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Fractional Complex Variables: Strong Local Fractional Complex Derivatives (LFCDs) of Non-Integer Rational Order

Review Article

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Abstract: Fractional complex variables focus on the fractional or non-integer order differential calculus of a complex variable. In fractional calculus, locality can narrow down pieces of a function where there may be better behavior in order to model in an analytic sense, as well as obtain more meaningful physical and/or geometric information. That's where we introduce the concepts of Strong Local Fractional Complex Derivatives or LFCDs. Strong LFCDs can "maximize" the opportunity that the piece of the function in a localized or local enough area is "well-behaved" (enough). We prove a theorem that shows where Strong LFCDs exist. Applications include index of stability in Complex or Real Fractional Advection Dispersion Equation (FADE).

MSC: 26A33

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1. Introduction to Fractional Complex Derivatives

We start by looking at derivatives of fractional or non-integer order s.t $\alpha \in \mathbb{Q}$

$$\frac{d^{\alpha}y}{dx^{\alpha}} \text{ or } \frac{d^{\alpha}f(x)}{dx^{\alpha}}, \frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}$$

Definition 1.1. *If for a function $f:[0,1]\to\mathbb{R}$, then $\exists$ the limit defining the derivative, where α is $0<\alpha<1$*

$$D^{\alpha}f(y)=\lim_{x\to y}\frac{d^{\alpha}(f(x)-f(y))}{d(x-y)^{\alpha}}$$

This is the Local Fractional Derivative (LFD) form: If for a function $f:[0,1]\to\mathbb{R}$, then $\exists$ a finite limit, where N is the largest integer for which Nth derivative of $f(x)$ at y exists and is finite, then we say that the LFD of order $\alpha: 1\leq\alpha\leq N$ at $x=y$ exists.

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$$D^{\alpha}f(y) = \lim_{x\to y}\left[\frac{d^{\alpha}\left[f(x) - \sum_{n=0}^{N}\frac{f^{(n)}(y)}{\Gamma(n+1)}f(x-y)^{n}\right]}{d[x-y]^{n}}\right]$$

Generally, fractional derivatives are not local in nature; however, if we localize some function f, we can use LDF to solve some physical models that integer order derivatives cannot really solve or explain.

Now, recall $W = f(z)$:

$$z = x + iy, Re(z) = x, Im(z) = y$$

$$w = u(x,y) + iv(x,y) = f(z)$$

$$\Rightarrow \lim_{z\to a+bi} f(z) = \lim_{(x,y)\to(a,b)} u(x,y) + i\lim_{(x,y)\to(a,b)} v(x,y)$$

where $z \in \mathbb{C}$, $u, v \in \mathbb{R}$

$\Rightarrow$ Let $z \in R$, then we have $z^{\alpha} \in R^{\alpha} \subseteq \mathbb{C}$.

And if $z^{\alpha}\ \exists \Rightarrow w^{\alpha}$.

$$\Rightarrow w^{\alpha} = f(z) = u(x,y) + i^{\alpha}v(x,y)$$

Let $f : F \to R^{\alpha}$ local function defined on a fractal set F of fractal dimension $\alpha, 0 \leq \alpha \leq 1$. If $\forall\varepsilon > 0, \exists$ some $\delta > 0$ s.t.

$$|f(z) - L| < \varepsilon^{\alpha} \Rightarrow 0 < |z - z_0| < \delta$$

The limit of $f(z)$ as $z \to z_0$ is L

$$\Rightarrow \lim_{z\to z_0} f(z) = L.$$

The function $f(z)$ is said to be local fractional continuous at z_0 if $f(z_0)$ is defined, and

$$\lim_{z\to z_0} f(z) = f(z_0).$$

A function $f(z)$ is deemed local fractional cont. on R^{α} if it is local fractional continuous $\forall$ point of its domain $\mathbb{C}_{\alpha}(R)$.

Let the complex function $f(z)$ be defined in a neighborhood of a point z_0. The local fractional complex derivative of $f(z)$ at z_0 denoted by

$$D^{\alpha}f(z), \left.\frac{d^{\alpha}}{dz^{\alpha}}f(z)\right|_{z=z_0} \text{ or } f^{(\alpha)}(z_0),$$

$$= \lim_{z\to z_0}\frac{\Gamma(1+\alpha)[f(z) - f(z_0)]}{(z-z_0)^{\alpha}}, 0 < \alpha \leqslant 1 \tag{1}$$

If this limit exists, then the function $f(z)$ is said to be local fractional analytic at z_0. If this limit exists $\forall z_0 \in R^\alpha$, then the function $f(z)$ is deemed to be local fractional analytic in R^α.

