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$\mathcal{I}.\vee_m$-sets and $\mathcal{I}.\wedge_m$-sets

Research Article

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Abstract: We define $\wedge_m$-sets and $\vee_m$-sets in minimal spaces and discuss their properties. Also define $\mathcal{I}.\wedge_m$-sets and $\mathcal{I}.\vee_m$-sets in ideal minimal spaces and discuss their properties. At the end of the paper, we characterize m-T_1-spaces using these new sets.

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1. Introduction

Ozbakir and Yildirim [6] introduced m-$\mathcal{I}_g$-closed sets in ideal minimal spaces and discussed their properties. In this paper, in Section 3, we define $\wedge_m$-sets and $\vee_m$-sets in minimal spaces and discuss their properties. In Section 4, we define $\mathcal{I}.\wedge_m$-sets and $\mathcal{I}.\vee_m$-sets in ideal minimal spaces and discuss their properties. In Section 5, we characterize m-T_1-spaces using these new sets.

2. Preliminaries

Definition 2.1 ([3])**.** *A subfamily $m_X \subset \wp(X)$ is said to be a minimal structure on X if $\emptyset$, $X \in m_X$. The pair (X, m_X) is called a minimal space (or an m-space). A subset A of X is said to be m-open if $A \in m_X$. The complement of an m-open set is called m-closed set. We set m-int(A)=$\cup\{U : U \subset A,\ U \in m_X\}$ and m-cl(A)=$\cap\{F : A \subset F,\ X-F \in m_X\}$.*

Lemma 2.2 ([3])**.** *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) m-cl(X−A)=X−m-int(A) and m-int(X−A)=X−m-cl(A),

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(2) If $X-A\in m_X$, then m-cl(A)=A and if $A\in m_X$, then m-int(A)=A,

(3) m-cl(∅)=∅, m-cl(X)=X, m-int(∅)=∅ and m-int(X)=X,

(4) If $A\subset B$, then m-cl(A)⊂m-cl(B) and m-int(A)⊂m-int(B),

(5) A⊂m-cl(A) and m-int(A)⊂A,

(6) m-cl(m-cl(A))=m-cl(A) and m-int(m-int(A))=m-int(A).

A minimal space (X, m_X) has the property [$\mathcal{U}$] if "the arbitrary union of m-open sets is m-open" [6]. (X, m_X) has the property [$\mathcal{I}$] if "the any finite intersection of m-open sets is m-open". [6].

Lemma 2.3 ([6]). *Let X be a nonempty set and m_X a minimal structure on X satisfying property [$\mathcal{U}$]. For a subset A of X, the following properties hold:*

(1) $A\in m_X$ if and only if m-int(A)=A,

(2) A is m-closed if and only if m-cl(A)=A,

(3) m-int(A)$\in m_X$ and m-cl(A) is m-closed.

An ideal $\mathcal{I}$ on a minimal space (X, m_X) is a non-empty collection of subsets of X which satisfies the following conditions.

(1) $A\in\mathcal{I}$ and $B\subset A$ imply $B\in\mathcal{I}$ and

(2) $A\in\mathcal{I}$ and $B\in\mathcal{I}$ imply $A\cup B\in\mathcal{I}$.

Definition 2.4 ([6]). *Let (X, m_X) be a minimal space with an ideal $\mathcal{I}$ on X and $(.)^*_m$ be a set operator from $\wp(X)$ to $\wp(X)$. For a subset $A\subset X$, A^*_m $(\mathcal{I}, m_X)=\{x\in X : U_m\cap A\notin\mathcal{I}$ for every $U_m\in\mu_m(x)\}$ where $\mu_m(x)=\{U_m\in m_X : x\in U_m\}$ is called the minimal local function of A with respect to $\mathcal{I}$ and m_X. We will simply write A^*_m for $A^*_m(\mathcal{I}, m_X)$.*

Definition 2.5 ([6]). *Let (X, m_X) be a minimal space with an ideal $\mathcal{I}$ on X. The set operator m-cl* is called a minimal ⋆-closure and is defined as m-cl$^*(A)=A\cup A^*_m$ for $A\subset X$. We will denote by $m^*_x(\mathcal{I}, m_X)$ the minimal structure generated by m-cl*, that is, $m^*_x(\mathcal{I}, m_X)=\{U\subset X :$ m-cl$^*(X-U)=X-U\}$. $m^*_x(\mathcal{I}, m_X)$ is called ⋆-minimal structure which is finer than m_X. The elements of $m^*_x(\mathcal{I}, m_X)$ are called m⋆-open and the complement of an m⋆-open set is called m⋆-closed. The interior of a subset A in (X, $m^*_x(\mathcal{I}, m_X)$) is denoted by m-int$^*(A)$.*

If $\mathcal{I}$ is an ideal on (X, m_X), then (X, m_X, $\mathcal{I}$) is called an ideal minimal space or ideal m-space.

Lemma 2.6 ([6], Theorem 2.1). *Let (X, m_X) be a minimal space with $\mathcal{I}$, $\mathcal{I}'$ ideals on X and A, B be subsets of X. Then*

(1) $A\subset B \Rightarrow A^*_m\subset B^*_m$,

(2) $\mathcal{I}\subset\mathcal{I}' \Rightarrow A^*_m(\mathcal{I}')\subset A^*_m(\mathcal{I})$,

(3) A^*_m=m-cl$(A^*_m)\subset$m-cl(A),

(4) $A^*_m\cup B^*_m\subset(A\cup B)^*_m$,

(5) $(A^*_m)^*_m \subset A^*_m$.

