International Journal of Mathematics And its Applications
Volume 3, Issue 1 (2015), 39–47.
ISSN: 2347-1557

Common Fixed Point Theorems for Weakly Compatible of Four Mappings in Generalized Fuzzy Metric Spaces

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Abstract : In this paper, common fixed point theorems for weakly compatible maps in complete $\mathcal{M}$-fuzzy metric spaces is proved.

Keywords : Fuzzy contractive mapping, complete $\mathcal{M}$-fuzzy metric.

AMS Subject Classification : 54E40, 54E35, 54H25.

1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [17] in 1965. Since, then to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [2] and Kramosil and Michalek [5] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and E-infinity theory which were given and studied by E_1 Naschie [1]. Many authors [2, 3, 5, 6] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. One should there exists a space between spaces. And one such generalization is generalized metric space or D-metric space initiated by Dhage in 1992. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. Rhoades generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, Sedghi and Shobe [8] introduced D^* metric space, as a probable modification of the definition of D-metric.

Using D^*-metric concept, Sedghi and Shobe defined $\mathcal{M}$-fuzzy metric space and proved common fixed point theorem in it.

In this paper we prove common fixed point theorems in complete $\mathcal{M}$-fuzzy metric space.

Definition 1.1. *A 3-tuple $(X, \mathcal{M}, *)$ is called $\mathcal{M}$-fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$.*

(FM-1) $\mathcal{M}(x, y, z, t) > 0$

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(FM-2) $\mathcal{M}(x,y,z,t) = 1$ *if and only if* $x = y = z$

(FM-3) $\mathcal{M}(x,y,z,t) = \mathcal{M}(p\{x,y,z\},t)$, *where* p *is a permutation function.*

(FM-4) $\mathcal{M}(x,y,a,t) * \mathcal{M}(a,z,z,s) \leq \mathcal{M}(x,y,z,t+s)$

(FM-5) $\mathcal{M}(x,y,z,.) : (0,\infty) \to [0,1]$ *is continuous*

(FM-6) $\lim_{t\to\infty} \mathcal{M}(x,y,z,t) = 1.$

Definition 1.2. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space and* $\{x_n\}$ *be a sequence in* X

(a) $\{x_n\}$ *is said to be converges to point* $x \in X$ *if* $\lim_{n\to\infty} \mathcal{M}(x_1,x_1,x_n,t) = 1$ *for all* $t > 0$

(b) $\{x_n\}$ *is called Cauchy sequence if* $\lim_{n\to\infty} \mathcal{M}(x_{n+p},x_{n+p},x_n,t) = 1$ *for all* $t > 0$ *and* $p > 0$

(c) A $\mathcal{M}$*-fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Lemma 1.3. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space. Then* $\mathcal{M}(x,y,z,t)$ *is non-decreasing with respect to* t*, for all* x,y,z *in* X.

Definition 1.4. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space.* $\mathcal{M}$ *is said to be continuous function on* $X^3\times(0,\infty)$ *if* $\lim_{n\to\infty}\mathcal{M}(x_n,y_n,z_n,t_n) = \mathcal{M}(x,y,z,t)$, *whenever a sequence* $\{(x_n,y_n,z_n,t_n)\} =$ *in* $X^3\times(0,\infty)$ *converges to a point* $(x,y,z,t) \in X^3\times(0,\infty)$. *i.e.,* $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$ *and* $\lim_{n\to\infty}\mathcal{M}(x,y,z,t_n) = \mathcal{M}(x,y,z,t)$.

Lemma 1.5. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space. Then* $\mathcal{M}$ *is continuous function on* $X^3\times(0,\infty)$.

Definition 1.6. *Let* A *and* S *be mappings from a* $\mathcal{M}$*-fuzzy metric space* $(X,\mathcal{M},*)$ *into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is,* $Ax = Sx$ *implies that* $ASx = SAx$.

Definition 1.7. *Let* A *and* S *be mappings from a* $\mathcal{M}$*-fuzzy metric space* $(X,\mathcal{M},*)$ *into itself. Then the mappings are said to be compatible if* $\lim_{n\to\infty}\mathcal{M}(ASx_n,SAx_n,SAx_n,t) = 1$ *for all* $t > 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Lemma 1.8. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space. If we define* $E_\lambda\mathcal{M} : X^3 \to \mathcal{R}^+\cup\{0\}$ *by* $E_{\lambda,\mathcal{M}}(x,y,z) = inf\{t > 0 : \mathcal{M}(x,y,z,t) > 1-\lambda\}$ *for every* $\lambda \in (0,1)$, *then*

(1) for each $\mu \in (0,1)$ *there exists* $\lambda \in (0,1)$ *such that* $E_{\mu,\mathcal{M}}(x_1,x_1,x_n) \leq E_{\lambda,\mathcal{M}}(x_1,x_2,x_2)$ $+ E_{\lambda,\mathcal{M}}(x_2,x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1},x_{n-1},x_n)$ *for any* $x_1,x_2,\ldots,x_n \in X$.

