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Soft Characteristic Interior Ideals in Semigroups

Research Article

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Abstract: Based on the concept of soft int-semigroups and soft ideals, we introduce the concept of soft fuzzy interior ideals and soft characteristic ideals of a semigroup. Also we give characterizations of them by specific soft sets.

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1. Introduction

Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings. Most of our real life problems in engineering, social and medical sciences, economics etc. involve imprecise data and their solution involves the use of mathematical principles based on uncertainty and imprecision. To handle such uncertainties a number of theories have been proposed for dealing with such systems in an effective way.

Some of these are probability, fuzzy sets, intuitionistic fuzzy sets, interval mathematics and rough sets etc. All these theories, however, are associated with an inherent limitation, which is the inadequacy of the parameterization tool associated with these theories. Molodtsov [7] introduced soft sets and established the fundamental results of the new theory. It is a general mathematical tool for dealing with objects which have been defined using a very loose and hence very general set of characteristics. A soft set is a collection of approximate descriptions of an object.

Each approximate description has two parts: a predicate and an approximate value set. In classical mathematics, we construct a mathematical model of an object and define the notion of the exact solution of this model. Usually the mathematical model is too complicated and we cannot find the exact solution. So, in the second step, we introduce the notion of approximate solution and calculate that solution. In the Soft Set Theory, we have the opposite approach to this problem.

The initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in Soft Set Theory makes this theory very convenient and

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easily applicable in practice. Soft set theory has potential applications in many fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Most of these applications have already been demonstrated in Molodtsov's paper [7]. Cagman et al. [2] applied this concept to the theory of groups. They studied the soft int-groups, which are different from the definition of soft groups in [1, 8]. This new approach is based on the inclusion relation and intersection of sets.

It brings the soft set theory, set theory and the group theory together. Some supplementary properties of soft int-groups and normal soft intgroups, analogues to classical group theory and fuzzy group theory are introduced in [4, 5, 9]. Recently, Ideal theory in semigroups based on soft int-semigroup is investigated in [10]. The concept of fuzzy soft interior ideal over an ordered semigroup is investigated in [6].

In this paper we introduce and investigate some properties of soft interior ideals of semigroups based on intersectional sets. We also consider characteristic soft interior ideal.

2. Preliminaries

Let S be a semigroup. By a subsemigroup of S we mean a nonempty subset A of S such that $A^2 \subseteq A$.

Definition 2.1 ([6]). *A subsemigroup A of a semigroup S is called an interior ideal of S if $SAS \subseteq A$.*

Throughout this paper, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U.

Definition 2.2 ([10]). *For any subset A of E, a soft set (α, A) over U is defined to be the set of ordered pairs*

$$(\alpha, A) := \{(x, \alpha(x)) : x \in E, \alpha(x) \in P(U)\}$$

where $\alpha : E \to P(U)$ such that $\alpha(x) = \emptyset$ if $x \notin A$.

The function α is called approximate function of the soft set (α, A).

Definition 2.3 ([10]). *Let $(\alpha, A); (\beta, B)$ be two soft sets. Then, (α, A) is a soft subset of (β, B), denoted by $(\alpha, A)\widetilde{\subseteq}(\beta, B)$, if $\alpha(x) \subseteq \beta(x)$ for all $x \in E$. (α, A) and (β, B) are called soft equal, denoted by $(\alpha, A) = (\beta, B)$, if and only if $\alpha(x) = \beta(x)$ for all $x \in E$.*

Definition 2.4 ([10]). *Let (α, A) and (β, B) be two soft sets. Then, union $(\alpha, A)\widetilde{\cup}(\beta, B)$ and intersection $(\alpha, A)\widetilde{\cap}(\beta, B)$ are defined by*

$$(\alpha \,\widetilde{\cup}\, \beta)(x) = \alpha(x) \cup \beta(x),$$

$$(\alpha \,\widetilde{\cap}\, \beta)(x) = \alpha(x) \cap \beta(x),$$

respectively.

