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Some Relationship Between Matrix Completion Problems

Research Article

Kalyan Sinha[1]*

1 Department of Mathematics, PSTDS Vidyapith, Chinsurah, Hooghly, West Bengal-712305, India.

Abstract: In this paper relationship between the solutions of the matrix completion problems for certain class of matrices is studied. Specifically the similarities and dissimilarities among the class P_0^+ and its subclasses completion problem are discussed.

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1. Introduction

A real $n \times n$ matrix B is a *P-matrix* (resp. *P_0-matrix*, *$P_{0,1}$-matrix*) if every principal minor of B is positive (nonnegative, nonnegative with strictly positive diagonal entries). A real $n \times n$ matrix $B = [b_{ij}]$ is a *Q-matrix* if for every $k \in \{1, 2, \ldots, n\}$, $S_k(B) > 0$, where $S_k(B)$ is the sum of all $k \times k$ principal minors of B. Again a real $n \times n$ matrix $B = [b_{ij}]$ is a *P_0^+-matrix* if for every $k \in \{1, 2, \ldots, n\}$, all $k \times k$ principal minors of B are nonnegative and at least one principal minor of each order is positive. The matrix B is *sign symmetric* (resp. *weakly sign symmetric*) if $b_{ij}b_{ji} > 0$ or $b_{ij} = 0 = b_{ji}$ (resp. $b_{ij}b_{ji} \geq 0$) for each pair of $i, j \in \{1, \ldots, n\}$.

A *partial matrix* is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A *pattern* for $n \times n$ matrices is a subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. For $\alpha \subseteq \{1, \ldots, n\}$, the principal submatrix $B(\alpha)$ is obtained from B by deleting all rows and columns not in α. A principal minor is the determinant of a principal submatrix. In this paper, a pattern does not include all diagonal positions.

For a given class Π of matrices (e.g., P, P_0-matrices) a *partial Π-matrix* is a partial matrix for which the specified entries satisfy the properties of a Π-matrix. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A Π-*completion* of a partial Π-matrix M is a completion of M which is a Π-matrix. For a particular class Π of matrices, a pattern has Π-*completion* if every partial Π-matrix specifying the pattern can be completed to a Π-matrix. Matrix completion problems for several classes of matrices have been studied by a number of authors (e.g., [3, 4] etc). Prof L. Hogben compared a large number of pairs of completion problems in her paper [5] in 2003. For a survey of matrix completion results one may see [2].

If X_1 and X_2 are classes of matrices with $X_1 \subseteq X_2$, the completion problem for the classes $X_i, i = 1, 2$ may be completely different. In general, it is impossible to conclude whether the completion of X_1 implies the completion of X_2 or vice versa.

* *E-mail: kalyansinha90@gmail.com*

In this paper, we examine the relationship between the solutions to the matrix completion problems for pair of related subclasses of P_0^+-matrices. In all these cases it is established that if a pattern has completion for larger classes, then it has also completion for smaller classes. But the converse is not true i.e. for a pattern a smaller class may have completion whether the larger class may not have completion.

1.1. Partial matrices

A partial P-matrix (resp. P_0, $P_{0,1}$-matrix) is matrix in which all fully specified principal minors are P-matrices (resp. P_0, $P_{0,1}$-matrices). Although every P_0^+-matrix is a P-matrix as well as P_0-matrix, but the definition of partial P_0^+-matrix is different from the other classes.

A partial P_0^+-matrix M is a partial matrix in which all fully specified principal minors are nonnegative and $S_k(M) > 0$ for every $k = 1, 2, \ldots, n$, whenever all $k \times k$ principal submatrices are fully specified. A $P_{0,1}^+$-matrix is a P_0^+-matrix with strictly positive diagonal entries. A partial $P_{0,1}^+$-matrix M_1 is a partial P_0^+-matrix in which all specified diagonal entries are strictly positive. Again, a *partial sign symmetric matrix (resp. weakly sign symmetric-matrix)* is a matrix in which fully specified principal submatrices are sign symmetric matrix (resp. weakly sign symmetric-matrix). A partial positive (nonnegative)-matrix is a partial matrix whose specified entries are positive (nonnegative).

