Exponentiated Flexible Weibull Extension Distribution

Research Article

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Abstract: In this paper, a new three parameter model is introduced. We called it the exponentiated flexible Weibull extension (EFW) distribution. Several properties of this distribution have been discussed. The maximum likelihood estimators of the parameters are derived. Two real data sets are analyzed using the new model, which show that the new model fits the data better than some other very well known models.

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1. Introduction

The Weibull distribution [16] is often used in the modeling of lifetimes of components of engineering applications, physical systems and many different fields. In previous years, many authors provided many extensions for the Weibull distribution and their applications. Mudholkar and et al. [9] proposed a three parameter model by exponentiating the Weibull Distribution and called it the exponentiated Weibull distribution. A three parameter modified Weibull extension is proposed by Xie et al. [15]. Sarhan et al. [13] has defined a new four parameter distribution referred to as exponentiated modified Weibull extension distribution by exponentiating the modified Weibull extension distribution which discussed by Xie et al. Bebbington et al. [3] introduced a new two parameter distribution referred to as a flexible Weibull extension, which has a hazard function that can be increasing, decreasing or bathtub shaped. A flexible Weibull extension distribution has cumulative distribution function (cdf) given by

$$F(x) = \left[1 - e^{-e^{\alpha x - \beta / x}} \right], \quad x > 0, \tag{1}$$

and its probability density function (pdf) takes the following form

$$f(x) = (\alpha + \frac{\beta}{x^2}) e^{\alpha x - \beta / x} e^{-e^{\alpha x - \beta / x}}, x > 0. \tag{2}$$

In this paper we propose a new three parameters model by exponentiating the flexible Weibull extension distribution as was done for the exponentiated weibull (EW) distribution by Mudholkar et al. We referred to it by the exponentiated flexible Weibull extension (EFW) distribution.

The paper is organized as follows. In Section 2, we present the EFW distribution, and provide its cumulative distribution function, the probability density function , the survival function and the hazard function. Some statistical properties such

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as the quantile , the median, the mode and the moments are obtained in Section 3. Section 4 discusses the distribution of the order statistics. Section 5 obtains the parameter estimation using MLE method. In Section 6 a numerical result are obtained by using two real data sets. Finally, a conclusion for the results is given in Section 7.

2. Exponentiated Flexible Weibull Extension Distribution

In this section, we introduce the exponentiated flexible Weibull extension distribution.

2.1. EFW Specifications

A non-negative random variable X has the EFW distribution with three parameters $\Omega = (\alpha, \beta, \theta)$, say EFW(Ω) if its cumulative distribution function is given by the following form

$$F(x) = \left[1 - e^{-e^{\alpha x - \beta / x}}\right]^{\theta}, \quad \alpha, \beta, \theta > 0, \ x > 0. \tag{3}$$

The two parameters α and β are scale parameters but θ is shape parameters. Since the cdf of EFW is in closed form, we can use it to generating simulated data by using the following formula

$$x = \frac{1}{2\alpha} \left\{ \left[\ln(-\ln(1 - U^{\frac{1}{\theta}}))\right] + \sqrt{\left[\ln(-\ln(1 - U^{\frac{1}{\theta}}))\right]^2 + 4\alpha\beta} \right\},$$

where U is a random variable which follows a standard uniform distribution on $(0, 1)$ interval.

The density function corresponding to (3) is

$$f(x) = \theta(\alpha + \frac{\beta}{x^2})e^{\alpha x - \beta / x}e^{-e^{\alpha x - \beta / x}} \left[1 - e^{-e^{\alpha x - \beta / x}}\right]^{\theta - 1}, \quad x > 0. \tag{4}$$

2.2. Survival and Hazard Rate Functions

If $X \sim$EFW(Ω), then the survival function and the hazard rate function of X are given respectively by

$$S(x) = 1 - F(x) = 1 - \left[1 - e^{-e^{\alpha x - \beta / x}}\right]^{\theta} \tag{5}$$

and

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta(\alpha + \frac{\beta}{x^2})e^{\alpha x - \beta / x}e^{-e^{\alpha x - \beta / x}} \left[1 - e^{-e^{\alpha x - \beta / x}}\right]^{\theta - 1}}{1 - \left[1 - e^{-e^{\alpha x - \beta / x}}\right]^{\theta}}. \tag{6}$$
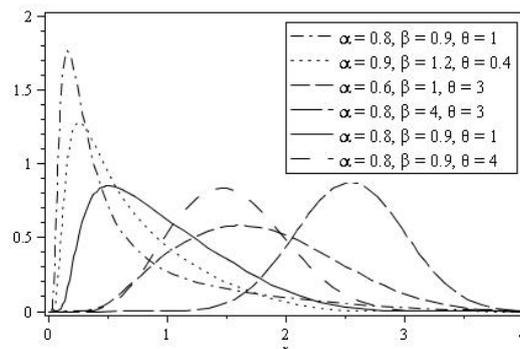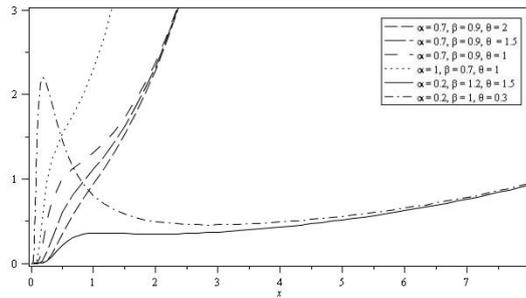

