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Oscillation of Second Order Quasi-linear Neutral Delay Differential Equations

Research Article

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Abstract: In this paper, by employing generalized Riccati transformation and some specific analytical skills, we will establish some new oscillation criteria for the second order quasi-linear neutral delay equation of the form

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0,$$

where $z(t) = x(t) + p(t)x(\tau(t)), -1 < p_1 \leq p(t) \leq 0,\ q(t) > 0, \alpha > 0$. The results obtained essentially improve and extent the results in the cited literature.

MSC: 34K11.

Keywords: Oscillation, second order, neutral delay differential equation, generalized Riccati transformation.

1. Introduction

Neutral delay equations appear in modelling of networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar, see the Euler equation in some variational problems, in the theory of automatic control and in neuromechanical systems in which inertia plays an important role. In the last few decades, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different classes of second order neutral delay differential equations [1-23]. Erbe et al.[3], Grammatikopoulos et al.[5], Györi et al.[6], Jiang et al.[8], Ladas et al.[10, 11], Li et al.[12], Sahiner[17], Yan[22] have studied the oscillation of the second order neutral differential equation

$$[x(t) - p(t)x(t-\tau)]'' + q(t)f(x(t-\sigma)) = 0, t \geq 0, \tag{1}$$

where $\tau, \sigma > 0, p, q \in C([0,\infty), (-\infty,+\infty)), q(t) \geq 0$, and $f \in C((-\infty,+\infty),(-\infty,+\infty)), xf(x) > 0, x \neq 0$. They all considered the case that $p(t) \leq 0$. The paper[5] studied the oscillation of second-order neutral delay differential equation

$$[x(t) + p(t)x(t-\tau)]'' + q(t)x(t-\sigma) = 0, t \geq 0, \tag{2}$$

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and obtained that if $\int^{\infty} q(s)(1-p(s-\tau))ds = \infty$, then the solutions of Equation (2) is oscillatory. Wong[20] studied the oscillation and nonoscillation for Equation (1) when $p(t) = p, 0 \leq p < 1$. Lin[14] investigated the oscillation and nonoscillation of Equation (1) when $0 \leq p(t) \leq p < 1$.

Recently, Xu and Meng[21], Ye and Xu[23] considered the oscillation of the equations

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + q(t)f(x(\sigma(t))) = 0, t \geq t_0, \tag{3}$$

where $y(t) = x(t) + p(t)x(\tau(t)), 0 \leq p(t) < 1$, $q(t) \geq 0, \alpha > 0$. After that, in 2012, Han et al.[7] considered Equation (eqn13) in the case $-1 \leq p(t) \leq p < 0$ and $f \in C(\mathbb{R}, \mathbb{R}), xf(x) \neq 0, x \neq 0$, and there exists a constant $L > 0$ such that $\frac{f(x)}{|x|^{\alpha-1}x} \geq L$, for $x \geq 0$. The authors obtained some conditions which guarantee that every solution x of Equation (3) oscillates. Grace and Lalli[4] studied the equation

$$(r(t)[x(t) + p(t)x(t-\tau)]^{'})^{'} + q(t)x(t-\sigma) = 0, t \geq 0, \tag{4}$$

and subject to

$$\frac{f(x)}{x} \geq k > 0, \int^{\infty} \frac{dt}{r(t)} = \infty,$$

and they gave an sufficient condition for the oscillation of Equation (4), that is, if there exists a function $\rho \in C^1([t_0, \infty], \mathbb{R})$ such that

$$\int^{\infty} [\rho(s)q(s)(1-p(s-\sigma)) - \frac{\rho^{'2}(s)r(s-\sigma)}{4k\rho(s)}]ds = \infty,$$

then Equation (4) is oscillatory. Paper[13] considered the oscillation of second-order Emden-Fowler neutral differential equation

$$[r(t)z'(t)]' + q(t)|x(\sigma(t))|^{\gamma-1}x(\sigma(t)) = 0, t \geq t_0, \tag{5}$$

where $z(t) = x(t) + p(t)x(\tau(t)), 0 \leq p(t) < 1$, $q(t) \geq 0, \gamma > 0$. The authors established some new oscillation results. Agarwal et al.[1], Chern et al.[2], Kusano et al.[9], Mirzov[16] and Sun and Meng[18] studied the oscillation of second order nonlinear delay differential equation