Proposition 2.7 ([6]). *The set operator m-cl* satisfies the following conditions:*

(1) $A \subset m\text{-}cl^*(A)$,

(2) $m\text{-}cl^*(\emptyset)=\emptyset$ *and* $m\text{-}cl^*(X)=X$,

(3) If $A \subset B$, *then* $m\text{-}cl^*(A) \subset m\text{-}cl^*(B)$,

(4) $m\text{-}cl^*(A) \cup m\text{-}cl^*(B) \subset m\text{-}cl^*(A \cup B)$.

Definition 2.8 ([6]). *A subset A of an ideal minimal space* $(X, m_X, \mathcal{I})$ *is m⋆-closed (resp. m⋆-dense in itself, m⋆-perfect) if* $A^*_m \subset A$ *(resp.* $A \subset A^*_m$, $A^*_m = A$*).*

Definition 2.9 ([6]). *A subset A of an ideal minimal space* $(X, m_X, \mathcal{I})$ *is m-*$\mathcal{I}$*-generalized closed (briefly, m-*$\mathcal{I}_g$*-closed) if* $A^*_m \subset U$ *whenever* $A \subset U$ *and U is m-open. A subset A of an ideal minimal space* $(X, m_X, \mathcal{I})$ *is said to be m-*$\mathcal{I}$*-generalized open (briefly, m-*$\mathcal{I}_g$*-open) if* $X-A$ *is m-*$\mathcal{I}_g$*-closed.*

Lemma 2.10 ([6]). *If* $(X, m_X, \mathcal{I})$ *is an ideal minimal space and* $A \subset X$, *then A is m-*$\mathcal{I}_g$*-closed if and only if* $m\text{-}cl^*(A) \subset U$ *whenever* $A \subset U$ *and U is m-open in X.*

Definition 2.11 ([6]). *A minimal space* (X, m_X) *is said to be m-*T_1 *if for any pair of distinct points x, y of X, there exist an m-open set containing x but not y and an m-open set containing y but not x.*

Lemma 2.12 ([6]). *Let* (X, m_X) *be a minimal space satisfying property [*$\mathcal{U}$*]. Then* (X, m_X) *is m-*T_1 *if and only if for each point* $x \in X$, *the singleton* $\{x\}$ *is m-closed.*

Lemma 2.13 ([6]). *Let* $(X, m_X, \mathcal{I})$ *be an ideal minimal space satisfying property [*$\mathcal{U}$*] and* $A \subset X$. *If* (X, m_X) *is an m-*T_1 *space, then A is m⋆-closed if and only if A is m-*$\mathcal{I}_g$*-closed.*

Lemma 2.14 ([5]). *Let* $(X, m_X, \mathcal{I})$ *be an ideal minimal space and* $A \subset X$. *If A is m-*$\mathcal{I}_g$*-closed then* $A^*_m - A$ *contains no nonempty m-closed set.*

3. $\wedge_m$-sets and $\vee_m$-sets

Definition 3.1. *Let A be a subset of a minimal space* (X, m_X). *We define subsets* $A^\wedge_m$ *and* $A^\vee_m$ *as follows:*

(1) $A^\wedge_m = \cap\{U : A \subset U \text{ and } U \text{ is m-open}\}$.

(2) $A^\vee_m = \cup\{F : F \subset A \text{ and } F \text{ is m-closed}\}$.

Lemma 3.2. *For subsets A, B and* A_i, $i \in \Delta$, *of a minimal space* (X, m_X) *the following properties hold:*

(1) $A \subset A^\wedge_m$.

(2) $A \subset B \Rightarrow A^\wedge_m \subset B^\wedge_m$.

(3) $(A^\wedge_m)^\wedge_m = A^\wedge_m$.

(4) If A is m-open then $A{=}A^{\wedge}_m$.

(5) $\cup\{(A_i)^{\wedge}_m : i{\in}\Delta\} \subset (\cup\{A_i : i \in \Delta\})^{\wedge}_m$.

(6) $(\cap\{A_i : i \in \Delta\})^{\wedge}_m \subset \cap\{(A_i)^{\wedge}_m : i{\in}\Delta\}$.

(7) $(X - A)^{\wedge}_m{=}X{-}A^{\vee}_m$.

Proof. (1), (2), (4), (6) and (7) are immediate consequences of Definition 3.1.

(3) From (1) and (2) we have $A^{\wedge}_m{\subset}(A^{\wedge}_m)^{\wedge}_m$. If x$\notin A^{\wedge}_m$, then there exists an m-open set U such that A⊂U and x∉U. Hence $A^{\wedge}_m{\subset}$U by Definition 3.1 and so x$\notin(A^{\wedge}_m)^{\wedge}_m$. Thus $(A^{\wedge}_m)^{\wedge}_m{\subset}A^{\wedge}_m$. Hence $(A^{\wedge}_m)^{\wedge}_m{=}A^{\wedge}_m$.

(5) Let A=∪{A$_i$: i∈Δ}. By (2) we have $\cup\{(A_i)^{\wedge}_m$: i∈Δ$\}{\subset}A^{\wedge}_m = (\cup\{A_i : i \in \Delta\})^{\wedge}_m$. □

Remark 3.3. *In Lemma 3.2, the equality in (5) and (6) does not hold as can be seen by the following example.*

Example 3.4. *Let* $X{=}\{a, b, c\}$ *and* $m_X{=}\{\emptyset, X, \{a\}, \{b\}\}$. *Take* $A{=}\{a\}$ *and* $B{=}\{b\}$, *then* $A^{\wedge}_m{\cup}B^{\wedge}_m{=}\{a\}{\cup}\{b\}{=}\{a, b\}$ *and* $(A{\cup}B)^{\wedge}_m{=}X$. *Hence* $A^{\wedge}_m{\cup}B^{\wedge}_m{=}\{a, b\}{\subsetneq}X{=}(A{\cup}B)^{\wedge}_m$. *Also take* $C{=}\{a, b\}$ *and* $D{=}\{b, c\}$, *then* $C^{\wedge}_m{\cap}D^{\wedge}_m{=}X$ *and* $(C{\cap}D)^{\wedge}_m{=}\{b\}$. *Hence* $(C \cap D)^{\wedge}_m{=}\{b\}{\subsetneq}X{=}C^{\wedge}_m{\cap}D^{\wedge}_m$.