Figure 1. The pdf of the EFW distribution at different values of its parameters

Figure 2. The hazard function of the EFW distribution at different values of its parameters.

3. Statistical Properties

In this section, we will derive some of statistical properties for the EFW, specially moments, modes, quantiles and median.

3.1. Quantile and Median of EFW

In this subsection, we will present the forms of the quantile, the mode and the median of EFW as closed forms. The quantile x_q of the EFW(Ω) is given by

$$x_q = \frac{1}{2\alpha} \left\{ \left[\ln(-\ln(1-q^{\frac{1}{\theta}})) \right] + \sqrt{\left[\ln(-\ln(1-q^{\frac{1}{\theta}})) \right]^2 + 4\alpha\beta} \right\}, \ 0 < q < 1. \tag{7}$$

Sitting $q = \frac{1}{2}$ in (7), we get the median of EFW(Ω) distribution as

$$Med(X) = \frac{1}{2\alpha} \left\{ \left[\ln(-\ln(1 - (\frac{1}{2})^{\frac{1}{\theta}})) \right] + \sqrt{\left[\ln(-\ln(1 - (\frac{1}{2})^{\frac{1}{\theta}})) \right]^2 + 4\alpha\beta} \right\}. \tag{8}$$

3.2. The Mode

In this subsection, we will derive the mode of the EFW (Ω) distribution by derivation its pdf with respect to x and equate it to zero. The mode is the solution the following equation with respect to x

$$(\alpha + \frac{\beta}{x^2}) \left[\frac{-2\beta}{x^3(\alpha + \frac{\beta}{x})^2} - e^{\alpha x - \beta/x} \left(1 - \frac{1}{e^{e^{\alpha x - \beta/x}} - 1} \right) + 1 \right] = 0. \tag{9}$$

It is not possible to get an analytic solution in x to (9) in the general case. It has to be obtained numerically by using methods such as fixed-point or bisection method.

3.3. The Moments

In this subsection, we will derive the rth moments of the EFW (Ω) distribution as infinite series expansion.

Theorem 3.1. *If $X \sim EFW(\Omega)$, then the rth moment of X is given by*

$$\mu^{(r)} = \theta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{\theta - 1}{j} \frac{(-1)^{i+j+k}(j+1)^k \beta^i (k+1)^{2i-r} \Gamma(r-i-1)}{k! \ i! \ \alpha^{r-i}}$$
$$\times \left[\frac{(r-i)(r-i-1)}{k+1} + \alpha\beta(k+1) \right]. \tag{10}$$

Proof. The rth moment of the positive random variable X with pdf $f(x)$ is given by

$$\mu^{(r)} = \int_0^\infty x^r f(x; \Omega) dx. \tag{11}$$

Substituting from(4) into (11), we get

$$
\begin{aligned}
\mu^{(r)} &= \theta \int_0^\infty x^r (\alpha + \frac{\beta}{x^2}) e^{\alpha x - \beta/x} e^{-e^{\alpha x - \beta/x}} \left[1 - e^{-e^{\alpha x - \beta/x}}\right]^{\theta - 1} dx. \\
&= \theta\alpha \int_0^\infty x^r e^{\alpha x - \beta/x} e^{-e^{\alpha x - \beta/x}} \left[1 - e^{-e^{\alpha x - \beta/x}}\right]^{\theta - 1} dx \\
&\qquad\qquad + \theta\beta \int_0^\infty x^{r-2} e^{\alpha x - \beta/x} e^{-e^{\alpha x - \beta/x}} \left[1 - e^{-e^{\alpha x - \beta/x}}\right]^{\theta - 1} dx.
\end{aligned}
$$

Let

$$I_1 = \int_0^\infty x^r e^{\alpha x - \beta/x} e^{-e^{\alpha x - \beta/x}} \left[1 - e^{-e^{\alpha x - \beta/x}}\right]^{\theta - 1} dx$$

and

$$I_2 = \int_0^\infty x^{r-2} e^{\alpha x - \beta/x} e^{-e^{\alpha x - \beta/x}} \left[1 - e^{-e^{\alpha x - \beta/x}}\right]^{\theta - 1} dx.$$