Thus, a partial sign symmetric (resp. weakly sign symmetric) P-matrix (resp. P_0, $P_{0,1}$, P_0^+, $P_{0,1}^+$-matrix) is partial P-matrix (resp. P_0, $P_{0,1}$, P_0^+, $P_{0,1}^+$-matrix) in which fully specified principal submatrices are sign symmetric matrix (resp. weakly sign symmetric-matrix). Again a partial positive (resp. nonnegative) P-matrix (resp. P_0, $P_{0,1}$, P_0^+, $P_{0,1}^+$-matrix) is partial P-matrix (resp. P_0, $P_{0,1}$, P_0^+, $P_{0,1}^+$-matrix) in which fully specified entries are positive (resp. nonnegative).

1.2. Pairs of Π/Π_0-classes

Definition 1.1 ([5]). *The classes of matrices X and X_0 are referred to as a pair of Π/Π_0-classes if*

(i) Any partial X-matrix is a partial X_0 matrix.

(ii) For any X_0-matrix A and $\epsilon > 0$, $A + \epsilon I$ is a X-matrix.

(iii) For any partial X-matrix A, there exist a $\delta > 0$ such that $A - \delta\widehat{I}$ is a partial X-matrix (where $\widehat{I}$ is a partial identity matrix specifying the same pattern as A)

Consider M be a partial P-matrix specifying a pattern N. Let I_M be the partial matrix specifying N with all specified diagonal entries 1 and off-diagonal entries 0. Since determinant of a matrix is a continuous function of its entries, there is $\epsilon > 0$ such that the partial matrix $M_0 = M - \epsilon I_M$ (i.e., one obtained by applying operations on the respective specified entries) is a partial P-matrix. Clearly, M_0 is a partial P_0^+-matrix specifying N. Thus the classes P and P_0^+-matrices are a pair of Π/Π_0-classes.

2. Relationship Between the Subclasses of P_0^+ and P-matrix Completion Problem

Theorem 2.1. *Any pattern that has P_0^+-completion also has P-completion.*

In Theorem 2.2 of [5], it is shown that for a pair of Π/Π_0-matrices, if a pattern has Π_0-completion, then it must also have Π-completion. Since P-matrix and P_0^+-matrix are a pair of Π/Π_0-classes, hence the result follows.

The following equivalent corollary is immediate.

Corollary 2.2. *Any pattern that does not have P-completion does not have P_0^+-completion.*

Also the following corollary which is similar to the corollary 2.3 in [5] hold for the subclasses of P_0^+ and P-matrices.

Corollary 2.3.

(i) Any pattern that has nonnegative P_0^+-completion also has nonnegative P-completion.

(ii) Any pattern that has sign symmetric P_0^+-completion also has sign symmetric P-completion.

(iii) Any pattern that has weakly sign symmetric P_0^+-completion also has weakly sign symmetric P-completion.

The proof of the above corollary is similar to the proof of the Theorem 2.1. However, the converse of the above Corollary 2.3 is not true for all cases. The following examples show this. Suppose $N_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)$ $,(2,3),(3,1),(3,2)\}$. The pattern N_2 has P-completion [3]. On the other hand, consider the partial P_0^+-matrix,

$$M_2 = \begin{bmatrix} 1 & 1 & ? \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

specifies N_2. Clearly M_2 cannot be completed to a P_0^+-matrix since $\det \widehat{M_2} = 0$ for any completion $\widehat{M_2}$ of M_2.

Consider $N_3 = \{(1,1),(2,1),(2,2)\}$. The pattern N_3 has sign symmetric (resp. nonnegative, weakly sign symmetric) P-completion ([4], [5]). But the pattern does not have sign symmetric (resp. nonnegative, weakly sign symmetric) P_0^+-completion. To see this, consider the partial sign symmetric (resp. nonnegative, weakly sign symmetric) P_0^+-matrix,

$$M_3 = \begin{bmatrix} 1 & ? \\ 2 & 0 \end{bmatrix}$$

specifies N_3. Since $\det \widehat{M_3} \leq 0$ for any sign symmetric (resp. nonnegative, weakly sign symmetric) completion $\widehat{M_3}$ of M_3, M_3 cannot be completed to a sign symmetric P_0^+-matrix.

3. Relationship Between the Subclasses of $P_{0,1}^+$ and P-matrix Completion Problem

The Theorem 2.1 can also be applied to find the relationship between the subclasses of $P_{0,1}^+$ and P-matrix completion problem. Since every $P_{0,1}^+$-matrix is a P_0^+-matrix, thus the hypothesis of Definition 1.1 is satisfied if we replace X_0-matrix as $P_{0,1}^+$-matrix and X-matrix as P-matrix. The following corollary which is similar to the corollary 2.9 in [5] hold for the subclasses of P_0^+ and P-matrices.