If $\exists$ a function

$$f(z) = u(x,y) + i^\alpha v(x,y) \tag{2}$$

The local fractional equations

$$\frac{\partial^\alpha u(x,y)}{\partial x^\alpha} - \frac{\partial^\alpha v(x,y)}{\partial y^\alpha} = 0 \tag{3}$$

$$\frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + \frac{\partial^\alpha v(x,y)}{\partial x^\alpha} = 0 \tag{4}$$

are local fractional Cauchy-Riemann Equations.

2. Strong (or Weak) Local Fractional Complex Derivatives

Theorem 2.1 (from Yang)**.** *Suppose that (2) is local fractional analytic in a region R^α. Then we have (3) and (4).*

Proof. Local fractional C-R Equations are sufficient equations/conditions that $f(z)$ be local functional analytic in region $R^\alpha \Rightarrow R^1$,where α is 1. The local fractional partial equations

$$\frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x,y)}{\partial y^{2\alpha}} = 0 \tag{5}$$

$$\frac{\partial^{2\alpha} v(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} v(x,y)}{\partial y^{2\alpha}} = 0 \tag{6}$$

are deemed local fractional Laplace Equations, denoted by

$$\nabla^\alpha u(x,y) = 0, \nabla^\alpha v(x,y) = 0 \ (\neg a, b)$$

$$\Rightarrow \nabla^\alpha = \partial^{2\alpha}/\partial x^{2\alpha} + \partial^{2\alpha}/\partial y^{2\alpha}$$

$$\nabla^\alpha = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \tag{7}$$

This is a local fractional Laplace operator. Suppose $\nabla^\alpha u(x,y) = 0$, then $u(x,y)$ is a local fractional harmonic function in R. When local may not be good enough, there may be cases where physically, geometrically - a strong local fractional complex derivative or strong LFCD may be needed. Hence, we have maybe the following: □

Theorem 2.2 (Theorem Proposition (Strong LFCDs))**.** *If $\exists$ LFCDs of non-integer order $\alpha \in \mathbb{Q}$ s.t. $\alpha : 1 \le \alpha \le 2$, then these are Strong LFCDs.*

Proof. If a function $f(z_0) \in \mathbb{C}$ domain is sufficiently smooth, and it meets Cauchy-Riemann conditions then $D^{\alpha}(f(z))$ at least exists $\forall \alpha \in \mathbb{Q}$ s.t. $1 \leq \alpha < \infty$. See previous Theorem.

Now, recall $w = f(z) = u(x, y) + iv(x, y)$ and $\exists$ partial derivatives for f with x and y so we have the following for $f(z_0)$:

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{(x_0,y_0)}$$

$$f'(z_0) = \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)_{(x_0,y_0)}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{(x_0,y_0)} = \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)_{(x_0,y_0)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; , \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the C-R Equations with $\alpha = 1$. By previous Theorem from Yang, $\exists$ local fractional Laplace equations (5) and (6)

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u}{\partial y^{2\alpha}} \; , \; \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} v}{\partial y^{2\alpha}}$$

with $\alpha = 1$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$$

This shows the existence of Laplace's Equation of order $\alpha = 2$ is well established and defined for a local enough second order $f(z)$ and we know C-R Equations of order 1 is also well-established and defined for a local first order $f(z)$, then $\Rightarrow$ we may have a strong or strong enough LCFDs which can exist for every $\alpha : 1 \leq \alpha \leq 2 \; \forall \alpha \in \mathbb{Q}$ and z_0 in $f(z_0) \in \mathbb{C}$.

Remark 2.3. *On the contrary, LFCDs with order $\alpha < 1$ and $\alpha > 2 \; \forall \alpha \in \mathbb{Q}$ are not strong (enough) or are even weak. As a consequence of Theorem for strong LFCDs proposition.*

For $\alpha = 1$, $\alpha = 2$, the strong case seems evident. Using Sobolev Spaces and The Sharp Trace Theorem where $\exists H^2$ over some half space in $\mathbb{C}$ called Ω and some boundary $\partial\Omega$ in $\mathbb{C}$. We have the following for the functions u, v as $u \rightarrow u|_{\partial\Omega}$ and $v \rightarrow v|_{\partial\Omega}$

$$H^2(\Omega) \rightarrow H^{\frac{3}{2}}(\partial\Omega).$$

$u \in \mathbb{C}^{\infty}(\Omega) \cap H^2(\Omega)$ and similar for $v \in \mathbb{C}^{\infty}(\Omega) \cap H^2(\Omega)$. This mapping extends to a unique continuous linear operator. Hence, it is onto.