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, t \geq t_0. \tag{6}$$

Especially, under the case $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt < \infty$, Sun and Meng[18] obtained some results which guarantee that every solution x of Equation (6) oscillates or tends to zero. Tiryaki[19] gave some oscillation criteria for the following equation

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + q(t)|x(\tau(t))|^{\beta-1}x(\tau(t)) = 0, t \geq t_0.$$

Liu et al.[15] considered generalized Emden-Fowler equation with neutral type delays:

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, t \geq t_0,$$

where $z(t) = x(t) + p(t)x(\tau(t)), \alpha \geq \beta > 0.$, In this paper, we use the generalized Riccati transformation and some specific analytical skills to establish some new suffecient conditions for the oscillation of second order quasi-linear neutral delay differential equation of the form

$$(r(t)|z^{'}(t)|^{\alpha-1}z^{'}(t))^{'} + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{7}$$

where $z(t)=x(t)+p(t)x(\tau(t)), -1<-p_1\leq p(t)\leq 0, q(t)>0, \alpha>0$.

Throughout this paper, we assume that

$(c_1) r,p,q\in C([0,\infty),[0,\infty)), r(t)>0, q(t)>0, for\ all\ t\in[0,\infty)$,

$(c_2) \sigma\in C([t_0,\infty),\mathbb{R}), \sigma(t)\leq t, \lim_{t\to\infty}\sigma(t)=\infty, \tau\in C([t_0,\infty),\mathbb{R}), \tau(t)\leq t, \lim_{t\to\infty}\tau(t)=\infty$,

$(c_3) f\in C(\mathbb{R},\mathbb{R}), xf(x)\neq 0, x\neq 0$, and there exists a constant $L>0$ such that $\frac{f(x)}{|x|^{\beta-1}x}\geq L$, for $x\neq 0$, where β is a positive constant.

2. Main Results

In this section, we will give some new oscillation criteria for Equation (7) which extend and improve some known results. First, in order to get our main results, we need to prove the following results:

Lemma 2.1. *Suppose that* $(c_1)-(c_3)$ *hold, and*

$$\int_{t_0}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(t)}dt=\infty. \tag{8}$$

If $x(t)$ *is an eventually positive solution of Equation (7), then there exists a* $t_*\geq t_0$ *such that*

$$z(t)>0, z'(t)>0, (r(t)(z'(t))^{\alpha})'<0,\ for\ t\geq t_* \tag{9}$$

or $lim_{t\to\infty}x(t)=0$.

Proof. Let $x(t)$ be an eventually positive solution of Equation (7). It follows from (c_2) that there exists $t_1\geq t_0$ such that $x(t)>0, x(\sigma(t))>0, x(\tau(t))>0$ for all $t\geq t_1$. In virtue of Equation (7) and (c_3), one can get that

$$(r(t)|z^{'}(t)|^{\alpha-1}z^{'}(t))^{'}=-q(t)f(x(\sigma(t)))\leq -Lq(t)(x(\sigma(t)))^{\beta}<0, t\geq t_1. \tag{10}$$