(2) The sequence $\{x_n\}n \in N$ *is convergent in* $\mathcal{M}$*-fuzzy metric space* $(X,\mathcal{M},*)$ *if and only if* $E_{\lambda,\mathcal{M}}(x_n,x_n,x) \to 0$*. Also the sequence* $\{x_n\}, n \in \mathcal{N}$ *is Cauchy sequence if and only if it is Cauchy with* $E_{\lambda,\mathcal{M}}$.

Lemma 1.9. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space. If* $\mathcal{M}(x_n,x_n,x_{n+1},t) \geq \mathcal{M}(x_0,x_0,x_1,k^nt)$ *for some* $k > 1$ *and for every* $n \in \mathcal{N}$*. Then sequence* $\{x_n\}$ *is a cauchy sequence.*

2 The Main Results

A class of implicit relation

Let Φ denotes a family of mappings such that each $\phi \in \Phi, \phi : [0,1]^3 \to [0,1]$ and ϕ is continuous and increasing in each co-ordinate variable. Also $\phi(s,s,s) > s$ for every $s \in [0,1)$.

Example 2.1. *Let* $\phi : [0,1]^3 \to [0,1]$ *be defined by*

(1) $\phi(x_1, x_2, x_3) = (\min\{x_i\})^h$ *for some* $0 < h < 1$.

(2) $\phi(x_1, x_2, x_3) = x_1^h$ *for some* $0 < h < 1$.

(3) $\phi(x_1, x_2, x_3) = \max\{x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}\}$, *where* $0 < \alpha_i < 1$ *for* $i = 1, 2, 3$.

In this paper, p is a positive real number and $\phi^{3p}(s,s,s) = [\phi(s,s,s)]^{3p}$ for every $s \in [0,1)$. Also, $\mathcal{M}(Sx, By, Bz, t) \vee \mathcal{M}(Ty, Ax, Ty, t) \vee \mathcal{M}(Tz, Ax, Tz, t) = \max\{\mathcal{M}(Sx, By, Bz, t), \mathcal{M}(Ty, Ax, Ty, t), \max(Tz, Ax, Tz, t)\}$. Our main result for a complete $\mathcal{M}$-fuzzy metric space X, as reads follows:

Theorem 2.1. *Let* A, B, S *and* T *be self mappings of complete* $\mathcal{M}$*-fuzzy metric space* $(X, \mathcal{M}, *)$ *satisfying the following conditions.*

(1) (A, S) *and* (S, T) *are weakly compatible pairs such that* $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$ *also* $A(X)$ *or* $B(X)$ *is a closed subset of* X.

(2) There exists $\psi, Q \in \Phi$ *such that for all* $x, y, z \in X$.

$$\begin{aligned}\mathcal{M}^{3p}(Ax, By, Bz, t) \geq a(s)\phi^{3p}(&\mathcal{M}(Sx, Ty, Tz, kt), \mathcal{M}(Ax, Sx, Sx, kt), \mathcal{M}(By, Ty, Tz, kt)\\ &+ b(s)\psi^p(\mathcal{M}^3(Sx, Ty, Tz, kt), \mathcal{M}(Sx, Ax, Ax, kt)\mathcal{M}(Ty, By, By, kt)\\ &\mathcal{M}(Tz, Bz, Bz, kt), \mathcal{M}(Sx, By, Bz, kt) \vee \mathcal{M}(Ty, Ax, Ty, kt) \vee \mathcal{M}(Tz, Ax, Tz, kt))\end{aligned}$$

for some $k > 1$ *where* $a, b : [0,1] \to [0,1]$ *are two continuous functions such that* $a(s) + b(s) = 1$ *for every* $S = \mathcal{M}(x, y, z, t)$. *Then* A, B *and* S, T *have a unique common fixed point in* X.

Proof. Let $x_0 \in X$ an arbitrary point as $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ there exists $x_1, x_2 \in X$ be $Ax_0 = Tx_1, Bx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{3n} = Ax_{3n} = Tx_{3n+1}, y_{3n+1} = Bx_{3n+1} = Sx_{3n+2}$, for $n = 0, 1, 2, \ldots$. Now, we prove $\{y_n\}$ is a Cauchy sequence. For simplicity, we get $d_n(t) = \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t), n = 0, 1, 2, \ldots$. Then we have