Then

$$\mu^{(r)} = \theta\alpha I_1 + \theta\beta I_2 \tag{12}$$

Since $0 < e^{-e^{\alpha x - \beta/x}} < 1$ for $x > 0$, we have

$$\left[1 - e^{-e^{\alpha x - \beta/x}}\right]^{\theta - 1} = \sum_{j=0}^\infty \binom{\theta - 1}{j} (-1)^j e^{-j e^{\alpha x - \beta/x}}. \tag{13}$$

Substituting from(13) into I_1, we get

$$I_1 = \sum_{j=0}^\infty \binom{\theta - 1}{j} (-1)^j \int_0^\infty x^r e^{\alpha x - \beta/x} e^{-(j+1) e^{\alpha x - \beta/x}} dx.$$

Using the series expansion of $e^{-(j+1) e^{\alpha x - \beta/x}}$, one gets

$$I_1 = \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{\theta - 1}{j} \frac{(-1)^{j+k} (j+1)^k}{k!} \int_0^\infty x^r e^{\alpha(k+1)x} e^{-\frac{\beta(k+1)}{x}} dx.$$

Using the series expansion of $e^{-\frac{\beta(k+1)}{x}}$, we have

$$I_1 = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{i=0}^\infty \binom{\theta - 1}{j} \frac{(-1)^{j+k+i} (j+1)^k \beta^i (k+1)^i}{k!\ i!} \int_0^\infty x^{r-i} e^{\alpha(k+1)x} dx.$$

By using the definition of gamma function in the form

$$\Gamma(z) = x^z \int_0^\infty e^{xt} t^{z-1} dt, \quad z, x > 0,$$

we have

$$I_1 = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{i=0}^\infty \binom{\theta - 1}{j} \frac{(-1)^{j+k+i} (j+1)^k \beta^i (k+1)^{2i-r-1}}{k!\ i!\ \alpha^{r-i+1}} \Gamma(r - i + 1). \tag{14}$$

Similary, we can obtain as follows

$$I_2 = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{i=0}^\infty \binom{\theta - 1}{j} \frac{(-1)^{j+k+i} (j+1)^k \beta^i (k+1)^{2i-r+1}}{k!\ i!\ \alpha^{r-i-1}} \Gamma(r - i - 1). \tag{15}$$

Substituting from (14) and (15) into (12) ,we find (10), which completes the proof. □

4.   Order Statistics

In this section, we present closed form expressions for the pdfs of the ith order statistic of the EFW distribution.

Let $X_1, X_2, ..., X_n$ be a simple random sample of size n from EFW distribution with cumulative distribution function $F(x; \Omega)$ and probability density function $f(x; \Omega)$ given by (3) and (4), respectively. Let $X_{1:n} \leq X_{2:n} \leq ...... \leq X_{2:n}$ denote the order statistics obtained from this sample. The probability density function of $X_{i:n}$ is given by

$$f_{i:n}(x; \Omega) = \frac{1}{B(i, n-i+1)} [F(x; \Omega)]^{i-1} [1 - F(x; \Omega)]^{n-1} f(x; \Omega), i = 1, 2, .., n, \tag{16}$$

where $B(.,.)$ is the beta function. Since $0 < F(x; \Omega) < 1$ for $x > 0$, by using the binomial series for $[1 - F(x; \Omega)]^{n-1}$, we can write (16) in the following form

$$f_{i:n}(x; \Omega) = \frac{1}{B(i, n-i+1)} f(x; \Omega) \sum_{k=0}^{n-i} \binom{n-i}{k} (-1)^k [F(x; \Omega)]^{i+k-1}. \tag{17}$$

Substituting from (3) and (4) into (17), we get

$$f_{i:n}(x; \alpha, \beta, \theta) = \sum_{k=0}^{n-i} \frac{(-1)^k n!}{k!(n-i-k)!(i-1)!(i+k)} f(x; \alpha, \beta, (i+k)\theta). \tag{18}$$

Thus $f_{i:n}(x; \alpha, \beta, \theta)$ defined in (18) is the weighted average of the EFW distribution with different shape parameters.

5.   Estimation and Inference

In this section, we discuss the estimation of the model parameters by using the method of maximum likelihood. Also the asymptotic confidence intervals of these parameters will be derived.