Therefore $r(t)|z^{'}(t)|^{\alpha-1}z^{'}(t)$ is a nonincreasing function and $z^{'}(t)$ is eventually of one sign. It follows that there exists $t_2\geq t_1$ such that $z(t)>0$ or $z(t)<0$ for $t\geq t_2$. If $z(t)>0$ for $t\geq t_2$, we claim that there exists a $t_*\geq t_2$ such that $z'(t)>0$ for $t\geq t_*$. Otherwise, there exists a $t_3\geq t_2$ such that $z'(t)<0$ for $t\geq t_3$. Since $r(t)|z^{'}(t)|^{\alpha-1}z^{'}(t)$ is a nonincreasing function, we have

$$-r(t)(-z'(t))^{\alpha}\leq -r(t_3)(-z'(t_3))^{\alpha}\doteq -K, \text{for } t\geq t_3,$$

which implies that

$$z'(t)\leq -K^{\frac{1}{\alpha}}(\frac{1}{r(t)})^{\frac{1}{\alpha}}, \text{for } t\geq t_3.$$

Integrating the above inequality from t_3 to t leads to

$$z(t)\leq z(t_3)-K^{\frac{1}{\alpha}}\int_{t_3}^{t}(\frac{1}{r(s)})^{\frac{1}{\alpha}}ds\to -\infty,\ t\to\infty,$$

a contradiction. Hence, there exists a $t_*\geq t_0$ such that (9) holds.

Now suppose that $z(t) < 0$. By equation $z(t) = x(t) + p(t)x(\tau(t))$, we assert $x(t)$ is bounded. If it is not true, there exists $\{k_n\}$ with $k_n \to \infty$ as $n \to \infty$ such that

$$x(k_n) = \max_{s\in[t_2,k_n]}\{x(s)\}, \quad \lim_{n\to\infty} x(k_n) = \infty.$$

In light of $\tau(t) \le t$ and $\tau(t) \to \infty$ as $t \to \infty$, there exists enough large n such that $k_n \ge \tau(k_n) > t_2$. It follows that

$$x(\tau(k_n)) \le \max_{s\in[t_2,k_n]}\{x(s)\} = x(k_n) = z(k_n) - p(k_n)x(\tau(k_n)) < -p(k_n)x(\tau(k_n)) < x(\tau(k_n)),$$

which is a contradiction. This implies that $x(t)$ is bounded. It follows that

$$\begin{aligned} 0 \ge \limsup_{t\to\infty} z(t) &\ge \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} p(t)x(\tau(t)) \\ &\ge \limsup_{t\to\infty} x(t) - p_1 \limsup_{t\to\infty} x(\tau(t)) \\ &= (1-p_1)\limsup_{t\to\infty} x(t) \\ &\ge (1-p_1)\liminf_{t\to\infty} x(t) \ge 0, \end{aligned}$$

which implies that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Lemma 2.2. *Suppose that $(c_1)-(c_3)$ and (8) hold. If $r'(t) \ge 0$, and*

$$\int_{t_0}^{\infty} q(t)(\sigma(t))^{\beta} dt = \infty, \tag{11}$$

further $x(t)$ is an eventually positive solution of Equation (7) such that (9) holds, then there exists a $T \ge t_0$ such that

$$z(t) > tz'(t), \ \text{for} \ \ t \ge T,$$

and $\frac{z(t)}{t}$ is strictly decreasing eventually.

The proof of Lemma 2.2 is similar to the proof of Lemma 2.2 in [7], we omit it.

Theorem 2.3. *Assume that $(c_1)-(c_3)$, (8) and (11) hold. Furthermore, $r'(t) \ge 0$. If there exists a positive function $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that for any $m \in (0,1]$,*

$$\int_{t_0}^{\infty} [L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta} - \frac{r(t)(\rho'_+(t))^{\lambda+1}}{m^{\lambda}(\lambda+1)^{\lambda+1}\rho^{\lambda}(t)}]dt = \infty,$$

where $\rho'_+(t) = \max\{0,\rho'(t)\}$, $\lambda = \min\{\alpha,\beta\}$, then every solution $x(t)$ of Equation (7) oscillates or $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of Equation (7). Without loss of generality, we assume that $x(t)$ is eventually positive. It follows that there exists a $t_1 \ge t_0$ such that $x(t) > 0, x(\sigma(t)) > 0, x(\tau(t)) > 0$ for all $t \ge t_1$. By Lemma 2.1, there exists a $t_2 \ge t_1$, such that (9) holds or $\lim_{t\to\infty} x(t) = 0$. Now suppose (9) holds. Define Riccati function

$$w(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(t)}, t \ge t_2. \tag{12}$$