Proposition 3.5. *Let* (X, m_X) *be a minimal space satisfying property [$\mathcal{U}$] and* A_i, $i{\in}\Delta$ *be subsets of X. Then* $\cup\{(A_i)^{\wedge}_m : i{\in}\Delta\}{=}(\cup\{A_i : i \in \Delta\})^{\wedge}_m$.

Proof. Let A=∪{A$_i$: i∈Δ}. If x$\notin\cup\{(A_i)^{\wedge}_m$: i∈Δ} then for each i∈Δ, there exists an m-open set U$_i$ such that A$_i$⊂U$_i$ and x∉U$_i$. If U=∪{U$_i$: i∈Δ} then U is m-open set by property [$\mathcal{U}$] with A⊂U and x∉U. Therefore x$\notin A^{\wedge}_m$. Hence $(\cup\{A_i : i \in \Delta\})^{\wedge}_m{\subset}\cup\{(A_i)^{\wedge}_m$: i∈Δ}. □

Lemma 3.6. *For subsets A, B and* A_i, $i{\in}\Delta$, *of a minimal space* (X, m_X) *the following properties hold:*

(1) $A^{\vee}_m{\subset}A$.

(2) $A{\subset}B \Rightarrow A^{\vee}_m{\subset}B^{\vee}_m$.

(3) $(A^{\vee}_m)^{\vee}_m{=}A^{\vee}_m$.

(4) If A is m-closed then $A{=}A^{\vee}_m$.

(5) $(\cap\{A_i : i \in \Delta\})^{\vee}_m \subset \cap\{(A_i)^{\vee}_m : i{\in}\Delta\}$.

(6) $\cup\{(A_i)^{\vee}_m : i{\in}\Delta\} \subset (\cup\{A_i : i \in \Delta\})^{\vee}_m$.

Proof. (1), (2), (4) and (6) are immediate consequences of Definition 3.1.

(3) From (1) and (2) we have $(A^{\vee}_m)^{\vee}_m{\subset}A^{\vee}_m$. If x$\in A^{\vee}_m$ then for some m-closed set F⊂A, x∈F. Then F$\subset A^{\vee}_m$ by Definition 3.1. Since F is m-closed, again by Definition 3.1, x$\in(A^{\vee}_m)^{\vee}_m$.

(5) Let A=∩{A$_i$: i∈Δ}. By (2) we have $(\cap\{A_i : i \in \Delta\})^{\vee}_m \subset \cap\{(A_i)^{\vee}_m$: i∈Δ}. □

Remark 3.7. *In Lemma 3.6, the equality in (5) and (6) does not hold as can be seen by the following example.*

Example 3.8. *In Example 3.4, take* $A=\{a, c\}$ *and* $B=\{b, c\}$, *then* $A^{\vee}_{m}\cap B^{\vee}_{m}=\{a, c\}\cap\{b, c\}=\{c\}$ *and* $(A\cap B)^{\vee}_{m}=\emptyset$. *Hence* $(A\cap B)^{\vee}_{m}=\emptyset\subsetneq\{c\}=A^{\vee}_{m}\cap B^{\vee}_{m}$. *Also take* $C=\{a\}$ *and* $D=\{c\}$, *then* $C^{\vee}_{m}\cup D^{\vee}_{m}=\emptyset$ *and* $(C\cup D)^{\vee}_{m}=\{a, c\}$. *Hence* $C^{\vee}_{m}\cup D^{\vee}_{m}=\emptyset\subsetneq\{a, c\}=(C\cup D)^{\vee}_{m}$.

Proposition 3.9. *Let* (X, m_X) *be a minimal space satisfying property* $[\mathcal{U}]$ *and* A_i, $i\in\Delta$ *be subsets of X. Then* $(\cap\{A_i : i\in\Delta\})^{\vee}_{m}=\cap\{(A_i)^{\vee}_{m} : i\in\Delta\}$.

Proof. Let $A=\cap\{A_i : i\in\Delta\}$. If $x\in\cap\{(A_i)^{\vee}_{m} : i\in\Delta\}$, then for each $i\in\Delta$, there exists a m-closed set F_i such that $F_i\subset A_i$ and $x\in F_i$. If $F=\cap\{F_i : i\in\Delta\}$ then F is m-closed by property $[\mathcal{U}]$ with $F\subset A$ and $x\in F$. Therefore $x\in A^{\vee}_{m}$. Hence $\cap\{(A_i)^{\vee}_{m} : i\in\Delta\}\subset(\cap\{A_i : i\in\Delta\})^{\vee}_{m}$. □

Definition 3.10. *A subset A of a minimal space* (X, m_X) *is said to be a*

(1) $\wedge_m$*-set if* $A=A^{\wedge}_{m}$.

(2) $\vee_m$*-set if* $A=A^{\vee}_{m}$.

Remark 3.11. $\emptyset$ *and X are* $\wedge_m$*-sets and* $\vee_m$*-sets.*

Theorem 3.12. *Let* (X, m_X) *be a minimal space satisfying property* $[\mathcal{U}]$. *Then the following hold.*

(1) Arbitrary union of $\wedge_m$*-sets is a* $\wedge_m$*-set.*

(2) Arbitrary intersection of $\vee_m$*-sets is a* $\vee_m$*-set.*

Proof.

