International Journal of Mathematics And its Applications
Volume 3, Issue 1 (2015), 83–88.
ISSN: 2347-1557

Cone Metric Spaces and Fixed Point Theorems of Contractive Mapping for c-Distance

A.K.Dubey†,[1], Rohit Verma‡ and Ravi Prakash Dubey*

† *Department of Mathematics, Bhilai Institute of Technology, Bhilai House, Durg, Chhattisgarh, India.*

‡ *Department of Mathematics, Shri Shankracharya Institute of Engineering and Technology, Chhattisgarh, India.*

* *Department of Mathematics, Dr. C.V.Raman University, Bilaspur, Chhattisgarh, India.*

Abstract : In this paper, we prove some fixed point theorems for different type of contractive conditions under c-distance in cone metric spaces. A new concept of the c-distance in cone metric space has been introduced recently in 2011. Our results generalize and extend some well-known results in the literature.

Keywords : Complete cone metric space, c-distance, fixed point.

AMS Subject Classification: 54H25,47H10.

1 Introduction and Preliminaries

The first important result on fixed points for contractive type mapping was the Banach's contraction principle by Banach in 1922. There are many generalizations of Banach's contraction mapping principle in the literature. In 2007, Huang and Zhang [6] generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for different type of contractive conditions. They introduced cone metric space without c-distance. In 2011, Cho et al.[14] introduced a new concept of the c-distance in cone metric spaces and proved some fixed point theorems in ordered cone metric spaces. This is more general than the classical Banach contraction mapping principle.

The aim of this paper is to extend and generalize the results of Fadail et al. [18] and proved some fixed point theorems on c-distance in cone metric space, under the continuity condition for maps. Before presenting our results, we recall some notations, definitions and examples needed in our subsequent discussions.

Definition 1.1. *Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that*

(1) P is nonempty set closed and $P \neq \{\theta\}$,

[1]Corresponding author E-Mail: anilkumardby70@gmail.com (A.K.Dubey)

(2) If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$,

(3) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in int\, P$, where $int\, P$ denotes the interior of P. A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \Rightarrow \| x \| \leqslant K \| y \| \tag{1.1}$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P.

Definition 1.2 ([6])**.** *Let X be a non-empty set and E be a real Banach space with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:*

(1) $\theta \prec d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, y) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.3 ([6])**.** *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X, and $x \in X$.*

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer $\mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > \mathbb{N}$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer $\mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 1.4 ([5])**.**

(1) If E is a real Banach space with a cone P and $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.

(2) If $c \in int\, P, \theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer $\mathbb{N}$ such that $a_n \ll c$ for all $n \geq \mathbb{N}$. Next, we give the definition of c-distance on a cone metric space (X, d) which is a generalization of w-distance of Kada et al.[10] with some properties.

Definition 1.5 ([14])**.** *Let (X, d) is a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:*

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,

(q2) $q(x, y) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 1.6 ([14]). *Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by $d(x,y) = |x-y|$ for all $x,y \in X$, then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y$ for all $x,y \in X$. Then q is a c-distance on X.*

Lemma 1.7 ([14]). *Let (X,d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X and $x,y,z \in X$. Suppose that u_n is a sequences in P converging to θ. Then the following hold:*

(1) If $q(x_n,y) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $y = z$,

(2) If $q(x_n,y_n) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $\{y_n\}$ converges to z,

(3) If $q(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X,

(4) If $q(y,x_n) \preceq u_n$ then $\{x_n\}$ is a Cauchy sequence in X,

Remark 1.8 ([14]).

(1) $q(x,y) = q(y,x)$ does not necessarily for all $x,y \in X$.

(2) $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

2 Main Result

In this section, we prove some fixed point theorems under c-distance in cone metric space over non normal cone with nonempty interior.

Theorem 2.1. *Let (X,d) be a complete cone metric space and q is a c-distance on X . Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition:*

$$q(fx,fy) \preceq a_1 q(x,y) + a_2 q(x,fx) + a_3 q(y,fy) + a_4[q(fx,y) + q(fy,x)] \tag{2.1}$$

for all $x,y \in X$, where a_1,a_2,a_3,a_4 are nonnegative real numbers such that $a_1 + a_2 + a_3 + 2a_4 < 1$. Then f has a fixed point $x^\star \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$converges to the fixed point. If $\vartheta = f\vartheta$ then $q(\vartheta,\vartheta) = \theta$. The fixed point is unique.

