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Symmetry Classifications and Reductions of (2+1)-dimensional Potential Burgers Equation

Research Article

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Abstract: We discuss the symmetries and reductions of the two-dimensional Potential Burgers Equation. We classify the one- and two-dimensional subalgebras of the symmetry algebra which is infinite-dimensional into conjugacy classes under the adjoint action of the symmetry group. Invariance under one-dimensional subalgebras provides the reductions to lower-dimensional partial differential equations (PDEs). Further reductions of these PDEs to second order ordinary differential equations (ODEs) are obtained through invariance under two dimensional subalgebras.

Keywords: A (2+1)-dimensional Potential Burgers Equation, Symmetry algebra, Conjugacy class.

1. Introduction

The Burgers Equation [1, 4, 5] $u_t = 2uu_x + u_{xx}$ has attracted much attention since it was first proposed by " Bateman ". Then Burgers gave some special solutions in 1940. Later on, Cole [2] and Hopf [3] independentely pointed out that any of the solutions of the Heat equation $\xi_t = \xi_{xx}$ can be mapped to a solution of the Burgers Equation

$$u_t + uu_x = \frac{\delta}{2} u_{xx}. \tag{1}$$

is the simplest second order non-linear PDE which balances the effect of non-linear convection and the linear diffusion. Hopf and Cole have shown that (1) may be linearized to the heat equation

$$\phi_t = \frac{\delta}{2} \phi_{xx} \tag{2}$$

via the Cole-Hopf transformation

$$u = -\delta \frac{\phi_x}{\phi}. \tag{3}$$

Here we consider a $(2+1)$-dimensional potential Burgers equation

$$u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy} = 0 \tag{4}$$

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where $A(t)$ and $B(t)$ are the arbitrary functions.

In this paper, a symmetry classification of (4) is presented using the Lie Group method. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [6–8] to successively reduce (4) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian and non-Abelian solvable subalgebras.

2. Derivation of Symmetries

In order to derive the symmetry generators of (4) and obtain the closed form solution for all $A(t)$ and $B(t)$, we consider the one parameter Lie point transformation that leaves (4) invariant. This transformation is given by

$$\bar{x}^i = x^i + \epsilon\, \xi^i(x, y, t, u) + O(\epsilon^2), \quad i = 1, 2, 3, 4. \tag{5}$$

where $\xi^i = \frac{\partial \bar{x}^i}{\partial \epsilon}|_{\epsilon=0}$ defines the symmetry generator associated with (5) given by

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}, \tag{6}$$

In order to determine the four components $\xi,\ \eta,\ \tau$, and ϕ, we prolong V to second order. This prolongation is given by

$$pr^{(2)} = pr^{(1)}V + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}.$$

In the above expression every coefficients of the prolonged generator is a function of $(x,\ y,\ t;\ u)$ can be determined by the formulae

$$\begin{aligned} \phi^i &= D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i} \\ \phi^{ij} &= D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij} \end{aligned}$$

Where D_i represents total derivative and the subscripts of u are the derivatives with respect to the respective coordinates. To proceed with reductions of (4) we now use the symmetry criterion for partial differential equations. For potential Burger's equation this criterion is expressed by the formula $V^{(1)}[u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy}] = 0$ whenever $u_t + uu_x + u_x^2 = A(t)u_{xx} + B(t)u_{yy}$. Using this symmetry criterion with (5) in mind immediately yields

$$\phi^t + 2\phi^x u_x + \phi u_x + u\phi^x - A(t)\phi^{xx} - B(t)\phi^{yy} - A_t u_{xx} - B_t u_{yy} = 0. \tag{7}$$

At this stage, we now calculate the expressions

$$\begin{aligned} \phi^x &= D_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{yx} + \tau u_{tx} \\ &= \phi_x + u_x\phi_u - \xi_x u_x - u_x^2\xi_u - \eta_x u_y - u_y u_x \eta_u - u_t\tau_x - u_t u_x \tau_u, \qquad (8) \\ \phi^t &= D_t(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xt} + \eta u_{yt} + \tau u_{tt} \\ &= \phi_t + u_t\phi_u - \xi_t u_x - u_x u_t \xi_u - \eta_t u_y - u_y u_t \eta_u - u_t\tau_t - u_t^2\tau_u, \qquad (9) \end{aligned}$$