Corollary 3.1.

(i) Any pattern that has $P_{0,1}^+$-completion also has P-completion.

(ii) Any pattern that has nonnegative $P_{0,1}^+$-completion also has nonnegative P-completion.

(iii) Any pattern that has sign symmetric $P_{0,1}^+$-completion also has sign symmetric P-completion.

(iv) Any pattern that has weakly sign symmetric $P_{0,1}^+$-completion also has weakly sign symmetric P-completion.

The proof of the Corollary 3.1 follows from the Theorem 2.1 by replacing role of P_0^+ to $P_{0,1}^+$. But the converse of the above Corollary 3.1 is not true which can be seen from the following Example 3.

The pattern $N_4 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(3,1)\}$ has (nonnegative, sign symmetric, weakly sign symmetric) P-completion [4] but does not has (nonnegative, sign symmetric, weakly sign symmetric) $P_{0,1}^+$-completion. The partial $P_{0,1}^+$-matrix,

$$M_4 = \begin{bmatrix} 1 & 1 & ? \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

specifies N_4 and cannot be completed to a (nonnegative, sign symmetric, weakly sign symmetric) $P_{0,1}^+$-matrix since $\det \widehat{M_4} = 0$ for any completion $\widehat{M_4}$ of M_4.

4. Relationship Between the Subclasses of $P_{0,1}^+$ and P_0^+-matrix Completion Problem

We now consider the following $P_{0,1}^+$-matrix,

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since the condition [iii] of the Definition 1.1 does not hold for the $P_{0,1}^+$-matrix M, thus the Theorem 2.1 cannot be applied to the subclasses of $P_{0,1}^+$ and P_0^+-matrices. However the following theorem is true for $P_{0,1}^+$ and P_0^+-matrices.

Lemma 4.1. *If a positive real number is added to a diagonal entry of $P_0^+(P_{0,1}^+)$-matrix, it remains $P_0^+(P_{0,1}^+)$-matrix.*

Let A be a $P_0^+(P_{0,1}^+)$-matrix (So, A is necessarily a P_0 matrix). Without loss of generality we may assume a positive number is added to $(1,1)$-entry, so let A' be obtained from A by adding $a > 0$ to a_{11}. Then $\det A' = \det A + \det A''$, where the first row of A'' is $(a, 0, 0, \dots, 0)$ and the remaining rows are the same as the corresponding rows of A, because the determinant is multi-linear function of the rows. $\det A'' = a \det A[2, 3, \dots, n] \geq 0$, so $\det A' \geq \det A$. The same argument can be applied to all principal minors, so for $P_0^+(P_{0,1}^+)$-matrices the result can be applied.

Theorem 4.2. *Any pattern that has P_0^+-completion also has $P_{0,1}^+$-completion.*

Let N be a pattern that has P_0^+-completion, and let M be a partial $P_{0,1}^+$-matrix specifying N. Clearly M is a partial P_0^+-matrix. Since N has P_0^+-completion, M can be completed to a P_0^+-matrix $\widehat{M}$. If $\widehat{M}$ is not a $P_{0,1}^+$-matrix, then one or more diagonal entries are 0. Since M was a partial $P_{0,1}^+$-matrix, any diagonal entry specified in M was positive. Let $D = [d_{ij}]$ be defined as follows:

$$d_{ij} = \begin{cases} 0, & \text{if } i = j, (i,i) \in N, \\ 1 & \text{if } i = j, (i,i) \notin N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widehat{M} + D$ completes M to a $P_{0,1}^+$-matrix, because it has a positive diagonal and, as a sum of a P_0^+-matrix and nonnegative diagonal matrix, $\widehat{M} + D$ is a P_0^+-matrix. Thus N has $P_{0,1}^+$-completion. But the converse of the Theorem 4.2 is not true i.e. a pattern that has $P_{0,1}^+$-completion may not have P_0^+-completion. Suppose $N_6 = \{(1,1),(1,2),(1,3),(2,2),(3,3)\}$. The pattern N_6 has $P_{0,1}^+$-completion but does not have P_0^+-completion.

