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Integral Representation of Linear Functionals on Function Spaces: Reisz-Markov Theorem

Research Article

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Abstract: Let μ be a regular Borel measure on X, where X is locally compact Hausdrof space. Let ϕ be defined on $\mathcal{L}(X)$ such that $\phi(f) = \int f d\mu$, where $f \in \mathcal{L}(X)$. $\phi(\alpha f + \beta g) = \int(\alpha f + \beta g)d\mu = \alpha \int f d\mu + \beta \int g d\mu = \alpha\phi(f) + \beta\phi(g)$ then ϕ is a positive linear functional on $\mathcal{L}(X)$. Thus every regular Borel measure defines a positive linear functional on $\mathcal{L}(X)$. Where $\mathcal{L}(X)$ is the α-algebra of μ-measurable functions on X. Here we wish to discuss the converse of this, that for every Positive linear functional ϕ on $\mathcal{L}(X)$ there exist a unique regular Borel measure on X such that $\phi(f) = \int f d\mu, \forall f \in \mathcal{L}(X)$. The result is known as Reisz Markov Theorem.

Keywords: Boral measure, μ-measurable, linear functional, σ-algebra.

1. Introduction

If ϕ be a real valued function of $\mathcal{L}(X)$ such that $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$ for all $f, g \in \mathcal{L}(X)$ and for all $\alpha, \beta \in R$. Then ϕ is called linear functional on $\mathcal{L}(X)$. If $\phi(f) \geq 0$ for $f \geq 0$ then ϕ is called a Positive Linear Functional on $\mathcal{L}(X)$. It is easy to say that a positive linear functional is monotone.

Definition 1.1. *Let $A \subset X$ and $f \in \mathcal{L}(X)$. If $C_A \leq f$ then we say A is contained in f and we write $A \subset f$.*

Remark 1.2. *If $f \in \mathcal{L}(X)$ and there exist a set $A \subset X$ such that $A \subset f$ then $f \geq C_A \Rightarrow f \geq 0$.*

Remark 1.3. *Let C is any compact set. Then by Baire Sandwich Theorem we can find an open set V such that $C \subset V$. Then there exist a function $f \in \mathcal{L}(X)$ such that $f = 1$ on C and $f = 0$ on $X - V$, $0 \leq f \leq 1$. Clearly $\chi_c \leq f \Rightarrow C \subset f$.Thus given any compact set there exist a function $f \in \mathcal{L}(X)$ such that $C \subset f$.*

Theorem 1.4. *Let ϕ be Positive linear functional on $\mathcal{L}(X)$. For a compact set C we define $\lambda(C) = \inf\{\phi(f)/C \subset f \in \mathcal{L}(X)\}$, then λ is a regular content.*

Proof.

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(1) Let C be any compact set and f be any function in $\mathcal{L}(X)$ such that $C \subset f$. Then $f \geq 0 \Rightarrow \phi(f) \geq 0 \Rightarrow \inf\{\phi(f)\} \geq 0 \Rightarrow \lambda(C) \geq 0$. By remark (1.3) there exist a function $g \in \mathcal{L}(X)$ such that $C \subset g \Rightarrow \lambda(C) \leq \phi(g)$ and ϕ is real valued. Hence $0 \leq \lambda(C) < \infty$ for every compact set C.

(2) Let C and D be two compact sets such that $C \subset D$. Let f be any function in $\mathcal{L}(X)$ such that $D \subset F$. Then $\chi_c \leq \chi_D \leq f \Rightarrow C \subset f \Rightarrow \lambda(C) \leq \phi(f) \Rightarrow \lambda(C) \leq \inf\{\phi(f)\} \Rightarrow \lambda(C) \leq \lambda(D)$ proves that λ Is monotone.

