\beta} d(x_1, x_2) + q^2 \\
&\leq q\left\{q d(x_0, x_1) + q\right\} + q^2 \\
&\leq q^2 d(x_0, x_1) + 2q^2
\end{aligned}$$

Continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} \in T_2(x_n)$ *or* $x_{n+1} \in T_1(x_n)$ and $d(x_{n+1}, x_n)$ $\leq q^n d(x_0, x_1) + nq^n$.

Let $0 \ll c$ be given, choose a natural number N_1 such that $q^n d(x_0, x_1) + nq^n \ll c$ for all $n \geq N_1$ this implies $d(x_{n+1}, x_n) \ll c$.

Therefore $\{x_n\}$ is a Cauchy sequence in (X, d) is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$. Choose a natural number N_2 such that

$$\begin{aligned}
d(x_n, p) &\ll \frac{c(1-\beta)}{2m(1+\beta)} \quad \text{and} \\
d(x_{n-1}, p) &\ll \frac{c(1-\beta)}{2m(\propto +\beta)} \quad \text{for all} \ \ n \geq N_2.
\end{aligned}$$

$$\begin{aligned}
d(T_1(p), p) &\leq d(p, x_n) + d(x_n, T_1(p)) \\
&\leq d(p, x_n) + H(T_2(x_{n-1}), T_1(p)) \\
&\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta[d(x_{n-1}, T_1(p)) + d(p, T_2(x_{n-1}))] \\
&\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta[d(x_{n-1}, T_1(p)) + d(p, x_n)] \\
&\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta[d(x_{n-1}, p) + d(p, T_1(p)) + d(p, x_n)] \\
d(T_1(p), p) &\leq \frac{\propto +\beta}{1-\beta} d(x_{n-1}, p) + \frac{1+\beta}{1-\beta} d(x_n, p) \ \text{ for all } \ n \geq N_2.
\end{aligned}$$

$d(T_1(p), p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(T_1(p), p) \in P$, and as $m \to \infty$, we get $\frac{c}{m} \to 0$ and P is closed $-d(T_1(p), p) \in P$ but $d(T_1(p), p) \in P$. Therefore $d(T_1(p), p) = 0$ and so $p \in T_1(p)$. Similarly it can be established that $p \in T_2(p)$. Hence $p \in T_1(p) \cap T_2(p)$. □

Corollary 2.2. *Let (X, d) be a complete cone metric space and let mappings $R, S : X \to C(X)$ satisfying the following conditions;*

(i) For each $x \in X, R(x), S(x) \in CB(X)$,

(ii) $H(R(x), S(y)) \leq ad(x, y) + bd(x, S(y)) + cd(y, R(x))$

where a, b, c are non-negative real numbers and $a + b + c < 1$. Then there exists $p \in X$ such that $p \in R(x) \cap S(x)$.

Proof. The symmetric property of d and the above inequality imply that

$$H(R(x), S(y)) \leq ad(x, y) + \frac{b + c}{2}[d(x, S(y)) + d(y, R(x))]$$

By substituting $R = T_1, S = T_2, a = \propto, \frac{b+c}{2} = \beta$ in Theorem (2.1), we obtain the required result. □

Theorem 2.3. *Let (X, d) be a complete cone metric space and let mappings $T_1, T_2 : X \to C(X)$ satisfying the following conditions;*

(i) For each $x \in X, T_1(x), T_2(x) \in CB(X)$,

(ii) $H(T_1(x), T_2(y)) \leq \propto d(x, y) + \beta[d(x, T_1(x)) + d(y, T_2(y))]$

where $\propto, \beta$ are non negative real numbers and $\propto + 2\beta < 1$. Then there exists $p \in X$ such that $p \in T_1(x) \cap T_2(x)$.

Proof. Let $x_0 \in X, T_1(x_0)$ is a non-empty closed bounded subset of X. Choose $x_1 \in T_1(x_0)$, for this x_1 by the same reason mentioned above $T_2(x_1)$ is non-empty closed bounded subset of X.
Since $x_1 \in T_1(x_0)$ and $T_1(x_0)$ and $T_2(x_1)$ are closed bounded subset of $X, \exists\, x_2 \in T_2(x_1)$ such that

$$d(x_1, x_2) \leq H(T_1(x_0), T_2(x_1)) + q,$$

where $q = max\left\{\frac{\propto+\beta}{1-\beta}, \frac{\propto+\beta}{1-\beta}\right\}$

$$\begin{aligned}
d(x_1, x_2) &\leq H(T_1(x_0), T_2(x_1)) + q \\
&\leqslant \propto d(x_0, x_1) + \beta[d(x_0, T_1(x_0)) + d(x_1, T_2(x_1))] + q \\
&\leqslant \propto d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + q \\
d(x_1, x_2) &\leq \frac{\propto + \beta}{1 - \beta} d(x_0, x_1) + q \\
d(x_1, x_2) &\leq qd(x_0, x_1) + q
\end{aligned}$$