$$\begin{aligned}d_{3n}^{3p}(t) &= \mathcal{M}^{3p}(y_{3n}, y_{3n+1}, y_{3n+1}, t)\\ &= \mathcal{M}^{3p}(Ax_{3n}, Bx_{3n+1}, Bx_{3n+1}, t)\\ &\geq a(s)\phi^{3p}(\mathcal{M}(Sx_{3n}, Tx_{3n+1}, Tx_{3n+1}, kt), \mathcal{M}(Ax_{3n}, Bx_{3n+1}, Bx_{3n+1}, kt),\\ &\mathcal{M}(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}, kt) + b(s)\psi^k(\mathcal{M}^3(Sx_{3n}, Tx_{3n+1}, Tx_{3n+1}, kt), \mathcal{M}(Sx_{3n}, Ax_{3n}, Ax_{3n}, kt)\\ &\mathcal{M}(Tx_{3n+1}, Bx_{3n+1}, Bx_{3n+1}, kt)\mathcal{M}(Bx_{3n+1}, Tx_{3n+1}, kt)\mathcal{M}(Sx_{3n}, Bx_{3n+1}, Bx_{3n+1}, kt)\vee\\ &\mathcal{M}(Tx_{3n+1}, Ax_{3n}, Tx_{3n+1}, kt) \vee\ \mathcal{M}(Tx_{3n+1}, Ax_{3n}, Tx_{3n+1}, kt))\\ &\geq a(s)\phi^{3p}(\mathcal{M}(y_{3n-1}, y_{3n}, y_{3n}, kt), \mathcal{M}(y_{3n}, y_{3n+1}, y_{3n+1}, kt), \mathcal{M}(y_{3n+1}, y_{3n}, y_{3n}, kt)\\ &\quad + b(s)\psi^p(\mathcal{M}^3(y_{3n-1}, y_{3n}, y_{3n}, kt), \mathcal{M}(y_{3n-1}y_n, y_n, kt)\mathcal{M}(y_n, y_{3n+1}, y_{3n+1}, kt)\\ &\quad \mathcal{M}(y_{3n+1}, y_{3n}, y_{3n}, kt)\mathcal{M}(y_{3n-1}, y_{3n+1}, y_{3n+1}, kt) \vee\ \mathcal{M}(y_{3n}, y_{3n}, y_{3n}, kt) \vee\ \mathcal{M}(y_{3n}, y_{3n}, y_{3n}, kt)).\end{aligned}$$

We prove that $d_{3n}(t) \geq d_{3n-1}(t)$. Now, if $d_{3n}(t) < d_{3n-1}(t)$ for some $n \in N$. Since ϕ and ψ are increasing functions, then

$$\begin{aligned}d_{3n}^{3p}(t) &\geq a(s)\phi^{3p}(d_{3n-1}(kt), d_{3n}(kt), d_{3n}(kt) + b(s)\psi^p(d_{3n-1}^3(kt), d_{3n-1}(kt)d_{3n}(kt)d_{3n}(kt), 1)\\ &\geq a(s)\phi^{3p}(d_{3n}(kt), d_{3n}(kt), d_{3n}(kt)) + b(s)\psi^p(d_{3n}^3(kt), d_{3n}^3(kt), 1)\\ &> a(s)d_{3n}^{3p}(kt) + b(s)d_{3n}^{3p}(kt)\\ &= d_{3n}^{3p}(kt).\end{aligned}$$

Hence we have $d_{3n}(t) > d_{3n}(kt)$ is a contradiction. Therefore $d_{3n}(t) \geq d_{3n-1}(t)$. Similarly, one can prove that $d_{3n+1}(t) \geq d_{3n}(t)$ for $n = 0, 1, 2, \ldots$.
Consequently, $\{d_n(t)\}$ is a $\{d_n(t)\}$ is a increasing sequence of non-negative real. Thus,

$$\begin{aligned} d_{3n}^{3p}(t) &\geq a(s)\phi^{3p}(d_{3n-1}(kt), d_{3n-1}(kt), d_{3n-1}(kt)) + b(s)\psi^p(d_{3n-1}^3(kt), d_{3n-1}^3(kt), 1) \\ &\geq a(s)d_{3n-1}^{3p}(kt) + b(s)d_{3n-1}^{3p}(kt) \\ &= d_{3n-1}^{3p}(kt). \end{aligned}$$

That is $d_{3n}(t) \geq d_{3n-1}(kt)$, similarly we have $d_{3n+1}(t) \geq d_{3n}(kt)$. Thus $d_n(t) \geq d_{n-1}(kt)$. That is $\mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) \geq \mathcal{M}(y_{n-1}, y_n, y_n, kt)$. So,

$$\mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) \geq \mathcal{M}(y_{n-1}, y_n, y_n, kt) \geq \cdots \geq \mathcal{M}(y_0, y_1, y_1, k^n t).$$

By Lemma 1.9 sequence $\{y_n\}$ is a Cauchy sequence, then it converges to $y \in X$. That is

$$\begin{aligned} \lim_{n\to\infty} y_n &= \lim_{n\to\infty} y_{3n} = \lim_{n\to\infty} y_{3n+1} \\ &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} Bx_{3n+1} \\ &= \lim_{n\to\infty} Sx_{3n} = \lim_{n\to\infty} Tx_{3n+1} = y. \end{aligned}$$