(3) Let C and D be any two compact sets and f and g be two functions in $\mathcal{L}(X)$ such that $C \subset f$ and $D \subset g$. $\chi_{c \cup D} = \chi_c + \chi_D - \chi_{c \cap D} \leq \chi_c + \chi_D \leq f + g \Rightarrow C \cup D \subset f + g$. Since $\chi_{A \cap B} \leq \chi_A + \chi_B$ for all A and B it follows that $C \cup D \subset f + g \Rightarrow \lambda(C \cup D) \leq \phi(f+g) \Rightarrow \lambda(C \cup D) \leq \phi(f) + \phi(g) \Rightarrow \lambda(C \cup D) \leq \inf\{\phi(f)\} + \inf\{\phi(g)\} \Rightarrow \lambda(C \cup D) \leq \lambda(C) + \lambda(D)$ Shows that λ is sub-additive.

(4) Let C and D be any disjoint compact sets. Let h be any functional in $\mathcal{L}(X)$ such that $C \cup D \subset h$. Let U and V be open disjoint sets such that $C \subset D$ and $D \subset V$. Let $f \in \mathcal{L}(X)$ such that $f = 1$ on C and $f = 0$ on $X - U$ and $0 \leq f \leq 1$. Let $g \in \mathcal{L}(X)$ such that $g = 1$ on D and $g = 0$ on $X - V$ & $0 \leq g \leq 1$. Since U and V are disjoint it follows immediately that

$$0 \leq f + g \leq 1 \Rightarrow 0 \leq h(f+g) \leq h \tag{1}$$

From $C \subset h$ and $C \subset f$ we get $C \subset hf \Rightarrow \lambda(C) \leq \phi(hf)$. By the same argument we get $\lambda(D) \leq \phi(hg) \Rightarrow \lambda(C) + \lambda(D) \leq \phi(hf) + \phi(hg) = \phi(hf + hg) \leq \phi(h)$. From (1) $\Rightarrow \lambda(C) + \lambda(D) \leq \inf\{\phi(h)\} \Rightarrow \lambda(C) + \lambda(D) \leq \lambda(C \cup D)$. From (3) we have $\lambda(C) + \lambda(D) \geq \lambda(C \cup D)$. Thus $\lambda(C) + \lambda(D) = \lambda(C \cup D)$ shows that λ is a content.

(5) Let C be ant compact set and $\epsilon > 0$, by the definition of infimum of λ we can find a function $f \in \mathcal{L}(X)$, such that $C \subset f$ and $\phi(f) < \lambda(C) + \epsilon$. As $\lambda \geq 0$ we have $0 < \lambda(C) + \epsilon < \lambda(C) + 2\epsilon \Rightarrow 0 < \frac{\lambda(C)+\epsilon}{\lambda(C)+2\epsilon} < 1$. Let $\alpha \in R$ such that $\frac{\lambda(C)+\epsilon}{\lambda(C)+2\epsilon} < \alpha < 1$. Define $U = \{f > \alpha\}$ and $D = \{f \geq \alpha\}$, then U is open and D is compact. Let $x \in C$ then $f(x) \geq \chi_c(x) = 1 > \alpha \Rightarrow f(x) > \alpha \Rightarrow x \in U$. Thus $C \subset U$ & $U \subset D \Rightarrow C \subset U \subset D \Rightarrow C \prec D$ and $D \subset \frac{1}{\alpha} f \Rightarrow \lambda(D) \leq \phi\left(\frac{1}{\alpha} f\right) = \frac{1}{\alpha}\phi(f) \leq \frac{1}{\alpha}\lambda(C) + \epsilon < \lambda(C) + 2\epsilon \Rightarrow \lambda(D) < \lambda(C) + 2\epsilon$. Shows that λ is regular.

□

Definition 1.5. *Let ϕ be any positive linear functional on $\mathcal{L}(X)$, Let λ be defined on compact sets by $\lambda(C) = \inf\{\phi(f)/c \subset f \in \mathcal{L}(X)\}$ then λ is a regular content.Then this λ is known as the content induced by ϕ.*

2. Main Result

Theorem 2.1. *Let ϕ be any positive linear functional on $\mathcal{L}(X)$, then there exist a unique regular Borel measure μ on X such that $\phi(f) = \int f d\mu$, $\forall f \in \mathcal{L}(X)$.*

For the proof of the theorem we are required the following Lemmas.