Obviously, $w(t)>0$. Differentiating (12), one can get that

$$w'(t)=\frac{\rho'(t)}{\rho(t)}w(t)+\rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\beta}(t)}-\rho(t)\frac{\beta r(t)(z'(t))^{\alpha+1}}{z^{\beta+1}(t)}. \tag{13}$$

It follows that there exists $t^*\geq t_2$ such that

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\lambda c_{\lambda}}{(\rho(t)r(t))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(t),t\geq t^*, \tag{14}$$

where

$$\lambda=\min\{\alpha,\beta\},\quad c_{\lambda}=\begin{cases}1,\alpha=\beta\\ \theta\in(0,1),\alpha\neq\beta.\end{cases}$$

In fact, if $\beta\geq\alpha$, by (10), we get that

$$(r(t)(z^{'}(t))^{\alpha})^{'}\leq -Lq(t)(x(\sigma(t)))^{\beta}\leq -Lq(t)(z(\sigma(t)))^{\beta}<0,t\geq t_1. \tag{15}$$

It follows from (11), (13), (15) and Lemma 2.2 that

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\alpha z^{\frac{\beta-\alpha}{\alpha}}(t)}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t),t\geq t_2.$$

When $\alpha=\beta$, it is obvious that

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t),t\geq t_2. \tag{16}$$

Now we consider $\beta>\alpha$. Since $z(t)>0,z'(t)>0$ for $t\geq t_2$, we obtain that $\lim_{t\to\infty}z(t)\doteq M\leq\infty$. If $M>1$, then there exists $t_3\geq t_2$ such that $z(t)>1$, for $t\geq t_3$. It follows that

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t),t\geq t_3.$$

If $M\leq 1$, then there exist $t_4\geq t_2$ and $\theta_1\in(0,M)$ such that $z(t)\geq\theta_1$ for $t\geq t_4$. Hence,

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\alpha\theta_1}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t),t\geq t_4.$$

It follows from the above inequalities that there exist $t_4^*\geq t_2$ and $\theta_1^*\in(0,1)$ such that

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\alpha\theta_1^*}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t),t\geq t_4^*. \tag{17}$$

If $\alpha>\beta$, by (11), (13), (15) and Lemma 2.2, we have

$$w'(t)\leq\frac{\rho'_+(t)}{\rho(t)}w(t)-L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta}-\frac{\beta}{(\rho(t)r(t))^{\frac{1}{\beta}}}(\frac{1}{z'(t)})^{\frac{\alpha-\beta}{\beta}}w^{\frac{\beta+1}{\beta}}(t),t\geq t_2.$$

In virtue of (9) and $r'(t) \geq 0$, we get that

$$(r(t)(z'(t))^{\alpha})' = r'(t)(z'(t))^{\alpha} + \alpha r(t)(z'(t))^{\alpha-1}z''(t) < 0, \ t \geq t_2,$$

which implies that $z''(t) < 0$, for $t \geq t_2$. It follows that $\frac{1}{z'(t)}$ is increasing function and $\lim_{t\to\infty}\frac{1}{z'(t)}$ exists or $\lim_{t\to\infty}\frac{1}{z'(t)} = \infty$. Using the similar technique in proof of the case $\beta \geq \alpha$, we obtain that there exist $\theta_2 \in (0,1)$ and $t_5 \geq t_2$ such that

$$w'(t) \leq \frac{\rho'_+(t)}{\rho(t)}w(t) - L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta} - \frac{\beta\theta_2}{(\rho(t)r(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t), t \geq t_5. \tag{18}$$