As $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$, we have

$$\begin{aligned} \mathcal{M}^{3p}(Au, Bx_{3n+1}, Bx_{3n+1}, t) \geq a(s)\phi^{3p}(&\mathcal{M}(Su, Tx_{3n+1}, Tx_{3n+1}, kt), \mathcal{M}(Au, Su, Su, kt), \\ &\mathcal{M}(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}, kt)) + b(s)\psi^p(\mathcal{M}^3(Su, Tx_{3n+1}, Tx_{3n+1}, kt), \\ &\mathcal{M}(Su, Au, Au, kt)\mathcal{M}(Tx_{3n+1}, Bx_{3n+1}, Bx_{3n+1}, kt) \\ &\mathcal{M}(Tx_{3n+1}, Bx_{3n+1}, Bx_{3n+1}, kt)\mathcal{M}(Su, Bx_{3n+1}, Bx_{3n+1}, kt) \\ &\vee \mathcal{M}(Tx_{3n+1}, Au, Tx_{3n+1}, kt) \vee \mathcal{M}(Tx_{3n+1}, Au, Tx_{3n+1}, kt). \end{aligned}$$

By continuous $\mathcal{M}$ and ϕ, on making $n \to \infty$ the above inequality, we get

$$\begin{aligned} \mathcal{M}^{3p}(Au, y, y, t) \geq a(s)\phi^{3p}(&\mathcal{M}(y, y, y, kt), \mathcal{M}(Au, y, y, kt), \mathcal{M}(y, y, y, kt) \\ &+ b(s)\psi^p(\mathcal{M}^3(y, y, y, kt), \mathcal{M}(y, Au, Au, kt)\mathcal{M}(y, y, y, kt)\mathcal{M}(y, y, y, kt), \mathcal{M}(y, y, y, kt) \\ &\vee\ \mathcal{M}(y, Au, y, kt) \vee\ \mathcal{M}(y, Au, Ty, kt)) \end{aligned}$$

hence we have

$$\begin{aligned} \mathcal{M}^{3p}(Au, y, y, t) \geq a(s)\phi^{3p}(&\mathcal{M}(Au, y, y, kt), \mathcal{M}(Au, y, y, kt), \mathcal{M}(Au, y, y, kt) \\ &+ b(s)\psi^p(\mathcal{M}^3(Au, y, y, kt), \mathcal{M}(Au, y, y, kt)\mathcal{M}(Au, y, y, kt)\mathcal{M}(Au, y, y, kt), 1). \end{aligned}$$

If $Au \neq y$, by above inequality we get

$$\begin{aligned} \mathcal{M}^{3p}(Au, y, y, t) &\geq a(s)\mathcal{M}^{3p}(Au, y, y, kt) + b(s)\mathcal{M}^{3p}(Au, y, y, kt) \\ &= \mathcal{M}^{3p}(Au, y, y, kt), \end{aligned}$$

which is contradiction. Hence $\mathcal{M}(Au, y, y, t) = 1$. Therefore $Au = y$. Thus $Au = Su = y$. As $A(X) \subseteq$

$T(X)$ there exists $v \in X$, such that $Tv = y$, so,

$$\begin{aligned}
\mathcal{M}^{3p}(y, Bv, Bv, t) &= \mathcal{M}^{3p}(Au, Bv, Bv, t) \\
&\geq a(s)\phi^{3p}(\mathcal{M}(Sv, Tv, Tv, kt), \mathcal{M}(Au, Su, Su, kt), \mathcal{M}(Bv, Tv, Tv, kt)) \\
&\quad + b(s)\psi^{p}(\mathcal{M}^{3}(Su, Tv, Tv, kt), \mathcal{M}(Su, Av, Av, kt)\mathcal{M}(Tv, Bv, Bv, kt) \\
&\quad \mathcal{M}(Tv, Bv, Bv, xt)\mathcal{M}(Su, Bv, Bv, kt) \vee\ \mathcal{M}(Tv, Av, Tv, kt) \vee\ \mathcal{M}(Tv, Au, Tv, kt)) \\
&= a(s)\phi^{3p}(1, 1, \mathcal{M}(Bv, y, y, kt)) + b(s)\psi^{p}(1, 1, 1).
\end{aligned}$$

We claim that $Bv = y$ for if $Bv \neq y$, then $\mathcal{M}(Bv, y, y, t) < 1$. On the above inequality we get

$$\begin{aligned}
\mathcal{M}^{3p}(y, Bv, Bv, t) &\geq a(s)\phi^{3p}(\mathcal{M}(y, Bv, Bv, kt), \mathcal{M}(y, Bv, Bv, kt), \mathcal{M}(y, Bv, Bv, kt) \\
&\quad + b(s)X^{p}(\mathcal{M}^{3}(y, Bv, Bv, kt), \mathcal{M}(y, Bv, Bv, kt)\mathcal{M}(y, Bv, Bv, kt)\mathcal{M}(y, Bv, Bv, kt), \\
&\quad \mathcal{M}(y, Bv, Bv, kt) \vee \mathcal{M}(y, Bv, Bv, kt) \vee \mathcal{M}(y, Bv, Bv, kt)) \\
&= a(s)\mathcal{M}^{3p}(y, Bv, Bv, kt) + b(s)\mathcal{M}^{3p}(y, Bv, Bv, kt) \\
&= \mathcal{M}^{3p}(y, Bv, Bv, kt) \text{ a contradiction.}
\end{aligned}$$

