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Transformation Formulas of Lauricella's Function of the Third kind of Several Variables

Research Article

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Abstract: In this research paper a transformation formulas for Lauricella's function of the third kind of several variables are established with the help of the Generalization of the Kummer's theorem on the sum of the series ${}_2F_1(-1)$ obtained by Lavoie et al. [4]. The presented results are generalizations of the will known result due to Srivastava [6].

MSC: 33C05, 33C20, 33C65, 33C70.

Keywords: Transformation formulas, Lauricella's function, Kummer's theorem, Hypergeometric functions.

1. Introduction

The Lauricella's function $F_C^{(n)}$ are defined and represented as follows [7]

$$F_C^{(n)}(a,b;c_1,\dots,c_n;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}\tag{1}$$

$|x_1|^{\frac{1}{2}}+\cdots+|x_n|^{\frac{1}{2}}<1$,

where $(a)_n$ denotes the Pochhammer's symbol defined by [7]

$$(a)_n=\begin{cases}1 & , \quad \text{if } n=0\\ a(a+1)\dots(a+n-1) & , \quad \text{if } n=1,2,3,\dots\end{cases}\tag{2}$$

$$=\frac{\Gamma(a+n)}{\Gamma(a)}, a\neq 0,-1,-2,\dots\tag{3}$$

Also, we note that

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+a)=2^a\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(1+\frac{1}{2}a\right)\tag{4}$$

$$(a)_{2n}=2^{2n}\left(\frac{1}{2}a\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n\tag{5}$$

$$\frac{\Gamma(a-n)}{\Gamma(a)}=\frac{(-1)^n}{(1-a)_n}\tag{6}$$

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$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n! \tag{7}$$

$$(2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n! \tag{8}$$

In the theory of hypergeometric series, classical summation theorems such as Watson, Dixon and Kummer for the series ${}_3F_2(1)$, ${}_2F_1(-1)$ and others have wide applications,see for example Bailey [1], Lavoie et al. [4] and Rainville [5]. In the present investigation, we shall require the following generalization of the classical Kummer's theorem for the series ${}_2F_1(-1)$ Lavoie et al.[4]:

$$\begin{aligned}{}_2F_1\left[\begin{matrix} a, b & ; & \\ & & -1 \\ 1+a-b+i & ; & \end{matrix}\right] &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma\left(1-b+\frac{1}{2}(i+|i|)\right)} \\ &\times\left\{\frac{A_i}{\Gamma\left(\frac{a}{2}+\frac{i}{2}+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)\Gamma\left(1+\frac{a}{2}-b+\frac{i}{2}\right)}+\frac{B_i}{\Gamma\left(\frac{a}{2}+\frac{i}{2}-\left[\frac{i}{2}\right]\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}-b+\frac{i}{2}\right)}\right\}\end{aligned} \tag{9}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients A_i and B_i are given respectively in Lavoie et al.[4]. When $i = j = 0$, (9) reduces immediately to the classical Kummer's theorem Rainville [5]

$${}_2F_1\left[\begin{matrix} a, b & ; & \\ & & -1 \\ 1+a-b & ; & \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma\left(\frac{1}{2}\right)}{2^a\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)} \tag{10}$$

2. Transformation Formulas

In this section, the following transformation formulas will be established:

$$\begin{aligned}&F_C^{(2r)}(a, b; c_1, c_1+i, c_2, c_2+i, \ldots, c_r, c_r+i; x_1, -x_1, x_2, -x_2, \ldots, x_r, -x_r) \\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r}(b)_{2m_1+\cdots+2m_r}}{(c_1)_{2m_1}\ldots(c_r)_{2m_r}}\frac{x_1^{2m_1}\ldots x_r^{2m_r}}{(2m_1)!\ldots(2m_r)!} \\ &\times I_1(c_1, i, 2m_1)\left\{\frac{A_i^{(1)}}{A_1(c_1, i, 2m_1)}+\frac{B_i^{(1)}}{B_1(c_1, i, 2m_1)}\right\}\times\cdots\times I_r(c_r, i, 2m_r)\left\{\frac{A_i^{(r)}}{A_r(c_r, i, 2m_r)}+\frac{B_i^{(r)}}{B_r(c_r, i, 2m_r)}\right\}+\ldots \\ &\cdots+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1}(b)_{2m_1+1+\cdots+2m_r+1}}{(c_1)_{2m_1+1}\ldots(c_r)_{2m_r+1}}\frac{x_1^{2m_1+1}\ldots x_r^{2m_r+1}}{(2m_1+1)!\ldots(2m_r+1)!} \\ &\times I_1(c_1, i, 2m_1+1)\left\{\frac{A_i^{'(1)}}{A_1(c_1, i, 2m_1+1)}+\frac{B_i^{'(1)}}{B_1(c_1, i, 2m_1+1)}\right\}\times\ldots \\ &\cdots\times I_r(c_r, i, 2m_r+1)\left\{\frac{A_i^{'(r)}}{A_r(c_r, i, 2m_r+1)}+\frac{B_i^{'(r)}}{B_r(c_r, i, 2m_r+1)}\right\},\ r = 1, 2, \ldots\end{aligned} \tag{11}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