5.1.   Maximum Likelihood Estimators

We will derive the maximum likelihood estimators(MLEs) of the unknown parameters α, β and θ. Assume that $x_1, x_2, ..., x_n$ be a random sample of size n from EFW(Ω), then the likelihood function l of this sample is

$$l = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \theta). \tag{19}$$

Substituting from (4) into (19), we get

$$l = \prod_{i=1}^{n} \left\{ \theta(\alpha + \frac{\beta}{x_i^2}) e^{\alpha x_i - \beta/x_i} e^{-e^{\alpha x_i - \beta/x_i}} \left[1 - e^{-e^{\alpha x_i - \beta/x_i}} \right]^{\theta-1} \right\}.$$

The log-likelihood function $L = ln(l)$ is given by

$$L = n \ln(\theta) + \alpha \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} e^{\alpha x_i - \beta/x_i} + \sum_{i=1}^{n} \ln(\alpha + \frac{\beta}{x_i^2})$$

$$+ (\theta - 1) \sum_{i=1}^{n} \ln(1 - e^{-e^{\alpha x_i - \beta/x_i}}). \tag{20}$$

The first partial derivatives of log of the likelihood L with respect to θ, α and β are obtained as follows

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n} \ln(1 - e^{-e^{\alpha x_i - \beta/x_i}}), \\
\frac{\partial L}{\partial \alpha} &= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{\alpha x_i - \beta/x_i} + \sum_{i=1}^{n} \frac{x_i^2}{\alpha x_i^2 + \beta} + (\theta - 1) \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i - \beta/x_i}}{e^{e^{\alpha x_i - \beta/x_i}} - 1}
\end{aligned}
$$

and

$$
\frac{\partial L}{\partial \beta} = -\sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \beta/x_i} + \sum_{i=1}^{n} \frac{1}{\alpha x_i^2 + \beta} - (\theta - 1) \sum_{i=1}^{n} \frac{\frac{1}{x_i} e^{\alpha x_i - \beta/x_i}}{e^{e^{\alpha x_i - \beta/x_i}} - 1}.
$$

The normal equations can be obtained by setting the first partial derivatives of L to zero's. That is, the normal equations take the following form:

$$
\frac{n}{\theta} + \sum_{i=1}^{n} \ln(1 - e^{-e^{\alpha x_i - \beta/x_i}}) = 0, \tag{21}
$$

$$
\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{\alpha x_i - \beta/x_i} + \sum_{i=1}^{n} \frac{x_i^2}{\alpha x_i^2 + \beta} + (\theta - 1) \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i - \beta/x_i}}{e^{e^{\alpha x_i - \beta/x_i}} - 1} = 0 \tag{22}
$$

and

$$
-\sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \beta/x_i} + \sum_{i=1}^{n} \frac{1}{\alpha x_i^2 + \beta} - (\theta - 1) \sum_{i=1}^{n} \frac{\frac{1}{x_i} e^{\alpha x_i - \beta/x_i}}{e^{\alpha x_i - \beta/x_i} - 1} = 0. \tag{23}
$$

The normal equations do not have explicit solutions and they have to be obtained numerically. From(21) we can be obtained the MLE of θ for a given α and β as the following form

$$
\overset{\wedge}{\theta} = \frac{-n}{\sum\limits_{i=1}^{n} \ln(1 - e^{-e^{\alpha x_i - \beta/x_i}})}. \tag{24}
$$

Substituting from (24) into (22) and (23), we get the MLE of α and β by solving the following system of two non-linear equations:

$$
\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{\hat{\alpha} x_i - \hat{\beta}/x_i} + \sum_{i=1}^{n} \frac{x_i^2}{\hat{\alpha} x_i^2 + \hat{\beta}} + (\hat{\theta} - 1) \sum_{i=1}^{n} \frac{x_i e^{\hat{\alpha} x_i - \hat{\beta}/x_i}}{e^{e^{\hat{\alpha} x_i - \hat{\beta}/x_i}} - 1} = 0, \tag{25}
$$

$$
-\sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\hat{\alpha} x_i - \hat{\beta}/x_i} + \sum_{i=1}^{n} \frac{1}{\hat{\alpha} x_i^2 + \hat{\beta}} - (\hat{\theta} - 1) \sum_{i=1}^{n} \frac{\frac{1}{x_i} e^{\hat{\alpha} x_i - \hat{\beta}/x_i}}{e^{e^{\hat{\alpha} x_i - \hat{\beta}/x_i}} - 1} = 0. \tag{26}
$$

5.2. Asymptotic Confidence Bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters α, β and θ when $\alpha > 0$, $\beta > 0$ and $\theta > 0$ [17].

The simplest large sample approach is to assume that the MLEs($\hat{\alpha}, \hat{\beta}, \hat{\theta}$) are approximately multivariate normal with mean (α, β, θ) and covariance matrix I_0^{-1}, see[6], where I_0^{-1} is the inverse of the observed information matrix which defined by

$$
I_0^{-1} = -\begin{pmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \beta} & \frac{\partial^2 L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 L}{\partial \beta \partial \alpha} & \frac{\partial^2 L}{\partial \beta^2} & \frac{\partial^2 L}{\partial \beta \partial \theta} \\ \frac{\partial^2 L}{\partial \theta \partial \alpha} & \frac{\partial^2 L}{\partial \theta \partial \beta} & \frac{\partial^2 L}{\partial \theta^2} \end{pmatrix}^{-1} = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \beta) & Cov(\hat{\alpha}, \hat{\theta}) \\ Cov(\hat{\beta}, \hat{\alpha}) & Var(\hat{\beta}) & Cov(\hat{\beta}, \hat{\theta}) \\ Cov(\hat{\theta}, \hat{\alpha}) & Cov(\hat{\theta}, \hat{\beta}) & Var(\hat{\theta}) \end{pmatrix}. \tag{27}
$$