Hence $Tv = Bv = Av = Su = y$. Since (A, S) is weak compatible, we get that $ASU = SAU$, that is $Ay = Sy$. Since (B, T) is weak compatible. We get $TBv = BTv$, that is $Ty = By$. If $Ay \neq y$, then $\mathcal{M}(Ay, y, y, t) < 1$ how ever

$$\begin{aligned}
\mathcal{M}^{3p}(Ay, y, y, t) &= \mathcal{M}^{3p}(Ay, Bv, Bv, t) \\
&\geq a(s)\phi^{3p}(Sy, Tv, Tv, kt), \mathcal{M}(Ay, Sy, Sy, kt), \mathcal{M}(Bv, Tv, Tv, kt) \\
&\quad + b(s)\psi^{p}\mathcal{M}^{3}(Sy, Tv, Tv, kt), \mathcal{M}(Sy, Ay, Ay, kt)\mathcal{M}(Tv, Bv, Bv, kt)\mathcal{M}(Tv, Bv, Bv, kt), \\
&\quad \mathcal{M}(Sy, Bv, Bv, kt) \vee\ \mathcal{M}(Tv, Ay, Tv, kt) \vee\ \mathcal{M}(Tv, Ay, Tv, kt)) \\
&= a(s)\phi^{3p}(\mathcal{M}(Ay, y, y, kt), 1, 1) + b(s)\psi^{p}(\mathcal{M}^{3}(Ay, y, y, kt), 1, \mathcal{M}(Ay, y, y, kt)) \\
&\geq a(s)\phi^{3p}(\mathcal{M}(Ay, y, y, kt), \mathcal{M}(Ay, y, y, kt), \mathcal{M}(Ay, y, y, kt)) + b(s)\psi^{p}(\mathcal{M}^{3}(Ay, y, y, kt), \\
&\quad \mathcal{M}^{3}(Ay, y, y, kt), \mathcal{M}^{3}(Ay, y, y, kt)) \\
&> a(s)(\mathcal{M}^{3p}(Ay, y, y, kt)) + b(s)\mathcal{M}^{3p}(Ay, y, y, kt) \\
&= \mathcal{M}^{3p}(Ay, y, y, kt), \text{ a contradiction.}
\end{aligned}$$

Thus $Ay = y$, hence $Ay = Sy = y$. Similarly, we prove that $By = y$, for if $By \neq y$.
Then $\mathcal{M}(By, y, y, kt) < 1$, how ever,

$$\begin{aligned}
\mathcal{M}^{3p}(y, By, By, t) &= \mathcal{M}^{3p}(Ay, By, By, t) \\
&\geq a(s)\phi^{3p}(\mathcal{M}(Sy, Ty, Ty, kt)), \mathcal{M}(Ay, Sy, Sy, kt), \mathcal{M}(By, Ty, Ty, kt)) \\
&\quad + b(s)\psi^{p}(\mathcal{M}^{3}(Sy, Ty, Ty, kt), \mathcal{M}(Sy, Ay, Ay, kt)\mathcal{M}(Ty, By, By, kt) \\
&\quad \mathcal{M}(Ty, By, By, kt), \mathcal{M}(Sy, By, By, kt) \vee \mathcal{M}(Ty, Ay, Ty, kt) \vee \mathcal{M}(Ty, Ay, Ty, kt)) \\
&= a(s)\phi^{3p}(\mathcal{M}(y, By, By, kt), \mathcal{M}(y, y, y, kt), \mathcal{M}(By, By, By, kt) \\
&\quad + b(s)\psi^{p}(\mathcal{M}^{3}(y, By, By, kt), 1, \mathcal{M}(y, By, By, kt)) \\
&\geq a(s)\phi^{3p}(\mathcal{M}(y, By, By, kt), \mathcal{M}(y, By, By, kt), \mathcal{M}(y, By, By, kt)) \\
&\quad + b(s)\psi^{p}(\mathcal{M}^{3}(y, By, By, kt), \mathcal{M}^{3}(y, By, By, kt), \mathcal{M}^{3}(y, By, By, kt)) \\
&> a(s)\mathcal{M}^{3p}(y, By, By, kt) + b(s)\mathcal{M}^{3p}(y, By, By, kt) \\
&= \mathcal{M}^{3p}(y, By, By, kt), \text{ a contradiction.}
\end{aligned}$$

Therefore, $Ay = By = Sy = Ty = y$. That is y is a common fixed pint of A, B, S and T.