Lemma 2.2. *Let ϕ be any positive linear functional on $\mathcal{L}(X)$, λ is the content induced by ϕ. And μ is the Borel measure induced by λ. Let C be any compact set and $\epsilon > 0$ then there exist a function $f \in \mathcal{L}(X)$ such that $c \subset f$ and $\phi(f) \leq \int f d\mu + \varepsilon$.*

Proof. By definition of λ we can find a function $f \in \mathcal{L}(X)$ such that $c \subset f$ and $\phi(f) < \lambda(c) + \varepsilon$.

$$\begin{aligned} &\Rightarrow \phi(f) < \mu(c) + \varepsilon && \text{[Because } \mu \text{ is extension of } \lambda \text{ as } \lambda \text{ is regular]} \\ &= \int \chi_c d\mu + \varepsilon \\ &\leq \int f d\mu + \varepsilon. && \text{[Because } \chi_c \leq f\text{]} \end{aligned}$$

$\square$

Lemma 2.3. *Let ϕ be any positive linear functional on $\mathcal{L}(X)$, g be a function which is finite linear combination of characteristic functions of mutually disjoint compact G_δ sets, $f \in \mathcal{L}(X)$ and $0 \leq g \leq f$ then $\int g d\mu \leq \phi(f)$ where μ is the regular Borel measure induced by λ and λ is the regular content induced by ϕ.*

Proof. Let $g = \sum_{i=1}^{n} \alpha_i \chi_{D_i}$, $\alpha_i > 0$, where D_i are mutually disjoint compact G_δ sets. We can find disjoint open sets $v_1, v_2, \ldots, v_n$ such that $D_i \subset v_i$. Let $h_i \in \mathcal{L}(X)$ such that $h_i = 1$ on D_i & $h_i = 0$ on $X - v_i$. It is easy to see that $\alpha_i \chi_{D_i} \leq h_i g$ and $h_i g \leq h_i f \Rightarrow \alpha_i \chi_{D_i} \leq h_i f \Rightarrow \chi_{D_i} \leq \frac{1}{\alpha_i} h_i f \Rightarrow D_i \subset \frac{1}{\alpha_i} h_i f \Rightarrow \lambda(D_i) \leq \phi\left(\frac{1}{\alpha_i} h_i f\right) = \frac{1}{\alpha_i}\phi(h_i f)$

$$\begin{aligned} &\Rightarrow \alpha_i \lambda(D_i) \leq \phi(h_i f) \Rightarrow \alpha_i \mu(D_i) \leq \phi(h_i f) && [\mu \text{ is an extension of } \lambda] \\ &\Rightarrow \sum_{i=1}^{n} \alpha_i \mu(D_i) \leq \sum_{i-1}^{n} \phi(h_i f) \Rightarrow \int g d\mu \leq \phi(\sum_{i=1}^{n} h_i f) = \phi(f \sum_{i=1}^{n} h_i) \leq \phi(f) && [\text{because } f(\sum_{i=1}^{n} h_i) \leq f] \end{aligned}$$

$\square$

Lemma 2.4. *Let ϕ be any positive linear functional on $\mathcal{L}(X)$, $f \in \mathcal{L}(X)$ and $f \geq 0$ then $\int f d\mu \leq \phi(f)$, where μ is the regular Borel measure is induced by λ and λ is the regular content induced by ϕ.*