(1) Let $\{A_i : i\in\Delta\}$ be a family of $\wedge_m$-sets. If $A=\cup\{A_i : i\in\Delta\}$, then by Proposition 3.5, $A^{\wedge}_{m}=\cup\{(A_i)^{\wedge}_{m} : i\in\Delta\}=\cup\{A_i : i\in\Delta\}=A$. Hence A is a $\wedge_m$-set.

(2) Let $\{A_i : i\in\Delta\}$ be a family of $\vee_m$-sets. If $A=\cap\{A_i : i\in\Delta\}$, then by Proposition 3.9, $A^{\vee}_{m}=\cap\{(A_i)^{\vee}_{m} : i\in\Delta\}=\cap\{A_i : i\in\Delta\}=A$. Hence A is a $\vee_m$-set.

□

Remark 3.13. *In Theorem 3.12, we cannot drop the property* $[\mathcal{U}]$. *It is shown in the following example.*

Example 3.14. *In Example 3.4,* $\{a\}$ *and* $\{b\}$ *are* $\wedge_m$*-sets but their union is not* $\wedge_m$*-set. Also,* $\{a, c\}$ *and* $\{b, c\}$ *are* $\vee_m$*-sets but their intersection is not* $\vee_m$*-set.*

Theorem 3.15. *Let* (X, m_X) *be a minimal space. Then the following hold.*

(1) Arbitrary intersection of $\wedge_m$*-sets is a* $\wedge_m$*-set.*

(2) Arbitrary union of $\vee_m$*-sets is a* $\vee_m$*-set.*

Proof.

(1) Let $\{A_i : i\in\Delta\}$ be a family of $\wedge_m$-sets. If $A=\cap\{A_i : i\in\Delta\}$, then by Lemma 3.2, $A^{\wedge}_{m}\subset\cap\{(A_i)^{\wedge}_{m} : i\in\Delta\}=\cap\{A_i : i\in\Delta\}=A$. Again by Lemma 3.2, $A\subset A^{\wedge}_{m}$. Hence A is a $\wedge_m$-set.

(2) Let $\{A_i : i\in\Delta\}$ be a family of $\vee_m$-sets. If $A=\cup\{A_i : i\in\Delta\}$, then by Lemma 3.6, $A_m^{\vee}\supset\cup\{(A_i)_m^{\vee} : i\in\Delta\}=\cup\{A_i : i\in\Delta\}=A$. Again by Lemma 3.6, $A_m^{\vee}\subset A$. Hence A is a $\vee_m$-set.

□

4. Generalized $\wedge_m$-sets and $\vee_m$-sets in Ideal Minimal Spaces

Definition 4.1. *A subset A of an ideal minimal space $(X, m_X, \mathcal{I})$ is said to be*

(1) $\mathcal{I}.\wedge_m$-set if $A_m^{\wedge}\subset F$ whenever $A\subset F$ and F is $m\star$-closed.

(2) $\mathcal{I}.\vee_m$-set if $X-A$ is an $\mathcal{I}.\wedge_m$-set.

Proposition 4.2. *Let $(X, m_X, \mathcal{I})$ be an ideal minimal space. Then the following hold:*

(1) Every $\wedge_m$-set is an $\mathcal{I}.\wedge_m$-set but not conversely.

(2) Every $\vee_m$-set is an $\mathcal{I}.\vee_m$-set but not conversely.

Example 4.3. *Let $X=\{a, b, c\}$, $m_X=\{\emptyset, X, \{a, b\}, \{b, c\}\}$ and $\mathcal{I}=\{\emptyset\}$. Then $\{a, c\}$ is $\mathcal{I}.\wedge_m$-set but not $\wedge_m$-set and $\{b\}$ is $\mathcal{I}.\vee_m$-set but not $\vee_m$-set.*

Proposition 4.4. *Every m-open set is $\mathcal{I}.\wedge_m$-set but not conversely.*

Proof. Let $A\subset F$ and F is $m\star$-closed. If A is m-open, then $A_m^{\wedge}=A\subset F$. Hence A is $\mathcal{I}.\wedge_m$-set. □

Example 4.5. *In Example 4.3, $\{b\}$ is $\mathcal{I}.\wedge_m$-set but not m-open set.*

Theorem 4.6. *A subset A of an ideal minimal space $(X, m_X, \mathcal{I})$ is an $\mathcal{I}.\vee_m$-set if and only if $U\subset A_m^{\vee}$ whenever $U\subset A$ and U is $m\star$-open.*

Proof. Suppose that $A\subset X$ is an $\mathcal{I}.\vee_m$-set and U is an $m\star$-open set such that $U\subset A$. Then $X-A\subset X-U$ and $X-U$ is $m\star$-closed. Since $X-A$ is an $\mathcal{I}.\wedge_m$-set, we have $(X-A)_m^{\wedge}\subset X-U$ and so $X-A_m^{\vee}\subset X-U$, by Lemma 3.2. Therefore, $U\subset A_m^{\vee}$. Conversely, assume that $U\subset A_m^{\vee}$ whenever $U\subset A$ and U is $m\star$-open. Suppose $X-A\subset F$ and F is $m\star$-closed. Then, $X-F\subset A$ and $X-F$ is $m\star$-open. Therefore, $X-F\subset A_m^{\vee}$ and so $X-A_m^{\vee}\subset F$. By Lemma 3.2, we have $(X-A)_m^{\vee}\subset F$. Hence $X-A$ is an $\mathcal{I}.\wedge_m$-set and so A is an $\mathcal{I}.\vee_m$-set □

Theorem 4.7. *Let A be an $\mathcal{I}.\vee_m$-set in an ideal minimal space $(X, m_X, \mathcal{I})$. Then for every $m\star$-closed set F such that $A_m^{\vee}\cup(X-A)\subset F$, $F=X$ holds.*