$$\begin{aligned}\phi^{xx} &= D_xD_x(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{xxx}+\eta u_{yxx}+\tau u_{txx},\\ &= D_x(\phi_x+u_x\phi_u-u_{xx}\xi-\xi_x u_x-u_x^2\xi_u-\eta u_{yx}-\eta_x u_y-u_yu_x\eta_u-\\ &\quad \tau u_{tx}-u_t\tau_x-u_tu_x\tau_u)+\xi u_{xxx}+\eta u_{yxx}+\tau u_{txx},\\ &= \phi_{xx}+(2\phi_{xu}-\xi_{xx})u_x-\eta_{xx}u_y-\tau_{xx}u_t+(\phi_u-2\xi_x)u_{xx}+(\phi_{uu}\\ &\quad -2\xi_{xu})u_x^2-3u_xu_{xx}\xi_u-\xi_{uu}u_x^3-2u_{yx}\eta_x-2u_{xy}\eta_x-2u_yu_x\eta_{xu}\\ &\quad -u_yu_{xx}\eta_u-2u_xu_{yx}\eta_u-u_x^2u_y\eta_{uu}-2u_{tx}\eta_x-2u_tu_x\tau_{xu}-2u_xu_{xt}\\ &\quad \tau_u-u_x^2u_t\tau_{uu},\end{aligned} \tag{10}$$

$$\begin{aligned}\phi^{yy} &= D_yD_y(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{xyy}+\eta u_{yyy}+\tau u_{tyy},\\ &= D_y(\phi_y+u_y\phi_u-u_{xy}\xi-\xi_y u_x-u_xu_y\xi_u-\eta u_{yy}-\eta_y u_y-u_y^2\eta_u\\ &\quad -\tau u_{ty}-u_t\tau_y-u_tu_y\tau_u)+\xi u_{xyy}+\eta u_{yyy}+\tau u_{tyy},\\ &= \phi_{yy}+(2\phi_{yu}-\xi_{yy})u_y-\xi_{yy}u_x-\tau_{yy}u_t+(\phi_u-2\eta_y)u_{yy}+(\phi_{uu}-2\eta_{yu})u_y^2\\ &\quad -3u_yu_{yy}\eta_u-\eta_{uu}u_y^3-2u_{yx}\xi_y-2u_yu_x\xi_{yu}-u_xu_{yy}\xi_u-2u_yu_{yx}\xi_u-u_y^2u_x\xi_{uu}-\\ &\quad 2u_{ty}\tau_y-2u_tu_y\tau_{yu}-2u_yu_{yt}\tau_u-u_y^2u_t\tau_{uu}-u_tu_{yy}\tau_u.\end{aligned} \tag{11}$$

The determining Equations are obtained from V are as follows

$$\tau = \tau(t), \tag{12}$$

$$\xi_u=\eta_u = 0, \tag{13}$$

$$\xi_y=\eta_x = 0, \tag{14}$$

$$\phi-\alpha(x,y,t)u-\beta(x,y,t) = 0, \tag{15}$$

$$-\eta_t-2B\phi_{yu}+B\eta_{yy} = 0, \tag{16}$$

$$-\xi_t+2\phi_x+\phi+u\tau_t-u\xi_x-2a\phi_{xu}+A\xi_{xx} = 0, \tag{17}$$

$$-A_t\tau+2A\xi_x-A\tau_t = 0, \tag{18}$$

$$-B_t\tau+2B\eta_y-B\tau_t = 0, \tag{19}$$

$$-2\xi_x+\tau_t+\phi_u = 0, \tag{20}$$

$$\phi_t+u\phi_x-A(t)\phi_{xx}-B(t)\phi_{yy} = 0. \tag{21}$$

3. Symmetry Analysis of $u_t+uu_x+u_x^2-A(t)u_{xx}-B(t)u_{yy}=0$

We consider $A(t)$ and $B(t)$ are arbitrary functions of t.

Thus the determining Equations are

$$\xi=\xi(x,t),\ \eta=\eta(y,t),\ \tau=\tau(t) \text{ and } \phi=\alpha(x,y,t)u+\beta(x,y,t).$$

After some manipulations one find that ξ and η becomes

$$\begin{aligned}\xi &= c_2t+c_1\\ \text{and }\ \eta &= c_3+yc_4.\end{aligned} \tag{22}$$

The remaining equations can then be used to determine τ and ϕ as

$$\begin{aligned} \tau &= c_5 \\ \text{and } \phi &= c_2. \end{aligned} \tag{23}$$

Where $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ are arbitrary constants. Thus, the Lie algebra of infinitesimal symmetries of the Potential Burgers Equation is spanned by the five vector fields

$$\begin{aligned} V_1 &= \partial_x, \\ V_2 &= t\partial_x + \partial_u, \\ V_3 &= \partial_y, \\ V_4 &= y\partial_y, \\ V_5 &= \partial_t. \end{aligned} \tag{24}$$

It is easy to check that the symmetry generators found in (24) form a closed Lie algebra whose commutation relations are given in the following table .