The second partial derivatives include in I_0 are given as follows

$$\frac{\partial^2 L}{\partial \theta^2} = -\frac{n}{\theta^2}, \qquad \frac{\partial^2 L}{\partial \theta \partial \alpha} = \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i - \beta/x_i} e^{-e^{\alpha x_i - \beta/x_i}}}{1 - e^{-e^{-\alpha x_i - \beta/x_i}}} \quad ,$$

$$\frac{\partial^2 L}{\partial \theta \partial \beta} = -\sum_{i=1}^{n} \frac{e^{\alpha x_i - \beta/x_i} e^{-e^{-\alpha x_i - \beta/x_i}}}{x_i \left[1 - e^{-e^{\alpha x_i - \beta/x_i}} \right]},$$

$$\frac{\partial^2 L}{\partial \alpha^2} = -\sum_{i=1}^{n} x_i^2 e^{\alpha x_i - \beta/x_i} - \sum_{i=1}^{n} \frac{x_i^4}{(\alpha x_i^2 + \beta)^2} + (\theta - 1)$$
$$\times \sum_{i=1}^{n} \frac{x_i^2 e^{\alpha x_i - \beta/x_i} \left[e^{e^{\alpha x_i - \beta/x_i}} \left(1 - e^{\alpha x_i - \beta/x_i} \right) - 1 \right]}{\left[e^{e^{\alpha x_i - \beta/x_i}} - 1 \right]^2},$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} e^{\alpha x_i - \beta/x_i} - \sum_{i=1}^{n} \frac{x_i^2}{(\alpha x_i^2 + \beta)^2} + (\theta - 1)$$
$$\times \sum_{i=1}^{n} \frac{e^{\alpha x_i - \beta/x_i} \left[1 - e^{e^{\alpha x_i - \beta/x_i}} \left(1 - e^{\alpha x_i - \beta/x_i} \right) \right]}{x_i^2 \left[e^{e^{\alpha x_i - \beta/x_i}} - 1 \right]^2},$$

$$\frac{\partial^2 L}{\partial \beta^2} = -\sum_{i=1}^{n} \frac{1}{x_i^2} e^{\alpha x_i - \beta/x_i} - \sum_{i=1}^{n} \frac{1}{(\alpha x_i^2 + \beta)^2} + (\theta - 1)$$
$$\times \sum_{i=1}^{n} \frac{e^{\alpha x_i - \beta/x_i} \left[1 - e^{e^{\alpha x_i - \beta/x_i}} \left(1 - e^{\alpha x_i - \beta/x_i} \right) \right]}{x_i^2 \left[e^{e^{\alpha x_i - \beta/x_i}} - 1 \right]^2}.$$

We can derive the $(1 - \delta)100\%$ confidence intervals of the parameters α, β and θ by using variance covariance matrix as in the following forms

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})} \quad , \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\beta})} \text{ and } \hat{\theta} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\theta})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$th percentile of the standard normal distribution.

6. Data Analysis

In this section we analyze two real data sets to illustrate that the EFW can be a good lifetime model, comparing with many known distributions such as flexible Weibull, Weibull, linear failure rate, exponentiated Weibull, generalized linear failure rate and generalized linear exponential distributions (FW,W,LFR,EW,GLFR,GLE). We have fitted all selected distributions in each example, we calculated the Kolmogorov Smirnov (K S) distance test statistic and its corresponding p-value, the log-likeihood values (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Bayesian information criterion (BIC) test statistic.

Example 6.1. *The data set in Table 1, gives the lifetimes of 50 devices that were provided by (Aarset, 1987)[1]. The MLEs of the unknown parameters and the Kolmogorov-Smirnov (K-S) test statistic with its corresponding p-value for the seven tested models are given in Table 2. The fitted survival and failure rate functions are shown in Fig. 3. and Fig. 4. respectively. The K-S test statistic value for EFW model is* 0.1433, *and the corresponding p-value is* 0.2617. *We observe that the EFW model has the lowest K-S value and the highest p-value for these data among all the models considered, which means that the new model fits the data better than the other six models.*

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18
18	21	32	36	40	45	46	47	50	55	60	63	63	67	67	67	67
72	75	79	82	82	83	84	84	84	85	85	85	85	85	86	86	

Table 1.   Life time of 50 devices, see Aarset(1987)[1].