$$\begin{aligned}&F_C^{(2r+1)}(a, b; c_1, c_1+i, c_2, c_2+i, \ldots, c_r, c_r+i, d; x_1, -x_1, x_2, -x_2, \ldots, x_r, -x_r, x) \\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r+m}(b)_{2m_1+\cdots+2m_r+m}}{(c_1)_{2m_1}\ldots(c_r)_{2m_r}(d)_m}\frac{x_1^{2m_1}\ldots x_r^{2m_r}x^m}{(2m_1)!\ldots(2m_r)!m!} \\ &\times I_1(c_1, i, 2m_1)\left\{\frac{A_i^{(1)}}{A_1(c_1, i, 2m_1)}+\frac{B_i^{(1)}}{B_1(c_1, i, 2m_1)}\right\}\times\cdots\times I_r(c_r, i, 2m_r)\left\{\frac{A_i^{(r)}}{A_r(c_r, i, 2m_r)}+\frac{B_i^{(r)}}{B_r(c_r, i, 2m_r)}\right\}+\ldots\end{aligned}$$

$$\cdots+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1+m}(b)_{2m_1+1+\cdots+2m_r+1+m}}{(c_1)_{2m_1+1}\ldots(c_r)_{2m_r+1}(d)_m}\frac{x_1^{2m_1+1}\ldots x_r^{2m_r+1}x^m}{(2m_1+1)!\ldots(2m_r+1)!m!}$$

$$\times I_1(c_1,i,2m_1+1)\left\{\frac{A_i^{'(1)}}{A_1(c_1,i,2m_1+1)}+\frac{B_i^{'(1)}}{B_1(c_1,i,2m_1+1)}\right\}\times\ldots$$

$$\cdots\times I_r(c_r,i,2m_r+1)\left\{\frac{A_i^{'(r)}}{A_r(c_r,i,2m_r+1)}+\frac{B_i^{'(r)}}{B_r(c_r,i,2m_r+1)}\right\},\ r=1,2,\ldots \tag{12}$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$, where

$$I_r(c_r,i,m_r)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c_r+i)\Gamma(c_r+m_r)}{2^{-m_r}\Gamma\left(c_r+m_r+\frac{1}{2}(i+|i|)\right)} \tag{13}$$

$$A_r(c_r,i,m_r)=\Gamma\left(-\frac{1}{2}m+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)\Gamma\left(\frac{1}{2}m_r+c_r+\frac{1}{2}i\right) \tag{14}$$

$$B_r(c_r,i,m_r)=\Gamma\left(-\frac{1}{2}m_r+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\Gamma\left(\frac{1}{2}m_r+c_r-\frac{1}{2}+\frac{1}{2}i\right) \tag{15}$$

The coefficients $A_i^{(r)},B_i^{(r)}$ to $A_i^{'(r)},B_i^{'(r)}$ can be obtained from the tables of A_i,B_i given in [4].

Proof of (11): Denoting the left hand side of (11) by S, expanding $F_C^{(2r)}$ in a power series and using the results [7].

$$(a)_{m+n}=(a)_m(a+m)_n \tag{16}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n) \tag{17}$$

$$(a)_{m-n}=\frac{(-1)^n(a)_m}{(1-a-m)_n},\ 0\le n\le m \text{ and } (m-n)!=\frac{(-1)^n m!}{(-m)_n},\ \ 0\le n\le m, \tag{18}$$

we get

$$S=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m_1+\cdots+m_r}(b)_{m_1+\cdots+m_r}}{(c_1)_{m_1}\ldots(c_r)_{m_r}}\frac{x_1^{m_1}\ldots x_r^{m_r}}{(m_1)!\ldots(m_r)!}f(c_i,i,m_1)\ldots f(c_r,i,m_r) \tag{19}$$

where

$$f(c_r,i,m_r)={}_2F_1\left[\begin{matrix}-m_r,1-c_r-m_r & ; & \\ & & -1\\ c_r+i & ; & \end{matrix}\right]$$