Proof. Let A be an $\mathcal{I}.\vee_m$-set in an ideal minimal space $(X, m_X, \mathcal{I})$. Suppose F is $m\star$-closed set such that $A_m^{\vee}\cup(X-A)\subset F$. Then $X-F\subset(X-A_m^{\vee})\cap A$. Since A is an $\mathcal{I}.\vee_m$-set and the $m\star$-open set $X-F\subset A$, by Theorem 4.6, $X-F\subset A_m^{\vee}$. Already, we have $X-F\subset X-A_m^{\vee}$ and so $X-F=\emptyset$ which implies that $F=X$. □

Corollary 4.8. *Let A be an $\mathcal{I}.\vee_m$-set in an ideal minimal space $(X, m_X, \mathcal{I})$. Then $A_m^{\vee}\cup(X-A)$ is $m\star$-closed if and only if A is a $\vee_m$-set.*

Proof. Let A be an $\mathcal{I}.\vee_m$-set. Suppose $A_m^{\vee}\cup(X-A)$ is m⋆-closed. Then by Theorem 4.7, $A_m^{\vee}\cup(X-A)=X$ and so $A\subset A_m^{\vee}$. By Lemma 3.6, we have $A_m^{\vee}\subset A$. Hence A is a $\vee_m$-set.

Conversely, suppose A is a $\vee_m$-set. Then $A_m^{\vee}\cup(X-A)=A\cup(X-A)=X$, which is m⋆-closed. □

Theorem 4.9. *Let A be a subset of an ideal minimal space (X, m_X, $\mathcal{I}$) satisfying property [$\mathcal{I}$] such that $A_m^{\vee}$ is a m⋆-closed set. If F=X, whenever F is m⋆-closed and $A_m^{\vee}\cup(X-A)\subset F$, then A is an $\mathcal{I}.\vee_m$-set.*

Proof. Let U be an m⋆-open set such that $U\subset A$. Since $A_m^{\vee}$ is m⋆-closed, $A_m^{\vee}\cup(X-U)$ is m⋆-closed. By hypothesis, $A_m^{\vee}\cup(X-U)=X$. This implies that $U\subset A_m^{\vee}$. Hence A is an $\mathcal{I}.\vee_m$-set. □

Theorem 4.10. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space. Then each singleton set in X is either m⋆-open or an $\mathcal{I}.\vee_m$-set.*

Proof. Suppose {x} is not m⋆-open. Then, X−{x} is not m⋆-closed. So the only m⋆-closed set containing X−{x} is X. Therefore, $(X-\{x\})_m^{\wedge}\subset X$ and so X−{x} is an $\mathcal{I}.\wedge_m$-set. Hence {x} is an $\mathcal{I}.\vee_m$-set. □

The set of all $\mathcal{I}.\vee_m$-sets is denoted by $D_{m\mathcal{I}}^{\vee}$ and the set of all $\mathcal{I}.\wedge_m$-sets is denoted by $D_{m\mathcal{I}}^{\wedge}$.

Definition 4.11. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space and $A\subset X$. Then $m\text{-}cl_{\mathcal{I}}^{\wedge}(A)=\cap\{U : A\subset U \text{ and } U\in D_{m\mathcal{I}}^{\wedge}\}$ and $m\text{-}int_{\mathcal{I}}^{\vee}(A)=\cup\{F : F\subset A \text{ and } F\in D_{m\mathcal{I}}^{\vee}\}$.*

Theorem 4.12. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space satisfying property [$\mathcal{U}$] and A_i, $i\in\Delta$ be subsets of X. Then the following hold.*

(1) If $A_i\in D_{m\mathcal{I}}^{\wedge}$ for all $i\in\Delta$, then $\cup\{A_i: i\in\Delta\}\in D_{m\mathcal{I}}^{\wedge}$.

(2) If $A_i\in D_{m\mathcal{I}}^{\vee}$ for all $i\in\Delta$, then $\cap\{A_i: i\in\Delta\}\in D_{m\mathcal{I}}^{\vee}$.

Proof.

(1) Let $A_i\in D_{m\mathcal{I}}^{\wedge}$ for all $i\in\Delta$. Suppose $\cup\{A_i : i\in\Delta\}\subset F$ and F is m⋆-closed. Then $A_i\subset F$ for all $i\in\Delta$. So $A_i^{\wedge}\subset F$ for all $i\in\Delta$. Therefore, $\cup\{(A_i)_m^{\wedge} : i\in\Delta\}\subset F$. By Proposition 3.5, $(\cup\{A_i : i\in\Delta\})_m^{\wedge}=\cup\{(A_i)_m^{\wedge} : i\in\Delta\}\subset F$. So $\cup\{A_i : i\in\Delta\}\in D_{m\mathcal{I}}^{\wedge}$.

(2) Let $A_i\in D_{m\mathcal{I}}^{\vee}$ for all $i\in\Delta$. Then, $X-A_i\in D_{m\mathcal{I}}^{\wedge}$ for all $i\in\Delta$. So, by (1), $\cup\{X-A_i : i\in\Delta\}\in D_{m\mathcal{I}}^{\wedge}$. $X-\cap\{A_i : i\in\Delta\}\in D_{m\mathcal{I}}^{\wedge}$ and so $\cap\{A_i : i\in\Delta\}\in D_{m\mathcal{I}}^{\vee}$.