Uniqueness: Let w be another common fixed point of A, B, S and T. That is $w = Aw = Sw = Bw = Tw$. If $\mathcal{M}(x, y, z, t) < 1$, then

$$\begin{aligned}
\mathcal{M}^{3p}(y, w, w, t) &= \mathcal{M}^{3p}(Ay, Bw, Bw, t) \\
&\geq a(s)\phi^{3p}(\mathcal{M}(Sy, Tw, Tw, kt), \mathcal{M}(Ay, Sy, Sy, kt)\mathcal{M}(Bw, Tw, Tw, kt)) \\
&\quad + b(s)\psi^{p}(\mathcal{M}^3(Sy, Tw, Tw, kt), \mathcal{M}(Sy, Ay, Ay, kt) \\
&\quad \mathcal{M}(Tw, Bw, Bw, kt), \mathcal{M}(Tw, Bw, Bw, kt), \mathcal{M}(Sy, Bw, Bw, kt) \\
&\quad \vee\ \mathcal{M}(Tw, Ay, Tw, kt) \vee \mathcal{M}(Tw, Ay, Aw, kt) \\
&= a(s)\phi^{3p}(\mathcal{M}(y, w, w, kt), 1, 1) + b(s)\psi^{p}(\mathcal{M}^3(y, w, w, kt), 1, \mathcal{M}(y, w, w, kt)) \\
&\geq a(s)\phi^{3p}(\mathcal{M}(y, w, w, kt), \mathcal{M}(y, w, w, kt), \mathcal{M}(y, w, w, kt) \\
&\quad + b(s)\psi^{p}(\mathcal{M}^3(y, w, w, kt), \mathcal{M}^3(y, w, w, kt), \mathcal{M}^3(y, w, w, kt)) \\
&> a(s)\mathcal{M}^{3p}(y, w, w, kt) + b(s)\mathcal{M}^{3p}(y, w, w, kt) \\
&= \mathcal{M}^{3p}(y, w, w, kt), \text{ a contradiction.}
\end{aligned}$$

Therefore y is the unique common fixed point of self-maps A, B, S and T. □

In the following theorem, function $\phi : [0, 1]^4 \to [0, 1]$, is continuous and increasing in each co-ordinate variable. Also $\phi(s, s, s, s) > s$ for every $s \in [0, 1)$.

Theorem 2.2. *Let A, B, S and T be self-mappings of a complete $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ satisfying that*

(1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ and $A(X)$ or $B(X)$ is a complete subset of X.

(2) $\mathcal{M}(Ax, By, Bz, t) \geq \phi(\mathcal{M}(Sx, Ty, Tz, kt), \mathcal{M}(Ax, Sx, Sx, kt), \mathcal{M}(By, Ty, Tz, kt), \mathcal{M}(Sx, By, Bz, kt)$ $\vee\mathcal{M}(Ty, Ax, Ty, kt) \vee\ \mathcal{M}(Tz, Ax, Tz, kt))$ for every x, y, z in $X, k > 1$ and $\phi \in \Phi$.

(3) The pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. As $A(X) \subseteq T(X), B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Ax_1, Bx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{3n} = Ax_{3n} = Tx_{3n+1}, y_{3n+1} = Bx_{3n+1} = Sx_{3n+2}$ for $n = 0, 1, 2, \ldots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $d_m(t) = \mathcal{M}(y_m, y_{m+1}, y_{m+1}, t)$, $t > 0$. We prove $\{d_{\mathcal{M}}(t)\}$ is increasing with respect to m. Let $m = 3n$, we have

$$\begin{aligned}
d_{3n}(t) &= \mathcal{M}(y_{3n}, y_{3n+1}, y_{3n+1}, t) \\
&= \mathcal{M}(Ax_{3n}, Bx_{3n+1}, Bx_{3n+1}, t) \\
&\geq \phi(\mathcal{M}(sx_{3n}, Tx_{3n+1}, Tx_{3n+1}, kt), \mathcal{M}(Ax_{3n}, Sx_{3n}, Sx_{3n}, kt), \mathcal{M}(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}, kt), \\
&\quad \mathcal{M}(Sx_{3n}, Bx_{3n+1}, Bx_{3n+1}, kt) \vee\ \mathcal{M}(Tx_{3n+1}, Ax_{3n}, Tx_{3n+1}, kt) \vee \mathcal{M}(Tx_{3n+1}, Ax_{3n}, Tx_{3n+1}, kt)) \\
&= \phi(\mathcal{M}(y_{3n-1}, y_{3n}, y_{3n}, kt), \mathcal{M}(y_{3n}, y_{3n-1}, y_{3n-1}, kt), \mathcal{M}(y_{3n+1}, y_{3n}, y_{3n}, kt), \\
&\quad \mathcal{M}(y_{3n-1}, y_{3n+1}, y_{3n+1}, kt) \vee \mathcal{M}(y_{3n}, y_{3n}, y_{3n}, kt) \vee \mathcal{M}(y_{3n}, y_{3n}, y_{3n}, kt)) \\
&= \phi(d_{3n-1}(kt), d_{3n-1}(kt), d_{3n}(kt), 1) \\
&\geq \phi(d_{3n-1}(kt), d_{3n}(kt), 1).
\end{aligned}$$