$\Rightarrow \alpha = \frac{3}{2}$ for functions or the fractional Laplace Equations for u, v exists, and moreover is Regular.

$\Rightarrow$ smooth u, v functions. How smooth? How strong? Recall that if a function u, v or f such as $f(z_0) \in \mathbb{C}$ domain is at least smooth enough and it meets Cauchy-Riemann conditions, then $D^{\alpha} f(z)$ at least exists $\forall \alpha \in \mathbb{Q}$ s.t. $1 \leq \alpha < \infty$. Well, $\alpha = \frac{3}{2} \Rightarrow$ existence and more. How strong? C-R $\Rightarrow$ strong. We use the Sharp Trace Theorem again. This time

$f \to f|_{\partial\Omega}$ and $\exists\Omega$ and $\partial\Omega$ in $\mathbb{C}$.

$$H^{\frac{3}{2}}(\Omega) \to H'(\partial\Omega)$$

$f \subset C^{\infty}(\Omega) \cap H^{\frac{3}{2}}(\Omega)$. This mapping is continuous linear operator. Hence, it is onto. $\Rightarrow \alpha = 1$ which meets C-R Equations condition.

Hence, $\alpha = \frac{3}{2}$ is strong as $\alpha = 1$ is clearly strong. Using similar Sharp-Trace Theorem analysis on other non-integer Rationals in α for $\alpha : 1 \leq \alpha \leq 2$ we would see other rational order α for functions f or u, v, to also be strong or at least strong enough. □

Applications include the Complex or Real Functional Advection Differential Equations or FADE.

Example 2.4.

$$\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial C(x,t)}{\partial x} + Q\frac{\partial^{\alpha} C(x,t)}{\partial x^{\alpha}}$$

α is the stability or indicator of turbulence. Fourier Transforms can be used to solve.

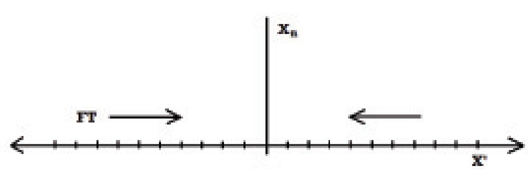
3. Appendix

Theorem 3.1 (Sharp Trace Theorem)**.** *$\exists$ a half space Ω in $\mathbb{R}^n$. $\exists$ a boundary of a half space $\partial\Omega$, also in $\mathbb{R}^n$. Let a function $u \to u_{\partial\Omega}$ mapping exists. Using Sobolev space $H(\Omega)$: we set*

$$H'(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

This mapping for $u \to u|_{\partial\Omega}$ extends a unique continuous linear operator so that it shows the function u is onto. $u \in C^{\infty}(\Omega) \cap H'(\Omega)$.

Proof. $\exists$ partitions of unity $u(x', x_n)$ s.t. defined $x_n > 0$



Let $u \in C^{\infty}(\bar{\Omega}) \cap H'(\bar{\Omega})$ be dense.

$$\hat{u}(\xi', x_n) = c\int_{\mathbb{R}^n} e^{i\xi' x'} u(x', x_n)dx', \text{(FT to } x' \text{ only)}$$

$$\frac{d}{dx_n}|\hat{u}|^2 = 2Re\left(\hat{u}\frac{\partial \hat{u}}{\partial x_n}\right)$$

then integrate over / from 0 to ∞ or $\int_0^\infty$

$$\Rightarrow = -|\hat{u}(\xi', 0)|^2 = 2Re\int_0^\infty \hat{u}\frac{d\hat{u}}{dx_n}dx_n$$

$$|\hat{u}(\xi', 0)|^2 \leq c\int_0^\infty A|\hat{u}|^2 + \left|\frac{\partial \hat{u}}{\partial x_n}\right|^2 \frac{1}{A}dx_n.$$

Now, choose $A = [1 + |\xi'|^2]^{\frac{1}{2}} = \langle \xi' \rangle$.

$$\therefore \langle \xi' \rangle^{\frac{1}{2}}|\hat{u}(\xi', 0)|^2 \leq \int_0^\infty \langle K\xi' \rangle|\hat{u}|^2 + \left|\frac{\partial \hat{u}}{\partial x_n}\right|^2 dx_n \quad (8)$$

Also, integrate (8) over ξ', $\xi' \in \mathbb{R}^{n-1}$

$$\Rightarrow \|u(\cdot, 0)\|^2_{H^{\frac{1}{2}}} \leq \int_0^\infty \int_0^\infty \left|(1 - \Delta_x)^{\frac{1}{2}}u\right|^2 + \left|\frac{\partial u^2}{\partial x_n}\right| dx_n.$$