Combining (16)–(18), we obtain that (14) holds. Let $A(x) = \frac{\rho'_+(t)}{\rho(t)}x - \frac{\lambda c_{\lambda}}{(\rho(t)r(t))^{\frac{1}{\lambda}}}x^{\frac{\lambda+1}{\lambda}}, x \geq 0$. By calculating, one can get that

$$A(x) \leq \frac{r(t)(\rho'_+(t))^{\lambda+1}}{(c_{\lambda})^{\lambda}(\lambda+1)^{\lambda+1}\rho^{\lambda}(t)}.$$

It follows that

$$w'(t) \leq -L\rho(t)q(t)(\frac{\sigma(t)}{t})^{\beta} + \frac{r(t)(\rho'_+(t))^{\lambda+1}}{(c_{\lambda})^{\lambda}(\lambda+1)^{\lambda+1}\rho^{\lambda}(t)}, t \geq t^*.$$

Integrating the above inequality from t^* to t leads to

$$w(t) \leq w(t^*) - \int_{t^*}^{\infty}[L\rho(t)q(s)(\frac{\sigma(s)}{s})^{\beta} - \frac{r(s)(\rho'_+(s))^{\lambda+1}}{(c_{\lambda})^{\lambda}(\lambda+1)^{\lambda+1}\rho^{\lambda}(s)}]ds \to -\infty \text{ as } t \to \infty,$$

which contradicts $w(t) > 0$. This completes the proof. □

Theorem 2.4. *Assume that* $(c_1) - (c_3)$, *(8) and (11) hold. Furthermore,* $r'(t) \geq 0$. *If there exists a positive function* $u \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that for any* $m \in (0,1]$,

$$\int_{t_0}^{\infty}[q(t)(\frac{\sigma(t)}{t})^{\beta} - \frac{1}{L}\frac{(u(t))^{\lambda+1}}{r^{\frac{1}{\lambda}}(t)}]\exp[(\lambda+1)m^{\lambda}\int_{t_0}^{t}\frac{u(s)}{r^{\frac{1}{\lambda}}(s)}ds]dt = \infty, \tag{19}$$

where $\lambda = \min\{\alpha,\beta\}$, *then every solution* $x(t)$ *of Equation (7) oscillates or* $\lim_{t\to\infty}x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of Equation (7). Without loss of generality, we assume that $x(t)$ is eventually positive. Then there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\sigma(t)) > 0, x(\tau(t)) > 0$ for all $t \geq t_1$. Lemma 2.1 ensures that there exists a $t_2 \geq t_1$, such that (9) holds or $\lim_{t\to\infty}x(t) = 0$. Suppose (9) holds. Define Riccati function

$$w(t) = \frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(t)}, t \geq t_2.$$

Then $w(t) > 0$, for $t \geq t_2$. Using the similar technique to prove inequality (14), we get that there exists a $t^{**} \geq t_2$ such that

$$\begin{aligned} w'(t) &\leq -Lq(t)(\frac{\sigma(t)}{t})^{\beta} - \frac{\lambda c_{\lambda}}{r^{\frac{1}{\lambda}}(t)}w^{\frac{\lambda+1}{\lambda}}(t) \\ &= [-Lq(t)(\frac{\sigma(t)}{t})^{\beta} + \frac{u^{\lambda+1}(t)}{r^{\frac{1}{\lambda}}(t)}] - \frac{1}{r^{\frac{1}{\lambda}}(t)}[u^{\lambda+1}(t) + \lambda c_{\lambda}w^{\frac{\lambda+1}{\lambda}}(t)], t \geq t^{**}. \end{aligned} \tag{20}$$

From Young's inequality, we have

$$u^{\lambda+1}(t) + \lambda c_{\lambda}w^{\frac{\lambda+1}{\lambda}}(t) = u^{\lambda+1}(t) + \lambda(c_{\lambda}^{\frac{\lambda}{\lambda+1}}w(t))^{\frac{\lambda+1}{\lambda}} \geq (\lambda+1)c_{\lambda}^{\frac{\lambda}{\lambda+1}}u(t)w(t), \ t \geq t^{**}.$$