Definition 2.5 ([10]). *A soft set (α, S) in a semigroup S is called a soft int-subsemigroup of S if $\alpha(xy) \supseteq \alpha(x) \cap \alpha(y)$ for all $x, y \in S$.*

Definition 2.6. *A soft int-subsemigroup (α, S) in a semigroup S is called a soft interior ideal of S if $\alpha(xay) \supseteq \alpha(x)$ for all $x, a, y \in S$.*

3. Main Results

In what follows, we take $E = S$, as a set of parameters, which is a semigroup unless otherwise stated. For a nonempty subset A of S, define a map $\chi_A : S \to P(U)$ as follows:

$$\chi_A(x) = \begin{cases} U, & \text{if } x \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then (χ_A, S) is a soft set over U, which is called the characteristic soft set [10]. The following theorem shows that the concept of a soft interior ideal in a semigroup is an extension of an interior ideal.

Theorem 3.1. *Let A be a nonempty subset of a semigroup S. Then, A is an interior ideal of S if and only if (χ_A, S) is a soft interior ideal of S.*

Proof. Assume that A is an interior ideal of S, that is, $SAS \subseteq A$. For any $x, y \in S$, if $x, y \notin A$ then $\chi_A(xy) \supseteq \chi_A(x) \cap \chi_A(y) = \emptyset$. If $x, y \in A$, then $xy \in A$ since A is a subsemigroup of S. Hence $\chi_A(xy) = U = \chi_A(x) \cap \chi_A(y)$. Therefore (χ_A, S) is a soft int-subsemigroup of S. To show that (χ_A, S) is a soft interior ideal of S, let $x, a, y \in S$. If $a \in A$, then $xay \in A$ since A is an interior ideal of S and so $\chi_A(xay) = U = \chi_A(a)$. If $a \notin A$, then $\chi_A(a) = \emptyset \subseteq \chi_A(xay)$. This proves that (χ_A, S) is a soft interior ideal of S.

Conversely, suppose that $(\chi_A, S))$ is a soft interior ideal of S. Let $x, y \in A$, then $\chi_A(x) = \chi_A(y) = U$. Hence $\chi_A(xy) = U = \chi_A(x) \cap \chi_A(y)$ which means that $xy \in A$. Therefore A is a subsemigroup of S. For any $x, y \in S$ and $a \in A$, we have $\chi_A(a) = U = \chi_A(xay)$ since it is a soft interior ideal of S. Hence $xay \in A$. Therefore A is an interior ideal of S. □

For a soft set (α, A) over U and a subset V of U, the V-inclusive set of (α, A), denoted by α^V, is defined to be the set [10],

$$\alpha^V = \{x \in A : V \subseteq \alpha(x)\}.$$

Proposition 3.2. *Let (α, S) be a soft interior ideals of S over U. Then α^V is an interior ideal of S for every $V \in P(U)$, provided $\alpha^V \neq \emptyset$.*

Proof. Let $V \subseteq U$ such that $\alpha^V \neq \emptyset$. Suppose $x, y \in \alpha^V$. Then $\alpha(x) \supseteq V$ and $\alpha(y) \supseteq V$. It follows $\alpha(xy) \supseteq \alpha(x) \cap \alpha(y) \supseteq V$, so that $xy \in \alpha^V$. Thus α^V is a subsemigroup of S. Now, what we need to show that α^V is an interior ideal is $S\alpha^V S \subseteq \alpha^V$. Assume that $x, a, y \in S$. If $a \in \alpha^V$, then $\alpha(a) \supseteq V$. Since (α, S) is a soft interior ideals of S, $\alpha(xay) \supseteq \alpha(a) \supseteq V$ It follows that $xay \in \alpha^V$, that is $S\alpha^V S \subseteq \alpha^V$. Therefore α^V is an interior ideal of S. □

Theorem 3.3. *Let A be an interior ideal of a semigroup S. Then for every $\emptyset \neq V \in P(U)$, there exists a soft interior ideal (α, S) of S such that $\alpha^V = A$.*

Proof. Let A be an interior ideal of S and (α, S) be a soft set in S defined by

$$\alpha(x) = \begin{cases} V, & x \in A; \\ \emptyset, & x \notin A. \end{cases}$$