Since ϕ is an increasing function. We claim that for every $n \in N, d_{3n}(kt) \geq d_{3n-1}(kt)$.
For if $d_{3n}(kt) < d_{3n-1}(kt)$. Then in inequality (2.1), we have

$$\begin{aligned} d_{3n}(t) &\geq \phi(d_{3n}(kt), d_{3n}(kt), d_{3n}(kt), d_{3n}(kt)) \\ &> d_{3n}(kt). \end{aligned}$$

That is $d_{3n}(t) > d_{3n}(kt)$ a contradiction. Hence $d_{3n}(kt) \geq d_{3n-1}(kt)$ for every $n \in N$ and for all $t > 0$. Similarly,we have $d_{3n+1}(kt) \geq d_{3n}(kt)$. Thus, $\{d_n(t)\}$ is an increasing sequence in $[0, 1]$. By inequality, (2.1) and $d_n(t)$ is an increasing sequence, we get,

$$\begin{aligned} d_{3n}(t) &\geq \phi(d_{3n-1}(kt), d_{3n-1}(kt), d_{3n-1}(kt), d_{3n-1}(kt)) \\ &\geq d_{3n-1}(kt). \end{aligned}$$

Similarly, we have $d_{3n+1}(t) \geq d_{3n}(kt)$. Thus $d_n(t) \geq d_{n-1}(kt)$. That is

$$\begin{aligned} \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t) &\geq \mathcal{M}(y_{n-1}, y_n, y_n, kt) \geq \dots \\ &\geq \mathcal{M}(y_0, y, y, k^n t). \end{aligned}$$

Hence by Lemma 1.9 $\{y_n\}$ is Cauchy and the completeness of $X, \{y_n\}$ converges of y in X. That is

$$\begin{aligned} &\lim_{n\to\infty} y_n = y \\ \Rightarrow &\lim_{n\to\infty} y_{3n} = \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} Tx_{3n+1} \\ &= \lim_{n\to\infty} y_{3n+1} = \lim_{n\to\infty} Bx_{3n+1} = \lim_{n\to\infty} Sx_{3n+1} = y \end{aligned}$$

As $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$. So we have

$$\begin{aligned} \mathcal{M}(Au, Bx_{3n+1}, Bx_{3n+1}, t) \geq \phi(&\mathcal{M}(Su, Tx_{3n+1}, Tx_{3n+1}, kt), \mathcal{M}(Au, Su, Su, kt), \\ &\mathcal{M}(Bx_{3n+1}, Bx_{3n+1}, Bx_{3n+1}, kt), \mathcal{M}(Su, Bx_{3n+1}, Bx_{3n+1}, kt)) \\ &\vee \mathcal{M}(Tx_{3n+1}, Au, Tx_{3n+1}, kt) \vee \ \mathcal{M}(Tx_{3n+1}, Au, Tx_{3n+1}, kt)). \end{aligned}$$

If $Au \neq y$, by continuous m and Φ, on making $n \to \infty$, the above inequality, we get

$$\begin{aligned} \mathcal{M}(Au, y, y, t) &\geq \phi(\mathcal{M}(y, y, y, kt), \mathcal{M}(Au, y, y, kt), \mathcal{M}(y, y, y, kt) \\ &\quad \mathcal{M}(y, y, y, kt) \vee \ \mathcal{M}(y, Au, y, kt) \vee \ \mathcal{M}(y, Au, y, kt)) \\ &\geq \phi(\mathcal{M}(Au, y, y, kt), \mathcal{M}(Au, y, y, kt), \mathcal{M}(Au, y, y, kt), \mathcal{M}(Au, y, y, kt)) \\ &> \mathcal{M}(Au, y, y, kt). \end{aligned}$$

That is $\mathcal{M}(Au, y, y, kt) > \mathcal{M}(Av, y, y, kt)$ which is a contradiction. Hence $\mathcal{M}(Av, y, y, t) = 1$, i.e, $Av = y$. Thus, $Au = Su = y$. As $A(X) \subseteq T(X)$, there exists $v \in X$, such that $Tv = y$. So,

$$\begin{aligned} \mathcal{M}(y, Bv, Bv, t) &= \mathcal{M}(Av, Bv, Bv, t) \\ &\geq \phi(\mathcal{M}(Su, Tv, Tv, kt), \mathcal{M}(Av, Sv, Sv, kt), \mathcal{M}(Bv, Tv, Tv, kt), \mathcal{M}(Su, Bv, Bv, kt) \\ &\quad \vee \mathcal{M}(Tv, Au, Tv, kt) \vee \ \mathcal{M}(Tv, Au, Tv, kt)) \\ &= \phi(1, 1, \mathcal{M}(Bv, y, y, kt), 1). \end{aligned}$$