□

Remark 4.13. *In Theorem 4.12, we cannot drop the property [$\mathcal{U}$]. It is shown in the following example.*

Example 4.14. *Let X={a, b, c}, m_X={∅, X, {a}, {b}} and $\mathcal{I}$={∅, {a}, {b}, {a, b}}. Then $D_{m\mathcal{I}}^{\wedge}$={∅, X, {a}, {b}}, take $A=\{a\}\in D_{m\mathcal{I}}^{\wedge}$ and $B=\{b\}\in D_{m\mathcal{I}}^{\wedge}$ but their union $A\cup B=\{a, b\}\notin D_{m\mathcal{I}}^{\wedge}$. Also, $\in D_{m\mathcal{I}}^{\vee}$={∅, X, {a, c}, {b, c}}, take $A=\{a, c\}\in D_{m\mathcal{I}}^{\vee}$ and $B=\{b, c\}\in D_{m\mathcal{I}}^{\vee}$ but their intersection $A\cap B=\{c\}\notin D_{m\mathcal{I}}^{\vee}$.*

Theorem 4.15. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space satisfying property [$\mathcal{U}$] and A, $B\subset X$. Then $m\text{-}cl_{\mathcal{I}}^{\wedge}$ is a kuratowski closure operator on X.*

Proof.

(1) Since $\emptyset_m^\wedge=\emptyset$, $\emptyset\in D^\wedge_{m\mathcal{I}}$ and so m-$cl^\wedge_{\mathcal{I}}(\emptyset)=\emptyset$.

(2) From the definition of m-$cl^\wedge_{\mathcal{I}}$(A), it is clear that A⊂m-$cl^\wedge_{\mathcal{I}}$(A).

(3) We have {U : A∪B⊂U, U∈$D^\wedge_{m\mathcal{I}}$}⊂{U : A⊂U, U∈$D^\wedge_{m\mathcal{I}}$}. So m-$cl^\wedge_{\mathcal{I}}$(A)⊂m-$cl^\wedge_{\mathcal{I}}$(A∪B). Similarly, m-$cl^\wedge_{\mathcal{I}}$(B)⊂m-$cl^\wedge_{\mathcal{I}}$(A∪B). Therefore, m-$cl^\wedge_{\mathcal{I}}$(A)∪m-$cl^\wedge_{\mathcal{I}}$(B) ⊂m-$cl^\wedge_{\mathcal{I}}$(A∪B). On the other hand, if x∉m-$cl^\wedge_{\mathcal{I}}$(A)∪m-$cl^\wedge_{\mathcal{I}}$(B), then x∉m-$cl^\wedge_{\mathcal{I}}$(A). So there exists $U_1\in D^\wedge_{m\mathcal{I}}$ such that A⊂U_1 but x∉U_1. Similarly, there exists $U_2\in D^\wedge_{m\mathcal{I}}$ such that B⊂U_2 but x∉U_2. Let U=U_1∪U_2. Then, by Theorem 4.12, U∈$D^\wedge_{m\mathcal{I}}$ such that A∪B⊂U but x∉U. So x∉m-$cl^\wedge_{\mathcal{I}}$(A∪B). Therefore, m-$cl^\wedge_{\mathcal{I}}$(A∪B) ⊂ m-$cl^\wedge_{\mathcal{I}}$(A)∪m-$cl^\wedge_{\mathcal{I}}$(B) which implies that m-$cl^\wedge_{\mathcal{I}}$(A∪B) = m-$cl^\wedge_{\mathcal{I}}$(A)∪m-$cl^\wedge_{\mathcal{I}}$(B).

(4) Now {U : A⊂U, U∈$D^\wedge_{m\mathcal{I}}$}={U : m-$cl^\wedge_{\mathcal{I}}$(A)⊂U, U∈$D^\wedge_{m\mathcal{I}}$} by the definition of m-$cl^\wedge_{\mathcal{I}}$ operator and so m-$cl^\wedge_{\mathcal{I}}$(A) = m-$cl^\wedge_{\mathcal{I}}$(m-$cl^\wedge_{\mathcal{I}}$(A)). Hence m-$cl^\wedge_{\mathcal{I}}$ is a kuratowski closure operator.

□

Remark 4.16. *In Theorem 4.15, we cannot drop the property [$\mathcal{U}$]. It is shown in the following example.*

Example 4.17. *In Example 4.14, take A={a} and B={b}. Then m-$cl^\wedge_{\mathcal{I}}$(A)∪m-$cl^\wedge_{\mathcal{I}}$(B) = {a}∪{b} = {a, b}⊊m-$cl^\wedge_{\mathcal{I}}$(A∪B)=X.*

Theorem 4.18. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space. Then X−m-$cl^\wedge_{\mathcal{I}}$(A) = m-$int^\vee_{\mathcal{I}}$(X−A) for every subset A of X.*

Proof. X−m-$cl^\wedge_{\mathcal{I}}$(A)=X−∩{U : A⊂U, U∈$D^\wedge_{m\mathcal{I}}$} = ∪{X−U : X−U⊂X−A, X−U∈$D^\vee_{m\mathcal{I}}$} = m-$int^\vee_{\mathcal{I}}$(X−A). □

Theorem 4.19. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space satisfying property [$\mathcal{U}$]. Then every singleton subset of X is an $\mathcal{I}.\wedge_m$-set if and only if G=$G^\vee_m$ holds for every m⋆-open set G.*

Proof. Suppose every singleton subset of X is an $\mathcal{I}.\wedge_m$-set. Let G be an m⋆-open set and y∈X−G. Since {y} is $\mathcal{I}.\wedge_m$-set, $\{y\}^\wedge_m$⊂X−G. Therefore, ∪{$\{y\}^\wedge_m$: y∈X−G}⊂ X−G. By Proposition 3.5, $(\cup\{\{y\} : y \in X-G\})^\wedge_m$=∪{$\{y\}^\wedge_m$: y∈X−G}⊂X−G and so $(X-G)^\wedge_m$⊂X−G. Therefore, $(X-G)^\wedge_m$=X−G. Since X−$G^\vee_m$=$(X-G)^\wedge_m$=X−G and so G=$G^\vee_m$.
Conversely, let x∈X and F be a m⋆-closed set containing x. Since X−F is m⋆-open, X−F=$(X-F)^\vee_m$=X−$F^\wedge_m$ and so F=$F^\wedge_m$. Therefore, $\{x\}^\wedge_m$⊂$F^\wedge_m$=F. Hence {x} is an $\mathcal{I}.\wedge_m$-set. □

Theorem 4.20. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space satisfying property [$\mathcal{U}$]. Then the following are equivalent.*

(1) Every m⋆-open set is a $\vee_m$-set,

(2) $D^\vee_{m\mathcal{I}}=\wp(X)$.