It follows from the above inequality and (20) that

$$w'(t)+\frac{1}{r^{\frac{1}{\lambda}}(t)}[(\lambda+1)c_\lambda^{\frac{\lambda}{\lambda+1}}u(t)w(t)]\le -Lq(t)(\frac{\sigma(t)}{t})^\beta+\frac{u^{\lambda+1}(t)}{r^{\frac{1}{\lambda}}(t)},$$

that is,

$$(\exp[(\lambda+1)c_\lambda^{\frac{\lambda}{\lambda+1}}\int_{t_2}^t\frac{u(s)ds}{r^{\frac{1}{\lambda}}(s)}]w(t))'\le -[Lq(t)(\frac{\sigma(t)}{t})^\beta-\frac{u^{\lambda+1}(t)}{r^{\frac{1}{\lambda}}(t)}]\exp[(\lambda+1)c_\lambda^{\frac{\lambda}{\lambda+1}}\int_{t_2}^t\frac{u(s)ds}{r^{\frac{1}{\lambda}}(s)}],$$

for all $t\ge t^{**}$. Integrating the above inequality from t^{**} to t, one can obtain by (19) that

$$\begin{aligned}\exp[(\lambda+1)c_\lambda^{\frac{\lambda}{\lambda+1}}&\int_{t_2}^t\frac{u(s)ds}{r^{\frac{1}{\lambda}}(s)}]w(t)\\ &\le \exp[(\lambda+1)c_\lambda^{\frac{\lambda}{\lambda+1}}\int_{t_2}^{t^{**}}\frac{u(s)ds}{r^{\frac{1}{\lambda}}(s)}]w(t^{**})\\ &\quad -L\int_{t^{**}}^t[q(v)(\frac{\sigma(v)}{v})^\beta-\frac{1}{L}\frac{u^{\lambda+1}(v)}{r^{\frac{1}{\lambda}}(v)}]\exp[(\lambda+1)c_\lambda^{\frac{\lambda}{\lambda+1}}\int_{t_2}^v\frac{u(s)ds}{r^{\frac{1}{\lambda}}(s)}]dv\\ &\to -\infty \text{ as } t\to\infty.\end{aligned}$$

It is a contradiction. This completes the proof. □

The following examples illustrate our main results.

Example 2.5. *Consider the neutral delay differential equation*

$$(|z'(t)|^{\alpha-1}z'(t))'+\frac{1}{t^{1+\frac{\beta}{2}}}|x(\sigma(t))|^{\beta-1}x(\sigma(t))=0, t\ge t_0, \tag{21}$$

where $z(t)=x(t)-\frac{1}{2}x(t-1)$, $\sigma(t)=\sqrt{t}\ln t$, $\alpha>0,\beta>0$.

Let $r(t)=1$, $p(t)=-\frac{1}{2}$, $q(t)=\frac{1}{t^{1+\frac{\beta}{2}}}$, $\sigma(t)=\sqrt{t}\ln t$, $\tau(t)=t-1$, $\rho(t)=t^\beta$. Theorem 2.3 ensures that every solution $x(t)$ of Equation (21) oscillates or $\lim_{t\to\infty}x(t)=0$.

Example 2.6. *Consider the second order neutral delay differential equation*

$$(x(t)-\frac{1}{3}x(\frac{t}{3}))''+\frac{1}{27}(e^{\frac{t}{6}}-e^{\frac{-t}{2}})x^3(\frac{t}{6})=0,\ t\ge t_0. \tag{22}$$

Let $\alpha=1$, $\beta=3$, $r(t)=1$, $p(t)=-\frac{1}{3}$, $q(t)=e^{\frac{t}{6}}-e^{\frac{-t}{2}}$, $\sigma(t)=\frac{t}{6}$, $\tau(t)=\frac{t}{3}$, $u(t)=\frac{1}{t}$. Theorem 2.4 ensures that every solution $x(t)$ of Equation (22) oscillates or $\lim_{t\to\infty}x(t)=0$.

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