Proof. Let v be the Baire measure restriction of μ then $f \in l^{|}(v)$. Let S^* denote the class of functions which are finite linear combination of characteristics function of mutually disjoint compact G_δ sets. As S^* is dense in $l^{|}(v)$. Hence for every natural number n there exist $g_n \in S^*$ such that$\|g_n - f\| < \frac{1}{n}$ and we can take $0 \leq g_n \leq f$ as $f \geq 0$.
$|\int(g_n - f)d\mu| \leq \int |g_n - f| d\mu = \|g_n - f\| \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} \int(g_n - f)d\mu \to 0 \Rightarrow \int g_n d\mu = \int[(g_n - f) + f]d\mu = \int[g_n - f]d\mu + \int f d\mu$. Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} \int g_n d\mu = \int f d\mu$, from lemma (2.3) $\Rightarrow \int g_n d\mu \leq \phi(f) \Rightarrow \lim_{n\to\infty} \int g_n d\mu \leq \phi(f) \Rightarrow \int f d\mu \leq \phi(f)$. $\square$

Lemma 2.5. *Let ϕ be any positive linear functional on $\mathcal{L}(X)$ and λ is the content induced by ϕ and μ the regular Boral measure induced by λ. Let $f \in \mathcal{L}(X)$ and $0 \leq f \leq 1$ then $\int f d\mu = \phi(f)$.*

Proof. As f is zero outside a compact set, Let C be a compact set such that $f = 0$ on $X - C$. Let $\epsilon > 0$. By lemma (2.2) there exist a function $g \in \mathcal{L}(X)$ such that $c \subset f$ and $\phi(g) \leq \int g d\mu + \epsilon$.
It is easy to verify that $f \leq g \Rightarrow g - f \geq 0$. Hence by lemma (2.4) we get $\int(g - f)d\mu \leq \phi(g - f) \Rightarrow \int g d\mu - \int f d\mu \leq \phi(g) - \phi(f) \Rightarrow \phi(f) - \int f d\mu \leq \phi(g) - \int g d\mu \Rightarrow \phi(f) - \int f d\mu < \epsilon$, taking $\epsilon \to 0$, we get $\phi(f) - \int f d\mu < 0 \Rightarrow$

$$\phi(f) \leq \int f d\mu \tag{2}$$

By lemma (2.4) we have

$$\int f d\mu \leq \phi(f) \tag{3}$$

Thus we have from (2) and (3), $\phi(f) = \int f d\mu$. Proved. □

Proof of Theorem (2.1) as follows.

Proof. Let λ is the regular content induced by ϕ and μ be the regular Borel measure is induced by λ.

Case(1): Let $f \in \mathcal{L}(X)$, $f \geq 0$, Since $f = 0$ outside a compact set and f is continuous on X, it follows that f is bounded on X. Let $M > 0$ be a constant such that $f \leq M$ on X. This gives $0 \leq f \leq M \Rightarrow 0 \leq \frac{1}{M} f \leq 1$ on X. Define $h = \frac{1}{M} f$, then $0 \leq h \leq 1$ & $h \in \mathcal{L}(X)$ from lemma (2.5) we obtain $\phi(h) = \int h d\mu \Rightarrow \phi\left(\frac{1}{M} f\right) = \int \frac{1}{M} f d\mu \Rightarrow \phi(f) = \int f d\mu$.

Case(2): Suppose f is any function in $\mathcal{L}(X)$, then f^+, f^- are non-negative functions belonging to $\mathcal{L}(X)$, then by case(1) $\phi(f^+) = \int f^+ d\mu$ and $\phi(f^-) = \int f^- d\mu \Rightarrow \phi(f^+) - \phi(f^-) = \int f^+ d\mu - \int f^- d\mu \Rightarrow \phi(f^+ - f^-) = \int f d\mu \Rightarrow \phi(f) = \int f d\mu$.

Uniqueness: Assume that μ_1 and μ_2 be two regular Borel measures such that $\phi(f) = \int f d\mu_1$ and $\phi(f) = \int f d\mu_2$ for all $f \in \mathcal{L}(X) \Rightarrow \int f d\mu_1 = \int f d\mu_2$ for all $f \in \mathcal{L}(X) \Rightarrow \mu_1 = \mu_2$. This proves the theorem. □

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