Let $x, y \in S$. If $x, y \in A$ then $xy \in A$, hence $\alpha(xy) = V = \alpha(x) \cap \alpha(y)$. If $x, y \notin A$ then $\alpha(x) = \emptyset = \alpha(y)$ and so $\alpha(xy) \supseteq \alpha(x) \cap \alpha(y)$. If exactly one of x and y belongs to A then exactly one of $\alpha(x)$ and $\alpha(y)$ is equal to $\emptyset$. Hence $\alpha(xy) \supseteq \alpha(x) \cap \alpha(y) = \emptyset$. Therefore (α, S) is a soft int-subsemigroup of S. Now let $x, \alpha, y \in S$, if $a \in A$ then $xay \in A$ since A is an interior ideal of S. Thus $\alpha(xay) = V = \alpha(x)$. If $a \notin A$ then $\alpha(x) = \emptyset$, and hence $\alpha(xay) \supseteq \emptyset = \alpha(x)$. Therefore (α, S) is a soft interior ideal of S. It is clear that $\alpha^V = A$. □

Theorem 3.4. *Let (α, S) be a soft interior ideal of S and $V_1, V_2 \in P(U)$. Then the two interior ideals $\alpha^{V_1}, \alpha^{V_2}$ (with $V_1 \subset V_2$) are equal if and only if there exist no $x \in S$ such that $V_1 \subseteq \alpha(x) \subset V_2$.*

Proof. Assume that $\alpha^{V_1} = \alpha^{V_2}$, for $V_1 \subset V_2$ and that there exist $x \in S$ such that $V_1 \subseteq \alpha(x) \subset V_2$. Let $y \in \alpha^{V_2} \Rightarrow \alpha(y) \supseteq V_2 \supset \alpha(x) \supseteq V_1$. Then α^{V_2} is a proper subset of α^{V_1}. This is a contradiction.
Conversely, suppose that there is no $x \in S$ such that $V_1 \subseteq \alpha(x) \subset V_2$. Then we have $V_1 \subset V_2$ implies that $\alpha^{V_1} \supseteq \alpha^{V_2}$. If $x \in \alpha^{V_1}$ then $V_1 \subseteq \alpha(x)$. Since $\alpha(x)$ is not a proper subset of V_2, we get $\alpha(x) \supseteq V_2$ or $x \in \alpha^{V_2}$ and hence $\alpha^{V_1} = \alpha^{V_2}$. This completes the proof. □

Let S and T be semigroups. By a homomorphism we mean a mapping $f : S \to T$ satisfying $f(xy) = f(x)f(y)$ for all $x, y \in S$. From now on, $Aut(S)$ will denote the set of all automorphisms of S.

Definition 3.5 ([3]). *An interior ideal A of a semigroup S is called a characteristic interior ideal of S if $f(A) = A$ for all $f \in Aut(S)$.*

Definition 3.6. *A soft interior ideal (α, S) of a semigroup S is called a soft characteristic interior ideal of S if $\alpha(f(x)) = \alpha(x)$ for all $x \in S$ and all $f \in Aut(S)$.*

Theorem 3.7. *If (α, S) is a soft characteristic interior ideal of a semigroup S, then α^V is a characteristic interior ideal of S for all $V \in P(U)$.*

Proof. Let $V \in Im(\alpha)$, where $Im(\alpha) = \{W \subseteq U : \alpha(x) = W, \text{ for } s \in S\}$, and $f \in Aut(S)$. Let $x \in \alpha^V$, since (α, S) is a soft characteristic interior ideal, we have $\alpha(f(x)) = \alpha(x) \supseteq V$. It follows that $f(x) \in \alpha^V$ and hence $f(\alpha^V) \subset \alpha^V$. Now let $x \in \alpha^V$ and $y \in S$ be such that $f(y) = x$. Then $\alpha(y) = \alpha(f(y)) = \alpha(x) \supseteq V$, whence $y \in \alpha^V$. Then $x = f(y) \in f(\alpha^V)$, so that $\alpha^V \subset f(\alpha^V)$. Thus α^V, $V \in Im(\alpha)$, is a characteristic interior ideal of S. □

Theorem 3.8. *Let (α, S) be a soft interior ideal of a semigroup S. If α^V is a characteristic interior ideal of S for every $V \subset U$, then (α, S) is a soft characteristic interior ideal of S.*