Separating (19) into its even and odd terms, we have

$$S=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r}(b)_{2m_1+\cdots+2m_r}}{(c_1)_{2m_1}\ldots(c_r)_{2m_r}}\frac{x_1^{2m_1}\ldots x_r^{2m_r}}{(2m_1)!\ldots(2m_r)!}$$

$$\times f(c_1,i,2m_1)f(c_2,i,2m_2)\ldots f(c_r,i,2m_r)$$

$$+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+1+2m_2+\cdots+2m_r}(b)_{2m_1+1+2m_2+\cdots+2m_r}}{(c_1)_{2m_1+1}(c_2)_{2m_r}\ldots(c_r)_{2m_r}}\frac{x_1^{2m_1+1}x^{2m_2}\ldots x_r^{2m_r}}{(2m_1+1)!(2m_2)!\ldots(2m_r)!}$$

$$\times f(c_1,i,2m_1+1)f(c_2,i,2m_2)\ldots f(c_r,i,2m_r)+\ldots$$

$$\cdots+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_{r-1}+2m_r+1}(b)_{2m_1+\cdots+2m_{r-1}+2m_r+1}}{(c_1)_{2m_1}\ldots(c_{r-1})_{2m_{r-1}}(c_r)_{2m_r+1}}\frac{x_1^{2m_1}\ldots x_{r-1}^{2m_{r-1}}x_r^{2m_r+1}}{(2m_1)!\ldots(2m_{r-1})!(2m_r+1)!}$$

$$\times f(c_1,i,2m_1)\ldots f(c_{r-1},i,2m_{r-1})f(c_r,i,2m_{r+1})$$

$$+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1}(b)_{2m_1+1+\cdots+2m_r+1}}{(c_1)_{2m_1+1}\ldots(c_r)_{2m_r+1}}\frac{x_1^{2m_1+1}\ldots x_r^{2m_r+1}}{(2m_1+1)!\ldots(2m_r+1)!}$$

$$\times f(c_1,i,2m_1+1)f(c_2,i,2m_2+1)\ldots f(c_r,i,2m_r+1). \tag{20}$$

Finally, in (20) if we use the result (9), then we obtain the right hand side of (11). This completes the proof of (11). The result (12) can be proved by the similar manner.

3. Special Cases

In (11) if we take $i=0$ and use the results (3)-(8), then after some simplification we obtain the following transformation formula: $F_C^{(2r)}(a,b;c_1,c_1,c_2,c_2,\ldots,c_r,c_r;x_1,-x_1,x_2,-x_2,\ldots,x_r,-x_r)$

$$= F\begin{array}{c}4:\ 0;\ \ldots\ ;0\\0:\ 3;\ \ldots\ ;3\end{array}\left[\begin{array}{ccccc}\frac{a}{2},\frac{a}{2}+\frac{1}{2},\frac{b}{2},\frac{b}{2}+\frac{1}{2} & : & -- & ;\ \ldots\ ; & -- & ; \\ & & & & & -4x_1^2,-4x_2^2,\ldots,-4x_r^2 \\ -- & : & c_1,\frac{c_1}{2},\frac{c_1}{2}+\frac{1}{2} & ;\ \ldots\ ; & c_r,\frac{c_r}{2},\frac{c_r}{2}+\frac{1}{2} & ;\end{array}\right], \tag{21}$$

$(r=1,2,\ldots)$

which for $r=1$, reduces immediately to a known result of Srivastava [6].

$$F_4[a,b;c_1,c_1;x_1,-x_1] = {}_4F_3\left[\begin{array}{cc}\frac{a}{2},\frac{a}{2}+\frac{1}{2},\frac{b}{2},\frac{b}{2}+\frac{1}{2} & ; \\ & -4x_1^2 \\ c_1,\frac{c_1}{2},\frac{c_1}{2}+\frac{1}{2} & ;\end{array}\right], \tag{22}$$

where $F\begin{array}{c}A:\ B';\ \ldots\ ;B^{(n)}\\C:\ D';\ \ldots\ ;D^{(n)}\end{array}[x_1,\ldots,x_n]$ is the generalized Kampé de Fériet function of several variables [2] and F_4 is Appell's function [7].