The model	MLE of the parameters	KS-value	p-value
$FW(\alpha, \beta)$	$\hat{\alpha} = 0.0122, \hat{\beta} = 0.7002$	0.4386	4.29×10^{-9}
$W(\sigma, c)$	$\hat{\sigma} = 44.913, \hat{c} = 0.949$	0.2397	0.0052
$LFR(a, b)$	$\hat{a} = 0.014, \hat{b} = 2.4 \times 10^{-4}$	0.1955	0.0370
$EW(\sigma, c, \theta)$	$\hat{\sigma} = 91.023, \hat{c} = 4.69, \hat{\theta} = 0.164$	0.1841	0.0590
$GLFR(a, b, \theta)$	$\hat{a} = 0.0038, \hat{b} = 3.04 \times 10^{-4}, \hat{\theta} = 0.533$	0.1620	0.1293
$GLE(a, b, c)$	$\hat{a} = 9.621 \times 10^{-3}, \hat{b} = 4.52 \times 10^{-4}, \hat{c} = 0.73$	0.1598	0.1391
$EFW(\alpha, \beta, \theta)$	$\hat{\alpha} = 0.0147, \hat{\beta} = 0.133, \hat{\theta} = 4.22$	**0.1433**	**0.2617**

Table 2.   The MLEs of the parameters, K-S test statistic and corresponding p-values for Aarset data.

The log-likeihood values (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Bayesian information criterion (BIC) test statistic for the siven tested models are given in Table 3. We observe that the EFW model has the lowest values of L, AIC, CAIC and BIC. This means that the EFW model fits the data better than the other six models.
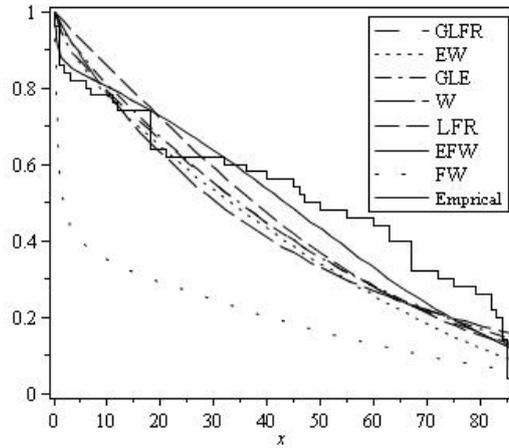
Figure 3.   The empirical and fitted sarvival functions of selected models for Aarset data.

The model	L	AIC	CAIC	BIC
$FW(\alpha, \beta)$	-250.81	505.620	505.88	509.448
$W(\sigma, c)$	-241.002	486.004	486.26	489.828
$LFR(a, b)$	-238.064	480.128	480.38	483.952
$EW(\sigma, c, \theta)$	-235.926	477.852	478.37	483.588
$GLFR(a, b, \theta)$	-233.145	472.290	472.81	478.026
$GLE(a, b, c)$	-229.114	464.228	464.75	469.964
$EFW(\alpha, \beta, \theta)$	$\mathbf{-226.989}$	**459.979**	**460.65**	**465.715**

**Table 3.**   The log-likelihood, AIC, CAIC and BIC values for Aarset data.

Substituting the MLEs of the unknown parameters into (27), we get estimation of the variance covariance matrix as the

following:

$$I_0^{-1} = \begin{pmatrix} 1.365 \times 10^{-6} & 1.141 \times 10^{-5} & 2.85 \times 10^{-4} \\ 1.141 \times 10^{-5} & 2.64 \times 10^{-3} & -1.75 \times 10^{-2} \\ 2.85 \times 10^{-4} & -1.75 \times 10^{-2} & 0.5054 \end{pmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters α, β and θ are given respectively as

$$[0.0125, 0.0170], \quad [0.0325, 0.2339], \quad [2.826, 5.613].$$
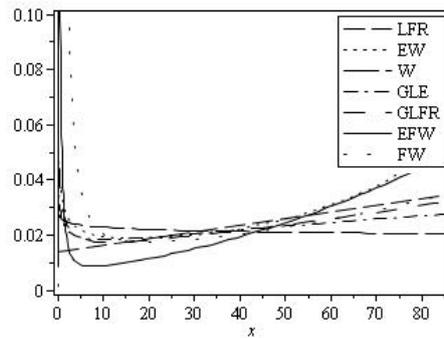
Figure 4. The fitted hazard functions of selected models for Aarset data.