Proof. (1)⇒(2) By Theorem 4.19, every singleton subset of X is an $\mathcal{I}.\wedge_m$-set. If any subset A of X is written as a union of singleton sets, then by Theorem 4.12(1), A is an $\mathcal{I}.\wedge_m$-set and so every subset of X is an $\mathcal{I}.\vee_m$-set. Therefore, $D^\vee_{m\mathcal{I}}$=℘(X).
(2)⇒(1) Let A be an m⋆-open set. By hypothesis, A is an $\mathcal{I}.\vee_m$-set and so by Lemma 3.2(1) and Theorem 4.6, A is a $\vee_m$-set. □

Theorem 4.21. *A subset A of an ideal minimal space (X, m_X, $\mathcal{I}$) is an m-$\mathcal{I}_g$-closed set if and only if m-cl*(A)⊂$A^\wedge_m$.*

Proof. Suppose that A⊂X is an m-$\mathcal{I}_g$-closed set. Let x∈m-cl*(A). Suppose x∉$A^{\wedge}_m$. Then there exists an m-open set U containing A such that x∉U. Since A is an m-$\mathcal{I}_g$-closed set, A⊂U and U is m-open implies that m-cl*(A)⊂U and so x∉m-cl*(A), a contradiction. Therefore, m-cl*(A)⊂$A^{\wedge}_m$.

Conversely, suppose m-cl*(A)⊂$A^{\wedge}_m$. If A⊂U and U is m-open, then $A^{\wedge}_m$⊂$U^{\wedge}_m$=U and so m-cl*(A)⊂$A^{\wedge}_m$⊂U. Therefore, A is m-$\mathcal{I}_g$-closed. □

Corollary 4.22. *A subset A of an ideal minimal space (X, m_X, $\mathcal{I}$) is m-$\mathcal{I}_g$-open if and only if $A^{\vee}_m$⊂m-int*(A).*

Proof. A⊂X is m-$\mathcal{I}_g$-open if and only if X−A is m-$\mathcal{I}_g$-closed if and only if m-cl*(X−A) ⊂$(X-A)^{\wedge}_m$ if and only if X−m-int*(A)⊂X−$A^{\vee}_m$ if and only if $A^{\vee}_m$⊂m-int*(A). □

Theorem 4.23. *If A is an m-$\mathcal{I}_g$-open set in an ideal minimal space (X, m_X, $\mathcal{I}$), then U=X whenever U is m-open and m-int*(A)∪(X−A)⊂U.*

Proof. Assume that A is m-$\mathcal{I}_g$-open in X. Let U be an m-open set in X such that m-int*(A)∪(X−A)⊂U, then X−U⊂X−(m-int*(A)∪(X−A)) = (X−m-int*(A))∩A = m-cl*(X−A)−(X−A). Since X−A is m-$\mathcal{I}_g$-closed, then by Lemma 2.14, $(X-A)^*_m$−(X−A) contains no nonempty m-closed set. But m-cl*(X−A)−(X−A)=$(X-A)^*_m$−(X−A) and so m-cl*(X−A)−(X−A) contains no nonempty m-closed set. Since X−U is m-closed, then X−U=∅ and so U=X. □

Corollary 4.24. *A $\wedge_m$-set A in an ideal minimal space (X, m_X, $\mathcal{I}$) is m-$\mathcal{I}_g$-closed if and only if A is m⋆-closed.*

Proof. Let A be a $\wedge_m$-set. If A is m-$\mathcal{I}_g$-closed, then by Theorem 4.21, m-cl*(A)⊂$A^{\wedge}_m$ and so m-cl*(A)⊂A which implies that A is m⋆-closed.

Conversely, it is clear, since every m⋆-closed set is m-$\mathcal{I}_g$-closed. □

Corollary 4.25. *An m-open set A in an ideal minimal space (X, m_X, $\mathcal{I}$) is m-$\mathcal{I}_g$-closed if and only if A is m⋆-closed.*

Proof. The proof follows from the fact that every m-open set is a $\wedge_m$-set. □

Theorem 4.26. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space and A⊂X. If $A^{\wedge}_m$ is an m-$\mathcal{I}_g$-closed set, then A is an m-$\mathcal{I}_g$-closed set.*

Proof. Let $A^{\wedge}_m$ be m-$\mathcal{I}_g$-closed set. By Theorem 4.21, m-cl*($A^{\wedge}_m$)⊂$(A^{\wedge}_m)^{\wedge}_m$=$A^{\wedge}_m$. Since A⊂$A^{\wedge}_m$ and so m-cl*(A)⊂m-cl*($A^{\wedge}_m$)⊂$A^{\wedge}_m$. Again, by Theorem 4.21, A is m-$\mathcal{I}_g$-closed. □

Remark 4.27. *The converse of Theorem 4.26 need not be true as shown by the following example.*

Example 4.28. *In Example 4.3, A={a} is m-$\mathcal{I}_g$-closed set but $A^{\wedge}_m$ is not m-$\mathcal{I}_g$-closed set.*