The one-parameter groups $g_i(\epsilon)$ generated by the V_i where $i = 1, 2, 3, 4, 5$ are

$$\begin{aligned} g_1(\epsilon) : (x, y, t; u) &\rightarrow (x + \epsilon, y, t, u)\ , \\ g_2(\epsilon) : (x, y, t; u) &\rightarrow (x + t\epsilon, y, t, u + \epsilon)\ , \\ g_3(\epsilon) : (x, y, t; u) &\rightarrow (x, y + \epsilon, t, u)\ , \\ g_4(\epsilon) : (x, y, t; u) &\rightarrow (x, y + y\epsilon, t, u)\ , \\ g_5(\epsilon) : (x, y, t; u) &\rightarrow (x, y, t + \epsilon, u)\ , \end{aligned}$$

where $exp(\epsilon V_i)\ (x, y, t; u) = (\bar{x}, \bar{y}, \bar{t}; \bar{u})$ and

(i)$g_1,\ g_2,\ g_3$ and g_4 are the space-invariant,

(ii)g_5 is time translation of the equation.

The symmetry generators found in Eq.(24) form a closed Lie Algebra whose commutator table is shown below.

Commutator table

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	0	0
V_2	0	0	0	0	$-V_1$
V_3	0	0	0	V_3	0
V_4	0	0	$-V_3$	0	0
V_5	0	V_1	0	0	0

The commutation relations of the Lie algebra L, determined by $V_1,\ V_2,\ V_3,\ V_4$ and V_5 are shown in the above table

$$[V_2, V_5] = -V_1 \text{ and } [V_5, V_2] = V_1$$

For this two-dimensional Lie Algebra the commutator table for V_i is a $(5 \otimes 5)$ table whose (i, j)th entry express the Lie Bracket $[V_i, V_j]$ given by the above Lie Algebra L. The table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the (i, j)th entry of the Commutator table and the related structure constants can be easily calculated from the above table are as follows:

$$C_{2,5,1} = -1, \text{ and } C_{5,2,1} = 1$$

The Lie Algebra L is Solvable. The Radical of L is $< V_1 > \oplus < V_2, V_3, V_4, V_5 >$

In the next section, we derive the reduction of (4) to PDEs with two independent variables and ODEs. These are five one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\} \ , \ L_{s,2} = \{V_2\} \ , \ L_{s,3} = \{V_3\} \ , \ L_{s,4} = \{V_4\} \ , \ L_{s,5} = \{V_5\}$$

and corresponding to each one-dimensional subalgebras we may reduce (4) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional Subalgebras. It is evident from the commutator table that there is only one two-dimensional solvable non-abelian Subalgebras. And there are eight two-dimensional Abelian Subalgebras namely,

$L_{NA,1} = \{V_3, V_4\}, \ L_{A,1} = \{V_1, V_2\}, \ L_{A,2} = \{V_1, V_3\}, \ L_{A,3} = \{V_1, V_4\},$
$L_{A,4} = \{V_1, V_5\}, \ L_{A,5} = \{V_2, V_3\}, \ L_{A,6} = \{V_2, V_4\}, \ L_{A,7} = \{V_3, V_5\},$
$L_{A,8} = \{V_4, V_5\}.$

3. Reductions of $u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy} = 0$ by One-dimensional Subalgebra

Case A:

The Subalgebra $V_1 = \partial_x$.

The characteristic equation associated with V_1 is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}.$$

On integrating the characteristic equation, we get these similarity variables,

$$y = s, \quad t = r \text{ and } u = w(r, s) \tag{25}$$

Using these similarity variables in Eq.(4) which can be transformed in to the form

$$w_r - B(r)w_{ss} = 0 \tag{26}$$

where $B(r)$ is a function of r.

Case B:

The Subalgebra $V_2 = t\partial_x + \partial_u$.

The characteristic equation associated with V_2 is

$$\frac{dx}{t} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{1} \ .$$

The characteristic equation with similarity variables,

$$y = s, \quad t = r \text{ and } w = tu - x. \tag{27}$$

Using these similarity variables in Eq.(4) which can be transformed in to the form

$$rw_r + 1 - r^2 B(r) w_{ss} = 0. \tag{28}$$

where $B(r)$ is a function of r.