In Fig. 5. we plot the profiles of the log-likelihood function of α, β and c for Aarset data. From Fig. 5. we show that the likelihood equations have a unique solution.
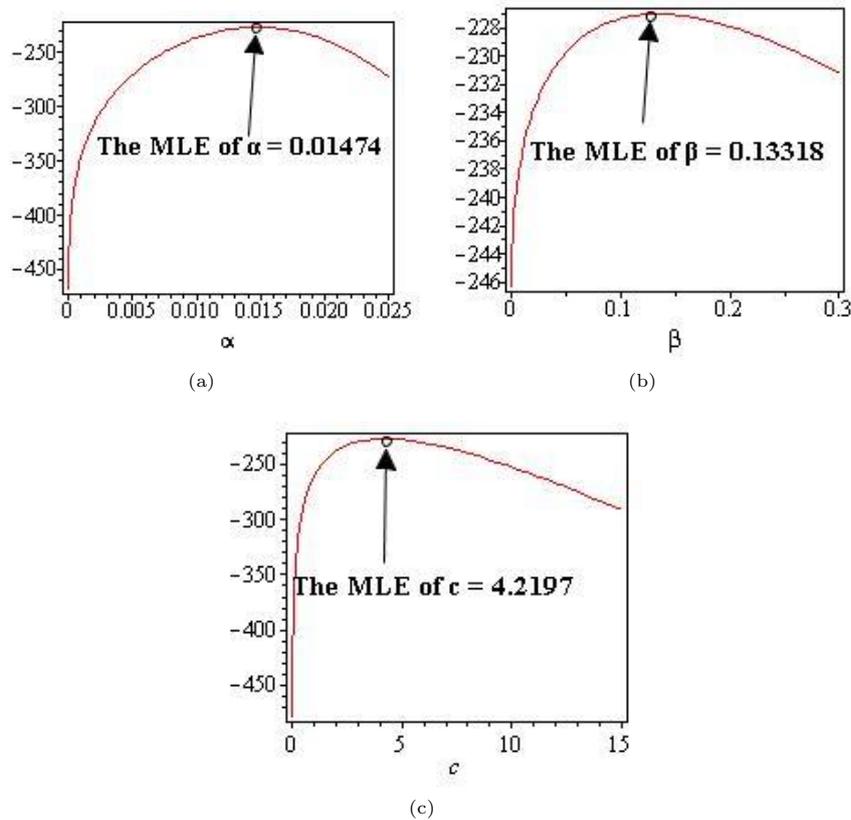
Figure 5. For Aarst data, (a) The profile of log-likelihood function of α. (b) The profile of log-likelihood function of β. (c) The profile of log-likelihood function of c.

Example 6.2. *Table 4, gives the lifetimes of 40 patients of leukemia from one of the government hospitals in Saudi Arabia that were studied by (Abuammoh et al, 1994)[2]. The fitted survival and failure rate functions are shown in Fig. 5. and Fig. 6. respectively. From Table 5 and Table 6, we find that the EFW model has the lowest K-S, L, AIC, CAIC and BIC values for this data among all the models considered. This means that the EFW model fits the data better than the other six models.*

115	181	255	418	441	461	516	739	743	789
807	865	924	983	1024	1062	1063	1165	1191	1222
1222	1251	1277	1290	1357	1369	1408	1455	1478	1549
1578	1578	1599	1603	1605	1696	1735	1799	1815	1852

Table 4. Lifetimes of 40 patients suffering from leukemia, see Abuammoh et al, (1994)[2].

The model	MLE of the parameters	KS-value	p-value
FW(α, β)	$\hat{\alpha} = 8.4 \times 10^{-4}, \hat{\beta} = 1.09 \times 10^3$	0.227	0.0275
W(σ, c)	$\hat{\sigma} = 9.501 \times 10^{-4}, \hat{c} = 4.229 \times 10^{-7}$	0.3585	4.143×10^{-5}
LFR(a, b)	$\hat{a} = 1143.3, \hat{b} = 1.055$	0.2680	0.0050
EW(σ, c, θ)	$\hat{\sigma} = 734.19, \hat{c} = 1.265, \hat{\theta} = 2.973$	0.1321	0.4494
GLFR(a, b, θ)	$\hat{a} = 2.102 \times 10^{-4}, \hat{b} = 1.39 \times 10^{-6}, \hat{\theta} = 1.55$	0.1183	0.5884
GLE(a, b, c)	$\hat{a} = 7.59 \times 10^{-5}, \hat{b} = 1.13 \times 10^{-6}, \hat{c} = 1.260$	0.1105	0.6727
EFW(α, β, θ)	$\hat{\alpha} = 8.482 \times 10^{-4}, \hat{\beta} = 33.17, \hat{\theta} = 8.21$	**0.1093**	**0.7373**

Table 5. The MLEs of the parameters, K-S test statistic and corresponding p-values for Abuammoh et al data.
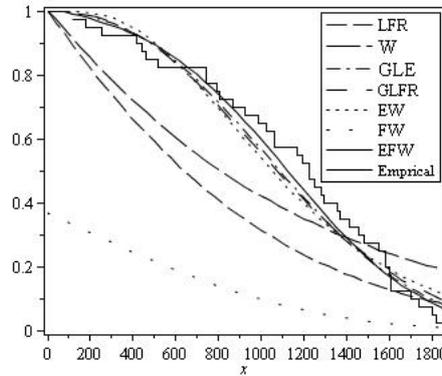

Figure 6. The empirical and fitted sarvival functions of selected models for Abuammoh et al data.