5. Characterizations of m-T_1-spaces

Definition 5.1. *Let (X, m_X, $\mathcal{I}$) be an ideal minimal space and A⊂X. Then $m^{\wedge}_{\mathcal{I}}$ is defined as follows: $m^{\wedge}_{\mathcal{I}}$={A⊂X : m-$cl^{\wedge}_{\mathcal{I}}$(X−A)=X−A}. $m^{\wedge}_{\mathcal{I}}$ is called ∧-minimal structure on X generated by m-$cl^{\wedge}_{\mathcal{I}}$. Each element of $m^{\wedge}_{\mathcal{I}}$ is called $m^{\wedge}_{\mathcal{I}}$-open and the complement of an $m^{\wedge}_{\mathcal{I}}$-open set is called $m^{\wedge}_{\mathcal{I}}$-closed. We observe that $m^{\wedge}_{\mathcal{I}}$ is always finer than $D^{\vee}_{m\mathcal{I}}$ and ${m^{\wedge}_{\mathcal{I}}}^c$ is always finer than $D^{\wedge}_{m\mathcal{I}}$.*

Theorem 5.2. *In an ideal minimal space (X, m_X, $\mathcal{I}$) satisfying property [$\mathcal{U}$], the following are equivalent.*

(1) (X, m_X, $\mathcal{I}$) is a m-T_1-space,

(2) Every $\mathcal{I}.\vee_m$-set is a $\vee_m$-set,

(3) Every $m_{\mathcal{I}}^{\wedge}$-open set is a $\vee_m$-set.

Proof. (1)⇒(2) Suppose there exists an $\mathcal{I}.\vee_m$-set A which is not $\vee_m$-set. Then $A_m^{\vee}\subsetneq$A. Therefore, there exists an element x∈A such that x$\notin A_m^{\vee}$. Then, {x} is not m-closed, the definition of $A_m^{\vee}$, a contradiction to Lemma 2.12. This proves (2).
(2)⇒(1) Suppose that (X, m_X, $\mathcal{I}$) is not a m-T_1-space. Then by Lemma 2.13 there exists an m-$\mathcal{I}_g$-closed set A which is not m⋆-closed. So, there exists an element x∈X such that x∈m-cl*(A) but x∉A. By Theorem 4.10, {x} is either m⋆-open or an $\mathcal{I}.\vee_m$-set. When {x} is m⋆-open, {x}∩A=∅, m-cl*(A)⊂m-cl*(X−{x}) = X−{x} which is a contradiction to the fact that x∈m-cl*(A). When {x} is an $\mathcal{I}.\vee_m$-set, by our assumption, {x} is a $\vee_m$-set and hence {x} is m-closed. Therefore, A⊂X−{x} and X−{x} is m-open. Since A is m-$\mathcal{I}_g$-closed, m-cl*(A)⊂X−{x}. This is also a contradiction to the fact that x∈m-cl*(A). Therefore, every m-$\mathcal{I}_g$-closed set is m⋆-closed and hence (X, m_X, $\mathcal{I}$) is a m-T_1-space.
(2)⇒(3) Suppose that every $\mathcal{I}.\vee_m$-set is a $\vee_m$-set. Then, a subset F is a $\mathcal{I}.\vee_m$-set if and only if F is a $\vee_m$-set. Let A∈$m_{\mathcal{I}}^{\wedge}$. Then A=m-$int_{\mathcal{I}}^{\vee}$(A)=∪{F: F⊂A and F∈$D_{m\mathcal{I}}^{\vee}$}=∪{F : F⊂A and F is a $\vee_m$-set}. Now $A_m^{\vee}$=$(m\text{-}int_{\mathcal{I}}^{\vee}(A))_m^{\vee}$=$(\cup\{F : F \subset A \text{ and } F = F_m^{\vee}\})_m^{\vee}$⊃∪{$F_m^{\vee}$: F⊂A and F=$F_m^{\vee}$}=∪{F : F⊂A and F=$F_m^{\vee}$}=∪{F : F⊂A and F∈$D_{m\mathcal{I}}^{\vee}$}=m-$int_{\mathcal{I}}^{\vee}$(A)=A. Always $A_m^{\vee}$⊂A and so $A_m^{\vee}$=A. Hence A is a $\vee_m$-set.
(3)⇒(2) Let A be an $\mathcal{I}.\vee_m$-set. Then, by definition of m-$int_{\mathcal{I}}^{\vee}$(A), m-$int_{\mathcal{I}}^{\vee}$(A)=A and so A∈$m_{\mathcal{I}}^{\wedge}$. By (3), A is a $\vee_m$-set. □

Corollary 5.3. *An ideal minimal space (X, m_X, $\mathcal{I}$) satisfying property [$\mathcal{U}$] is a m-T_1-space if and only if every singleton set is either m⋆-open or m-closed.*

Proof. Assume that (X, m_X, $\mathcal{I}$) is a m-T_1-space. Let x∈X. Suppose {x} is not m⋆-open. By Theorem 4.10, it is an $\mathcal{I}.\vee_m$-set. Since X is a m-T_1-space, by Theorem 5.2(2), {x} is a $\vee_m$-set and hence is m-closed.
Conversely, suppose (X, m_X, $\mathcal{I}$) is not a m-T_1-space. Then, there exists an $\mathcal{I}.\vee_m$-set A which is not a $\vee_m$-set, by Theorem 5.2(2). So there exists an element x∈A such that x$\notin A_m^{\vee}$. If {x} is m-closed, then $A_m^{\vee}$ contains the m-closed set {x}, which is not possible. If {x} is m⋆-open, then the m⋆-closed set X−{x} contains $A_m^{\vee}$∪(X−A). By Theorem 4.7, X−{x}=X, which is not possible. So {x} is neither m⋆-open nor m-closed, which is contradiction to our assumption. Therefore, (X, m_X, $\mathcal{I}$) is a m-T_1-space. □

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