This is the $\|$ trace of $u\|_{H^{\frac{1}{2}}(\partial\Omega)}$ or

$$\|u_T\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|u\|^2_{H'\Omega}$$

$$\Rightarrow \text{ onto } \rightarrow \text{ by for } H'(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$$

where $T : R \rightarrow Y$, R is in x, and dense(ness) is

$$(1 - \Delta_x)u = (1 - \Delta^5)u \Rightarrow (1 - \Delta^5)u = (1 + |\xi|^2)^u.$$

Next, use Cauchy Sequence. Prove using $\{Tn_k\}$ converges in Y, where we define $tn = \lim T_{n_k}$.

Next we prove the onto of function m mapping f. Let $f \in H^{\frac{1}{2}}(\partial\Omega)$. Define $v(\xi', \xi_n) = v(\xi) = \hat{f}(\xi')\frac{\langle \xi' \rangle}{\langle \xi \rangle}$.

$$\Rightarrow \langle |\xi'| \rangle v(\xi) = \langle \xi' \rangle^{\frac{1}{2}} \cdot \hat{f}(\xi') = \frac{\langle \xi' \rangle^{\frac{1}{2}}}{\langle \xi \rangle}$$

$$\Rightarrow \langle \xi \rangle^2|v(\xi)|^2 \leq \left|\langle \xi' \rangle^{\frac{1}{2}}\hat{f}(\xi)\right|^2 \cdot \frac{\langle \xi' \rangle}{\langle \xi \rangle^2}$$

$$\Rightarrow \int |\langle \xi \rangle| \cdot |v(\xi)|^2 = \int_{\mathbb{R}^n} |\langle \xi' \rangle^{\frac{1}{2}}\hat{f}(\xi')|^2 \otimes \left(\int_{-\infty}^a \frac{\langle \xi' \rangle}{\langle \xi' \rangle + \xi_n^2}d\xi_n d\xi'\right)$$

$$\int |\langle \xi \rangle v(\xi)|^2 d\xi = \pi\int |\langle \xi' \rangle^{\frac{1}{2}}f|^2 d\xi'$$

$$(**) = \pi \|f\|^2_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Let $u(x)$ be a function s.t. $u \in \mathbb{R}^n$. The on a FT (Fourier Transform) is continuous

$$\Rightarrow \|u\|_{H'(\Omega)} \leq c\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} < \infty$$

where as an example:

$$\hat{u}(\xi', 0) = Const.\hat{f}(\xi')$$

$$Lx_n = 0$$

□

Remark 3.2 (on $(**)$)**.**

$$\pi = \int_{-\infty}^{a} \frac{\langle\xi'\rangle}{\langle\xi'\rangle + \xi_n^2} d\xi_n.$$

Theorem 3.3 (General Trace Theorem)**.** *Let Ω be open and the boundary $\partial\Omega \in \mathbb{C}^\infty$ be continuous, s.t.*

$$\left(\frac{\partial}{\partial x}\right)^j_{\partial\Omega} : H^{k,p}(\Omega) \to H^{l,p}(\partial\Omega)$$

where $l = k - j - \frac{1}{p} > 0$ and $p : 1 < p < \infty$. This mapping is continuous and onto.

References

[1] Anglin, *Analytical PDE Theory*, CreateSpace.

[2] Evans, *Partial Differential Equations*, AMS Press.

[3] Gorenflo and Mainardi, *Essentials of Fractional Calculus*, PaPhySto Center, January (2000).

[4] Gorenflo and Mainardi, *Fractional Calculus and Stable Probability Distributions*, Archive of Mechanics(Warszawa), (3)(1998), 377-388.

[5] Kleinz and Osler, *A Childs Garden of Fractional Derivatives*, The College of Mathematics Journal, 31(2)(2000), 82-87.

[6] Podlubny, *The Laplace Transform Method for Linear Differential Equations of the Fractional Order*, Slovak Academy of Sciences, Institute of Experimental Physics, 94(2)(1994).

[7] Strauss, *Partial Differential Equations: An Introduction*, Wiley, (2007).

[8] Wunsch, *Complex Variables with Applications*, Pearson Addison-Wesley, (2005).

[9] Yang, *Local Fractional Functional Analysis and Its Applications*, Asian Academic Publisher Limited, (2011).

[10] Zwillinger, *Handbook of Differential Equations*, Academic Press,(1998).