Proof. Choose $x_o \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \ldots, x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$q(x_n,x_{n+1}) = q(fx_{n-1},fx_n) \tag{2.2}$$

$$\begin{aligned}&\preceq a_1 q(x_{n-1},x_n) + a_2 q(x_{n-1,},fx_{n-1}) + a_3 q(x_n,fx_n) + a_4[q(fx_{n-1},x_n) + q(fx_n,x_{n-1})]\\ &= a_1 q(x_{n-1},x_n) + a_2 q(x_{n-1,},x_n) + a_3 q(x_n,x_{n+1}) + a_4[q(x_n,x_n) + q(x_{n+1},x_{n-1})].\end{aligned}$$

So

$$\begin{aligned} q(x_{n,},x_{n+1}) &\preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_{n-1},x_n)\\ &= hq(x_{n-1},x_n), \text{where } h = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} < 1.\end{aligned} \tag{2.3}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_{n,}, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq (h^n + h^{n+1} + \cdots + h^{m-1}) q(x_0, x_1) \\ &\preceq \frac{h^n}{1-h} q(x_0, x_1) \end{aligned} \tag{2.4}$$

Thus, Lemma 1.7 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$. Since f is continuous, then $x^\star = lim\, x_{n+1} = lim\, f(x_n) = f(lim\, x_n) = f(x^\star)$. Therefore $x^\star$ is a fixed point of f. Suppose that $\vartheta = f\vartheta$, then we have

$$\begin{aligned} q(\vartheta, \vartheta) &= q(f\vartheta, f\vartheta) \\ &\preceq a_1 q(\vartheta, \vartheta) + a_2 q(\vartheta, f\vartheta) + a_3 q(\vartheta, f\vartheta) + a_4[q(f\vartheta, \vartheta) + q(f\vartheta, \vartheta)] \\ &= (a_1 + a_2 + a_3 + 2a_4) q(\vartheta, \vartheta) \end{aligned} \tag{2.5}$$

Since $a_1 + a_2 + a_3 + 2a_4 < 1$, Lemma 1.4 shows that $q(\vartheta, \vartheta) = \theta$. Finally, suppose that, there is another fixed point of $y^\star$ of f, then we have

$$\begin{aligned} q(x^\star, y^\star) &= q(fx^\star, fy^\star) \\ &\preceq a_1 q(x^\star, y^\star) + a_2 q(x^\star, fx^\star) + a_3 q(y^\star, fy^\star) + a_4[q(fx^\star, y^\star) + q(fy^\star, x^\star)] \\ &= (a_1 + 2a_4) q(x^\star, y^\star) \\ &\preceq (a_1 + a_2 + a_3 + 2a_4) q(x^\star, y^\star). \end{aligned}$$

Since $a_1 + a_2 + a_3 + 2a_4 < 1$, then by Lemma 1.4 we have $q(x^\star, y^\star) = \theta$ and also we have $q(x^\star, x^\star) = \theta$. Hence by Lemma 1.7 (1), $x^\star = y^\star$. Therefore the fixed point is unique. ☐

Remark 2.2.

(1) *Put $a_4 = 0$ in Theorem 2.1, we get the result of Theorem 3.3 of Fadail et al. [18].*

(2) *If we put $a_1 = a_4 = 0$ and $a_2 = a_3$ in Theorem 2.1, we get the result of Corollary 3.4 of Fadail et al. [18].*

Theorem 2.3. *Let (X, d) be a complete cone metric space and q is a c-distance on X . Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition:*

$$q(fx, fy) \preceq a_1 q(x, y) + a_2[q(x, fy) + q(y, fx)] + a_3[q(x, fx) + q(y, fy)] \tag{2.6}$$

for all $x, y \in X$, where a_1, a_2, a_3 are nonnegative real numbers such that $a_1 + 2a_2 + 2a_3 < 1$. Then f has a fixed point $x^\star \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $\vartheta = f\vartheta$ then $q(\vartheta, \vartheta) = \theta$. The fixed point is unique.

Proof. Choose $x_o \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2 x_0, \ldots x_{n+1} = fx_n = f^{n+1} x_0$. We have the following:

$$q(x_n, x_{n+1}) = q(fx_{n-1}, fx_n) \tag{2.7}$$

$$\begin{aligned} &\preceq a_1 q(x_{n-1}, x_n) + a_2[q(x_{n-1,}, fx_n) + q(x_n, fx_{n-1})] + a_3[q(x_{n-1}, fx_{n-1}) + q(x_n, fx_n)] \\ &= a_1 q(x_{n-1}, x_n) + a_2[q(x_{n-1,}, x_{n+1}) + q(x_n, x_n)] + a_3[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ &\preceq (a_1 + a_2 + a_3) q(x_{n-1}, x_n) + (a_2 + a_3) q(x_n, x_{n+1}). \end{aligned}$$

So

$$
\begin{aligned}
q(x_{n,}, x_{n+1}) &\preceq \frac{a_1+a_2+a_3}{1-a_2-a_3} q(x_{n-1}, x_n) \\
&= hq(x_{n-1}, x_n), \text{ where } h = \frac{a_1+a_2+a_3}{1-a_2-a_3} < 1.
\end{aligned} \tag{2.8}
$$

Let $m > n \geq 1$. Then it follows that

$$
\begin{aligned}
q(x_{n,}, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\
&\preceq (h^n + h^{n-1} + \cdots + h^{m-1}) q(x_0, x_1) \\
&\preceq \frac{h^n}{1-h} q(x_0, x_1)
\end{aligned} \tag{2.9}
$$