We claim that $Bv = y$. For if $Bv \neq y$. Then $\mathcal{M}(Bv, y, y, t) < 1$. On the above inequality, we get

$$\begin{aligned} \mathcal{M}(y, Bv, Bv, t) &\geq \phi(\mathcal{M}(y, Bv, Bv, kt), \mathcal{M}(y, Bv, Bv, kt), \mathcal{M}(y, Bv, Bv, kt), \mathcal{M}(y, Bv, Bv, kt)) \\ &> \mathcal{M}(y, Bv, Bv, kt) \text{ a contradiction.} \end{aligned}$$

Hence $Tv = Bv = Av = Su = y$. Since (A, S) is weakly compatible, we get that $ASU = SAU$, that is $Ay = Sy$. Since (B, T) is weakly compatible, we get that $TBv = BTv$ that is $Ty = By$. If $Ay \neq y$, then $\mathcal{M}(Ay, y, y, t) < 1$. However,

$$\begin{aligned}
\mathcal{M}(Ay, y, y, t) &= \mathcal{M}(Ay, Bv, Bv, t)\\
&\geq \phi(\mathcal{M}(Sy, Tv, Tv, kt), \mathcal{M}(Ay, Sy, Sy, kt), \mathcal{M}(Bv, Tv, Tv, kt), \mathcal{M}(Sy, Bv, Bv, kt)\\
&\vee\ \mathcal{M}(Tv, Ay, Tv, kt) \vee\ \mathcal{M}(Tv, Ay, Tv, kt))\\
&\geq \phi(\mathcal{M}(Ay, y, y, kt), 1, 1, \mathcal{M}(y, Ay, y, kt))\\
&\geq \phi(\mathcal{M}(Ay, y, y, kt), \mathcal{M}(Ay, y, y, kt), \mathcal{M}(Ay, y, y, kt), \mathcal{M}(Ay, y, y, kt))\\
&> \mathcal{M}(Ay, y, y, kt)) \text{ a contradiction.}
\end{aligned}$$

Thus $Ay = y$, hence $Ay = Sy = y$. Similarly, we prove that $By = y$. For if $By \neq y$. Then $\mathcal{M}(By, y, y, t) < 1$, how ever

$$\begin{aligned}
\mathcal{M}(y, By, By, t) &= \mathcal{M}(Ay, By, By, t)\\
&\geq \phi(\mathcal{M}(Sy, Ty, Ty, kt), \mathcal{M}(Ay, Sy, Sy, kt), \mathcal{M}(By, Ty, Ty, kt), \mathcal{M}(Sy, By, By, kt)\\
&\quad\ \vee\ \mathcal{M}(Ty, Ay, Ty, kt) \vee\ \mathcal{M}(Ty, Ay, Ty, kt))\\
&\geq \phi(\mathcal{M}(y, By, By, kt), \mathcal{M}(y, By, By, kt), \mathcal{M}(y, By, By, kt), \mathcal{M}(y, By, By, By, kt))\\
&> \mathcal{M}(y, By, By, kt) \text{ a contradiction.}
\end{aligned}$$

Therefore, $Ay = By = Sy = Ty = y$, ie., y is a common fixed point of A, B, S and T.

Uniqueness Let w be another common fixed point of A, B, S and T.

That is $w = Bw = Aw = Sw = Tw$. If $\mathcal{M}(x, y, z, t) < 1$, then

$$\begin{aligned}
\mathcal{M}(y, w, w, t) &= \mathcal{M}(Ay, Bw, Bw, t)\\
&\geq \phi(\mathcal{M}(Sy, Tw, Tw, kt), \mathcal{M}(Ay, Sy, Sy, kt), \mathcal{M}(Bw, Tw, Tw, kt), \mathcal{M}(Sy, Bw, Bw, kt)\\
&\quad\ \vee\ \mathcal{M}(Tw, Ay, Tw, kt) \vee\ \mathcal{M}(Tw, Ay, Tw, kt))\\
&= \phi(\mathcal{M}(y, w, w, kt), 1, 1, \mathcal{M}(y, w, w, kt) \vee\ \mathcal{M}(w, y, w, kt) \vee\ \mathcal{M}(w, y, w, kt))\\
&\geq \phi(\mathcal{M}(y, w, w, kt), \mathcal{M}(y, w, w, kt), \mathcal{M}(y, w, w, kt), \mathcal{M}(y, w, w, kt))\\
&> \mathcal{M}(y, w, w, kt) \text{ a contradiction.}
\end{aligned}$$

Therefore y is the unique common fixed point of self-maps A, B, S and T. □

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