Consider the partial P_0^+-matrix,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ ? & 0 & ? \\ ? & ? & 1 \end{bmatrix}$$

specifies N_6 and cannot be completed to a P_0^+-matrix since $\det \widehat{A_1} = 0$ for any completion $\widehat{A_1}$ of A_1. On the other hand, consider the partial $P_{0,1}^+$-matrix $A_2 = [a_{ij}]$ as follows:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ? & a_{22} & ? \\ ? & ? & a_{33} \end{bmatrix}.$$

Clearly A_2 specifies N_6 and by definition of partial $P_{0,1}^+$-matrix, all diagonals are positive. Consider a completion B_2 of A_2 as follows:

$$B_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

Now, B_2 is a $P_{0,1}^+$-matrix completion of A_2. The following corollary which is similar to the Theorem 2.12 in [5] hold for the subclasses of P_0^+ and $P_{0,1}^+$-matrices.The proof of Theorem 4.2 can also be true to the following subclasses:

Corollary 4.3. *(i) Any pattern that has nonnegative P_0^+-completion also has nonnegative $P_{0,1}^+$-completion.*

(ii) Any pattern that has sign symmetric P_0^+-completion also has sign symmetric $P_{0,1}^+$-completion.

(iii) Any pattern that has weakly sign symmetric P_0^+-completion also has weakly sign symmetric $P_{0,1}^+$-completion.

However the converse of the Corollary 4.3 is not true. Consider the pattern $N_7 = \{(1,1),(2,2),(1,2)\}$. Now, the partial $P_{0,1}^+$-matrix

$$A_3 = \begin{bmatrix} a_{11} & a_{12} \\ ? & a_{22} \end{bmatrix},$$

specifies N_7. Since A_3 is a partial $P_{0,1}^+$-matrix, thus $a_{ii} > 0$ for $i = 1, 2$. The matrix A_3 can be completed to a (nonnegative, sign symmetric, weakly sign symmetric) $P_{0,1}^+$-matrix for suitable choice of unspecified entries. On the other hand, consider the partial P_0^+-matrix,

$$A_4 = \begin{bmatrix} 1 & 0 \\ ? & 0 \end{bmatrix},$$

specifies N_7. Now for any completion B_4 of A_4, $\det B_4 = 0$. Hence, A_4 cannot be completed to a P_0^+-matrix.

5. Relationship Between Asymmetric Patterns of P-matrices and $P_{0,1}^+$-matrices

In the previous sections, for classes $X_1 \subseteq X_2$, all the results inferred X_1-completion of a pattern from X_2-completion of a pattern. For certain special patterns it is also possible to infer completion of a pattern for the larger class from completion for the smaller class.

A pattern Q is *asymmetric* if $(i,j) \in Q$ implies $(j,i) \notin Q$.

Theorem 5.1. *Any asymmetric pattern that has P-completion must have $P^+_{0,1}$-completion.*

Let N' be an asymmetric pattern that has P-completion, and let M be a partial $P^+_{0,1}$-matrix specifying N'. Since the pattern is asymmetric, there are no fully specified principal submatrices of size large than 1, and the size 1 matrices are P-matrices. Thus M is a partial P-matrix and can be completed to a P-matrix $\widehat{M}$. Clearly $\widehat{M}$ is a $P^+_{0,1}$-matrix that completes M.

The above Theorem 5.1 can be applied to the other subclasses of P-matrices and $P^+_{0,1}$-matrices. Thus the following corollary is also true.

Corollary 5.2.

(i) Any asymmetric pattern that has nonnegative P-completion also has nonnegative $P^+_{0,1}$-completion.

(ii) Any asymmetric pattern that has sign symmetric P-completion also has sign symmetric $P^+_{0,1}$-completion.

(iii) Any asymmetric pattern that has weakly sign symmetric P-completion also has weakly sign symmetric $P^+_{0,1}$-completion.

Also from the Theorem 5.1, the following corollary can easily derived.

Corollary 5.3. *Any asymmetric pattern has $P^+_{0,1}$-completion.*

Any asymmetric pattern has P-completion [1]. Thus by Theorem 5.1, every asymmetric pattern has $P^+_{0,1}$-completion.

6. Conclusion

In this paper, the relationship between the the subclasses of P, P^+_0,$P^+_{0,1}$-matrices are discussed. Although the set of both P^+_0-matrices ($P^+_{0,1}$-matrices) is the intersection of the set of Q-matrices with the set of P_0-matrices, but the relationship between Q-matrices and other classes say P,P_0-matrices etc is still remains unsolved.

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