Proof. Let $f \in Aut(S)$ and $x \in S$ be such that $\alpha(x) = V$. Then $x \in \alpha^V$ and $x \notin \alpha^W$ for all $W \supset V$. Since $f(\alpha^V) = \alpha^V$ by hypothesis, we get $f(x) \in \alpha^V$ and hence $\alpha(f(x)) \supseteq V$. Let $W = \alpha(f(x))$. If is possible, let $W \supset V$, then $f(x) \in \alpha^W = f(\alpha^W)$. Since f if injective, it follows that $x \in \alpha^W$, which is a contradiction. Hence $\alpha(f(x)) = V = \alpha(x)$, which means that (α, S) is a soft characteristic interior ideal of S. □

Definition 3.9. *Let (α, S) be a soft set over U. For a subset V of U. Define a soft set $(\alpha^{\cap}, S)$ over U by*

$$\alpha^{\cap}(x) = \alpha(x) \cap V, \forall x \in S.$$

Lemma 3.10. *If (α, S) is a soft interior ideal of a semigroup S, then so $(\alpha^{\cap}, S)$.*

Proof. Let $x, a, y \in S$ and let $V \in P(U)$. If $\alpha(x) \subseteq \alpha(y)$, then $\alpha(xy) \supseteq \alpha(x) \cap \alpha(y) = \alpha(x)$. This implies that

$$\alpha^{\cap}(xy) = \alpha(xy) \cap V \supseteq \alpha(x) \cap V = \alpha^{\cap}(x), \text{ and}$$
$$\alpha^{\cap}(y) = \alpha(y) \cap V \supseteq \alpha(x) \cap V = \alpha^{\cap}(x)$$

Hence $\alpha^{\cap}(xy) \supseteq \alpha^{\cap}(x) \cap \alpha^{\cap}(y)$. The argument is similar if $\alpha(x) \supseteq \alpha(y)$. Therefore $(\alpha^{\cap}, S)$ a soft int-subsemigroup of S. Also $(\alpha^{\cap}, S)$ a soft interior ideal since $\alpha(xay) \supseteq \alpha(a)$ implies

$$\alpha^{\cap}(xay) = \alpha(xay) \cap V \supseteq \alpha(a) \cap V = \alpha^{\cap}(a).$$

This completes the proof. □

Let (α, S) be a soft characteristic interior ideal of a semigroup S, and $x \in S$, $f \in Aut(S)$. Then $\alpha(f(x)) = \alpha(x)$, which implies that $\alpha^{\cap}(f(x)) = \alpha(f(x)) \cap V = \alpha^{\cap}(x)$. Thus we have the following theorem.

Theorem 3.11. *If (α, S) is a soft characteristic interior ideal of a semigroup S, then so $(\alpha^{\cap}, S)$.*

Definition 3.12 ([10]). *Let (α, S) be a soft set of S over U, for a subset V of U with $\alpha^V \neq \emptyset$, we define a soft set (α^*, S) in S defined by*

$$\alpha^*(x) = \begin{cases} \alpha(x), & x \in \alpha^V; \\ W, & otherwise. \end{cases}$$

where $W \subset U$ with $W \subset \alpha(x)$.

Theorem 3.13. *If (α, S) is a soft characteristic interior ideal of a semigroup S, then so (α^*, S).*

Proof. From theorem 24 [10], (α^*, S) is a soft int-subsemigroup of S. Let $x, a, y \in S$. If $a \in \alpha^V$, then $xay \in \alpha^V$ since α^V is an interior ideal of S by Proposition 3.2. Hence we have

$$\alpha^*(xay) = \alpha(xay) \supseteq \alpha(a) = \alpha^*(a).$$

If $a \notin \alpha^V$, then $\alpha^*(a) = W$. Thus

$$\alpha^*(xay) \supseteq W = \alpha^*(a).$$

Therefore (α^*, S) is a soft interior ideal of S. Now α^V is a characteristic interior ideal of S by Theorem 3.7. Let $x \in S$, $f \in Aut(s)$. If $x \in \alpha^V$, then $f(x) \in \alpha^V$. It follows that

$$\alpha^*(f(x)) = \alpha(f(x)) = \alpha(x) = \alpha^*(x).$$

If $x \notin \alpha^V$, then $f(x) \notin \alpha^V$ and hence $\alpha^*(f(x)) = W = \alpha^*(x)$. This proves that (α^*, S) is a soft interior ideal of S. □

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