Similarly, in (12) if we take $i=0$ and use the results (3)-(8), then we have

$$F_C^{(2r+1)}(a,b;c_1,c_1,c_2,c_2,\ldots,c_r,c_r,d;x_1,-x_1,x_2,-x_2,\ldots,x_r,-x_r,x) = F\begin{array}{c}2:0;\ldots;0;0\\0:3;\ldots;3;1\end{array}$$
$$\left[\begin{array}{cccccc}(a:2,\ldots,2,1),(b:2,\ldots,2,1) & : & -- & ;\ldots; & -- & ;-; \\ & & & & & \frac{-x_1^2}{4},\ldots,\frac{-x_r^2}{4},x \\ -- & : & (c_1:1),(\frac{c_1}{2}:1),(\frac{c_1}{2}+\frac{1}{2}:1) & ;\ldots; & (c_r:1)(\frac{c_r}{2}:1),(\frac{c_r}{2}+\frac{1}{2}:1) & ;d;\end{array}\right] \tag{23}$$

$(r=1,2,\ldots)$

where F-function on the R.H.S. of (23) is the generalized Lauricella function of several variables [7].

In (12), if we take $r=1$, then we get the following extension formulas of $F_C^{(3)}$

$$\begin{aligned}&F_C^{(3)}(a,b;c_1,c_1+i,d;x_1,-x_1,x)\\&=\sum_{m_1=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m_1+m}(b)_{2m_1+m}x_1^{2m_1}x^m}{(c_1)_{2m_1}(d)_m(2m_1)!m!}\times I_1(c_1,i,2m_1)\left\{\frac{A_i^{(1)}}{A_1(c_1,i,2m_1)}+\frac{B_i^{(1)}}{B_1(c_1,i,2m_1)}\right\}\\&+\sum_{m_1=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m_1+m+1}(b)_{2m_1+m+1}x_1^{2m_1+1}x^m}{(c_1)_{2m_1+1}(d)_m(2m_1+1)!m!}\times I_1(c_1,i,2m_1+1)\left\{\frac{A_i^{'(1)}}{A_1(c_1,i,2m_1+1)}+\frac{B_i^{'(1)}}{B_1(c_1,i,2m_1+1)}\right\}\end{aligned} \tag{24}$$

The coefficients $A_i^{(1)},B_i^{(1)}$ and $A_i^{'(1)},B_i^{'(1)}$ can be obtained from the coefficients A_i,B_i given in [4] by replacing a,b by $-2m_1,1-c_1-2m_1$ and $-2m_1-1,-c_1-2m_1$ respectively.

Also $I_1(c_1,i,2m_1), A_1(c_1,i,2m_1), B_1(c_1,i,2m_1), I_1(c_1,i,2m_1+1), A_1(c_1,i,2m_1+1)$ and $B_1(c_1,i,2m_1+1)$ can be obtained from (13), (14) and (15) by replacing m_1 by $2m_1$ and $2m_1+1$ respectively.

Finally, in (24), if we take $i=0$, then we get

$$F_A^{(3)}(a,b;c_1,c_1,d;x_1,-x_1,x) = X \begin{matrix} 2:0;0 \\ 0:3;1 \end{matrix} \left[\begin{matrix} a,b & : & - & ;-; & \\ & & & & -\frac{x_1^2}{4}, x \\ - & : & c_1, \frac{c_1}{2}, \frac{c_1}{2}+\frac{1}{2} & ;d; & \end{matrix} \right] \tag{25}$$

where $X \begin{matrix} A:B;D \\ E:G;H \end{matrix}$ is double hypergeometric series of Exton [3]

$$X \begin{matrix} A:B;B' \\ C:D;D' \end{matrix} \left(\begin{matrix} (a):(b);(b'); & \\ & x,y \\ (c):(d);(d'); & \end{matrix} \right) = \sum_{m,n=0}^{\infty} \frac{((a))_{2m+n}((b))_m((b'))_n x^m y^n}{((c))_{2m+n}((d))_m((d'))_n m!n!} \tag{26}$$

The other special cases of (24) for $i=\pm1,\pm2,\pm3,\pm4,\pm5$ can also be obtained in terms of Exton's double hypergeometric series.

4. Conclusion

We conclude our present investigation by remarking that the results established in this paper can be applied to obtain a large number of transformation formulas for the third kind of Lauricella's functions of several variables.

Further, in the formula (11), if we take $r=2$, then we can obtain new extension formulas of Lauricella's function of four variables $F_C^{(4)}(a,b;c_1,c_1+i,c_2,c_2+i;x_1,-x_1,x_2,-x_2)$. Also many special cases of this extension formulas can also be obtained in terms of Kampé de Fériet function of two variables.

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