Thus, Lemma 1.7 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$. Since f is continuous, then $x^\star = lim\, x_{n+1} = lim\, f(x_n) = f(lim\, x_n) = f(x^\star)$. Therefore $x^\star$ is a fixed point of f. Suppose that $\vartheta = f\vartheta$, then we have

$$
\begin{aligned}
q(\vartheta, \vartheta) &= q(f\vartheta, f\vartheta) \\
&\preceq a_1 q(\vartheta, \vartheta) + a_2[q(\vartheta, f\vartheta) + q(\vartheta, f\vartheta)] + a_3[q(\vartheta, f\vartheta) + q(\vartheta, f\vartheta)] \\
&= (a_1 + 2a_2 + 2a_3) q(\vartheta, \vartheta).
\end{aligned} \tag{2.10}
$$

Since $a_1 + 2a_2 + 2a_3 < 1$, Lemma 1.4 shows that $q(\vartheta, \vartheta) = 0$. Next we prove that the uniqueness of the fixed point. Suppose that, there is another fixed point of $y^\star$ of f, then we have the following:

$$
\begin{aligned}
q(x^\star, y^\star) &= q(fx^\star, fy^\star) \\
&\preceq a_1 q(x^\star, y^\star) + a_2[q(x^\star, fy^\star) + q(y^\star, fx^\star)] + a_3[q(x^\star, fx^\star) + q(y^\star, fy^\star) \\
&= (a_1 + 2a_2) q(x^\star, y^\star) \\
&\preceq (a_1 + 2a_2 + 2a_3) q(x^\star, y^\star).
\end{aligned} \tag{2.11}
$$

Since $a_1 + 2a_2 + 2a_3 < 1$, then by Lemma 1.4 we have $q(x^\star, y^\star) = \theta$ and also we have $q(x^\star, x^\star) = \theta$. Hence by Lemma 1.7 (1), $x^\star = y^\star$. Therefore the fixed point is unique. □

References

[1] A. Azam and M. Arshad, *Common fixed points of generalized contractive maps in cone metric spaces*, Bulletin Iranian Mathematical Society, 35(2)(2009), 255-264.

[2] D.Ilic and V.Rakocevic, *Common fixed points for maps on cone metric space*, Journal of Mathematical Analysis and Applications, 341(2)(2008), 876-882.

[3] D.Ilic and V.Rakocevic, *Quasi-contraction on a cone metric space*, Applied Mathematics Letters, 22(5)(2009), 728-731.

[4] D. Wardowski, End points and fixed points of set-valued contractions in cone metric spaces, Non linear Analysis Vol.71, no.1-2, PP.512-516, 2009.

[5] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory and Applications, 2009(2009), Article ID 643840.

[6] L.G.Huang and X.Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, 332(2)(2007), 1468-1476.

[7] M. Abbas ans G. Jungck, *Common fixed point results for non commuting mappings without continuity in cone metric spaces*, Journal Mathematical Analysis and Applications, 341(1)(2008), 416-420.

[8] M.Abbas and B.E.Rhoades, *Fixed and Periodic point results in cone metric spaces*, Applied Mathematics Letters, 22(4)(2009), 511-515.

[9] M. Dordevic, D. Doric, Z. Kadelburg, S. Radenovic and D. Spasic, *Fixed point results under c-distance in tvs-cone metric spaces*, Fixed Point Theory and Applications, 2011(2011), Article 29.

[10] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Mathematica Japonica, 44(2)(1996), 381-391.

[11] Sh.Rezapour and R.Hamlbarani, *Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, 345(2)(2008), 719-724.

[12] S. Jankovic, Z. Kadelburg and S. Radenovic, *On cone metric space: a survey*, Nonlinear Analysis, 74(7)(2011), 2591-2601.

[13] W. Sintunavarat, Y.J. Cho, and P. Kumam, *Common fixed point theorems for c-distance in ordered cone metric spaces*, Computers and Mathematics with Applications, 62(4)(2011), 1969-1978.

[14] Y.J. Cho, R. Saadati and S.H. Wang, *Common fixed point theorems on generalized distance in ordered cone metric spaces*, Computers & Mathematics with Applications, 61(4)(2011), 1254-1260.

[15] Z. Kadelburg, S. Radenovic and V. Rakocevic, *Topological vector space-valued cone metric spaces and fixed point theorems*, Fixed Point Theory and Applications, 2010(2010), Article ID 170253.

[16] Z. Kadelburg and S. Radenovic, *Coupled fixed point results under tvs-cone metric spaces and w-cone-distance*, Advances Fixed Point Theory, 2(1)(2012).

[17] Z. Kadelburg and S. Radenovic, *Some common fixed point results in non-normal cone metric spaces*, Journal of Nonlinear Science and its Applications, 3(3)(2010), 193-202.

[18] Z.M Fadail, A.G.B. Ahmad and Z. Golubovic, *Fixed Point Theorems of Single-valued Mapping for c-Distance in Cone Metric Spaces*, Abstract and Applied Analysis, 2012(2012), Article ID 826815.