To show that the likelihood equations have a unique solution, we plot the profiles of the log-likelihood function of α, β and c for Abuammoh et al data in Fig. 8.

The model	L	AIC	CAIC	BIC
FW(α, β)	-310.18	624.358	624.68	627.74
W(σ, c)	-319.87	643.747	644.06	647.12
LFR(a, b)	-318.46	640.916	641.24	644.30
EW(σ, c, θ)	-308.93	623.866	624.53	628.93
GLFR(a, b, θ)	-305.34	616.677	617.35	621.75
GLE(a, b, c)	-304.11	614.222	614.89	619.29
EFW(α, β, θ)	$\mathbf{-302.28}$	**610.57**	**611.24**	**615.36**

Table 6. The log-likelihood, AIC, CAIC and BIC values for Abuammoh et al data.
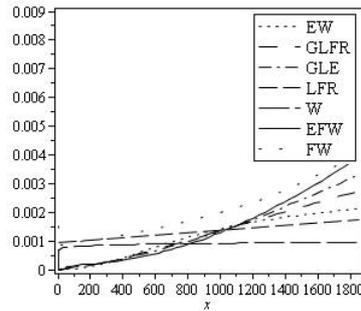
Figure 7.   **The fitted hazard functions of selected models for Abuammoh et al data.**

Substituting the MLEs of the unknown parameters into (27), we get estimation of the variance covariance matrix as the following:

$$I_0^{-1} == \begin{pmatrix} 3.815 \times 10^{-9} & -5.750 \times 10^{-4} & 6.939 \times 10^{-5} \\ -5.750 \times 10^{-4} & 1192.164 & -48.885 \\ 6.939 \times 10^{-5} & -48.885 & 4.2845 \end{pmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters α, β and θ are given respectively as

$$[7.271 \times 10^{-4}, 9.693 \times 10^{-4}], \quad [0, 100.84], \quad [4.1563, 12.270].$$

(a)

(b)

(c)
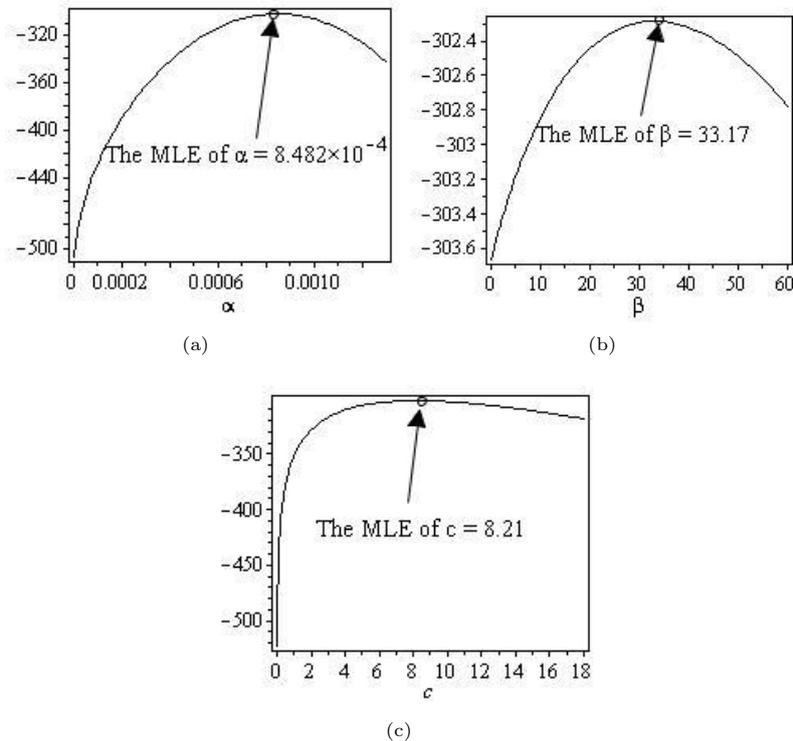
Figure 8. For Abuammoh et al data, (a) The profile of log-likelihood function of α. (b) The profile of log-likelihood function of β. (c) The profile of log-likelihood function of c.

7.   Conclusions

In this paper, we propose a new three parameter model we called it the exponentiated flexible Weibull extension distribution. Some statistical properties of this distribution have been derived and discussed. The quantile, median, and mode of EFW

are derived in closed forms. The distribution of the order statistics are discussed. The maximum likelihood estimators of the parameters are derived and we obtained the observed Fisher information matrix. Two real data sets are analyzed using the new distribution and it is compared with the flexible Weibull, Weibull, linear failure rate, exponentiated Weibull, generalized linear failure rate and generalized linear exponential distributions. It is evident from the comparisons that the new distribution is the best distribution for fitting these data sets compared to other distributions considered here.

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