For this $x_2, T_1(x_2)$ is a non-empty closed bounded subset of X. Since $x_2 \in T_2(x_1)$ and $T_2(x_1)$ and $T_1(x_2)$ are closed bounded subset of $X, \exists\, x_3 \in T_1(x_2)$ such that

$$\begin{aligned}
d(x_2, x_3) &\leq H(T_1(x_2), T_2(x_1)) + q^2 \\
&\leqslant \propto d(x_1, x_2) + \beta[d(x_2, T_1(x_2)) + d(x_1, T_2(x_1))] + q^2 \\
&\leqslant \propto d(x_1, x_2) + \beta[d(x_2, x_3) + d(x_1, x_2)] + q^2 \\
d(x_2, x_3) &\leq \frac{\propto + \beta}{1 - \beta} d(x_1, x_2) + q^2 \\
&\leq q\{qd(x_0, x_1) + q\} + q^2 = q^2 d(x_0, x_1) + 2q^2
\end{aligned}$$

Continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} \in T_2(x_n)\, or\, x_{n+1} \in T_1(x_n)$ and $d(x_{n+1}, x_n) \leq q^n d(x_0, x_1) + nq^n$. Let $0 \ll c$ be given, choose a natural number N_1 such that $q^n d(x_0, x_1) + nq^n \ll c$ for all $n \geq N_1$ this implies $d(x_{n+1}, x_n) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence in (X, d) is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$. Choose a natural number N_2 such that

$$\begin{aligned}
d(x_n, p) &\ll \frac{c(1-\beta)}{2m} \quad \text{and} \\
d(x_{n-1}, p) &\ll \frac{c(1-\beta)}{2m \propto} \quad \text{for all} \quad n \geq N_2. \\
d(T_1(p), p) &\leq d(p, x_n) + d(x_n, T_1(p)) \\
&\leq d(p, x_n) + H(T_2(x_{n-1}), T_1(p)) \\
&\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta[d(x_{n-1}, T_2(x_{n-1})) + d(p, T_1(p))] \\
&\leq d(p, x_n) + \propto d(x_{n-1}, p) + \beta[d(x_{n-1}, x_n) + d(p, T_1(p))] \\
d(T_1(p), p) &\leq \frac{1}{1-\beta} d(x_n, p) + \frac{\propto}{1-\beta} d(x_{n-1}, p) + \frac{\beta}{1-\beta} d(x_{n-1}, x_n) \text{ for all } n \geq N_2.
\end{aligned}$$

$d(T_1(p), p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(T_1(p), p) \in P$ and as $m \to \infty$, we get $\frac{c}{m} \to 0$ and P is closed $-d(T_1(p), p) \in P$, but $d(T_1(p), p) \in P$. Therefore $d(T_1(p), p) = 0$ and so $p \in T_1(p)$.
Similarly it can be established that $p \in T_2(p)$. Hence $p \in T_1(p) \cap T_2(p)$. □

Corollary 2.4. *Let (X, d) be a complete cone metric space and let mappings $R, S : X \to C(X)$ satisfying the following conditions;*

(i) For each $x \in X, R(x), S(x) \in CB(X)$,

(ii) $H(R(x), S(y)) \leq ad(x, y) + bd(x, R(x)) + cd(y, S(y))$

where a, b, c are non-negative real numbers and $a + b + c < 1$. Then there exists $p \in X$ such that $p \in R(x) \cap S(x)$.

Proof. Proof is similar to the proof of Corollary 2.4. □

References

[1] L.G.Huang and X.Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, 332(2)(2007), 1468-1476.

[2] Sh.Rezapour and R.Hamlbarani, *Some notes on the paper-Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, 345(2)(2008), 719-724.

[3] H. Mohebi, *Topical functions and their properties in a class of ordered Banach spaces, in continuous optimization*, Current Trends and Modern Applications, II(2005), 343-361.

[4] H. Mohebi, H. Sadeghi and A.M. Rubinov, *Best approximation in a class of normed spaces with star-shaped cone*, Numer. Funct. Anal. Optim., 27(3-4)(2006), 411-436.

[5] R.K. Bose and R.N. Mukherjee, *Common fixed points of some multivalued mappings*, Tamkang J. Math.,8(2)(1977), 245-249.

[6] Seong Hoon Cho and Mi Sun Kim, *Fixed point theorems for general contractive multivalued mappings*, J. Appl. Math. Informatics, 27(2009), 343-350.

[7] Robert M. Brooks and Klaus Schmitt, *The contraction mapping principle and some applications*, Electronic J. of Differential equations, Monograph 09(2009).

[8] A.K. Dubey and A. Narayan, *Cone Metric Spaces and Fixed Point Theorems for Pair of contractive maps*, Mathematica Aeterna, 2(10)(2012), 839-845.

[9] A.K. Dubey, Rita Shukla and R.P. Dubey, *An extension of the paper Cone Metric Spaces and Fixed Point Theorems of contractive mappings*, Int. J. of Applied Mathematical Research, 2(1)(2013), 84-90.

[10] R. Krishna Kumar and M. Marudai, *Generalization of a Fixed Point Theorem in Cone Metric Spaces*, Int. J. of Math. Analysis, 5(11)(2011), 507-512.

[11] Ismat Beg, Akbar Azam and Muhammad Arshad, *Common Fixed Points for Maps on Topological Vector Space Valued Cone Metric Spaces*, Int. J. of Mathematics and Mathematical Sciences, (2009), Article ID 560264, 8 pages, doi:10.1155/2009/560264.