Case C:

The Subalgebra $V_3 = \partial_y$.

The characteristic equation associated with V_3 is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0} .$$

Following the standard procedure we integrate the characteristic equation to get these similarity variables,

$$x = s, \quad t = r \text{ and } u = w(r, s). \tag{29}$$

Using these similarity variables in Eq.(4) we obtain that

$$w_r + ww_s + w_s^2 - A(r) w_{ss} = 0. \tag{30}$$

where $A(r)$ is a function of r.

Case D:

The Subalgebra $V_4 = y\partial_y$.

The characteristic equation associated with V_4 is

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{0} .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$s = x, \quad t = r \text{ and } u = w(r, s). \tag{31}$$

Using these similarity variables in Eq.(4) we get

$$w_r + ww_s + w_s^2 - A(r) w_{ss} = 0. \tag{32}$$

where $A(r)$ is a function of r.

Case E:

The Subalgebra $V_5 = \partial_t$.

The characteristic equation associated with V_5 is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0} .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$s = x, \quad y = r \text{ and } u = w(r, s) . \tag{33}$$

Using these similarity variables in Eq.(4) can be recast into the form

$$ww_s + w_s^2 - A(r) w_{ss} - B(r) w_{rr} = 0. \tag{34}$$

where $A(r)$ and $B(r)$ are functions of r.

4. Reductions of $u_t+uu_x+u_x^2-A(t)u_{xx}-B(t)u_{yy}=0$ by Two-dimensional Abelian Subalgebra

Case 1:

The subalgebra V_1 and V_3.

In this case, we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our purpose we begin reduction with V_1. The Transformed Equation is given in (26). At this stage, we express V_3 in terms of the similarity variables defined in (24). It is straight-forward to note that V_3 in the new variables takes the form

$$\tilde{V_3} = \partial s. \tag{35}$$

The characteristic equation for $\tilde{V_3}$ is

$$\frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0}.$$

Integrating this equation as before leads to new variables $\alpha = r,\ \beta(\alpha) = w$, which reduces (26) to a first order differential equations

$$\beta^{'} = 0. \tag{36}$$

Case 2:

The subalgebra V_3 and V_5

In this case, we find that the given generators commute $[V_3, V_5] = 0$. Thus either of V_3 or V_5 can be used to start the reduction with. For our purpose we begin reduction with V_3. The transformed equation is given in Eq(30). At this stage, we express V_5 in terms of the similarity variables defined in (24). It is straight-forward to note that V_5 in the new variables takes the form

$$\tilde{V_5} = \partial r. \tag{37}$$

The characteristic equation for $\tilde{V_5}$ is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}.$$

Integrating this equation as before leads to new variables $\alpha = s,\ \beta(\alpha) = w$, which reduces (30) to a second order differential equations

$$\beta\beta^{'} + \beta^{'2} - A\beta^{''} = 0. \tag{38}$$

Similarly all the reductions are given in the form of Appendix.

Appendix

Algebra	Reduction
$[V_1, V_2] = 0$	$\beta' = 0$
$[V_1, V_3] = 0$	$\beta' = 0$
$[V_1, V_4] = 0$	$\beta' = 0$
$[V_1, V_5] = 0$	$-B\beta'' = 0$
$[V_2, V_3] = 0$	$\alpha\beta' + 1 = 0$
$[V_2, V_4] = 0$	$\alpha\beta' + 1 = 0$
$[V_3, V_5] = 0$	$\beta\beta' + \beta'^2 - A\beta'' = 0$
$[V_4, V_5] = 0$	$\beta\beta' + \beta'^2 - A\beta'' = 0$

Also, the reductions of $u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy} = 0$ by two-dimensional Non-abelian Subalgebra satisfies the equation.

5. Conclusions

In this Paper,

(i) A $(2+1)$-dimensional potential Burgers equation $u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy} = 0$ where $A(t)$ and $B(t)$ are functions of t, is subjected to Lie's classical method.

(ii) Equation (4) admits a two-dimensional symmetry group.

(iii) It is established that the symmetry generators form a closed Lie algebra.

(iv) Classifications of Symmetry algebra of (4) into one- and two-dimensional abelian subalgebras are carried out.

(v) Systematic reductions to (1+1)-dimensional PDE and then to first- or second order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras.

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