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Strongly Prime Labeling For Some Graphs

Research Article

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Abstract: A graph $G = (V, E)$ with n vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding, n such that the label of each pair of adjacent vertices are relatively prime. A graph G which admits prime labeling is called a prime graph and a graph G is said to be a strongly prime graph if for any vertex, v of G there exists a prime labeling, f satisfying, $f(v) = 1$. In this paper we prove that the graphs corona of triangular snake, corona of quadrilateral snake, corona of ladder graph and a graph obtained by attaching P_2 at each vertex of outer cycle of prism D_n by $(D_n; P_2)$, helm, gearwheel are strongly prime graphs.

MSC: 54C08.

Keywords: Prime Labeling, Prime Graph, Strongly Prime Graph.

1. Introduction

We begin with simple, finite, undirected and non trivial graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The set of vertices adjacent to a vertex u of G is denoted by $N(u)$. For all other standard terminology and notations we refer to Bondy and Murthy [3]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. *If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.*

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. An enormous body of literature has grown around in graph labeling in last five decades. A systematic study of various applications of graph labeling is carried out in Bloom and Golomb [2]. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs. For detailed survey on graph labeling we refer to A Dynamic Survey of Graph Labeling by Gallian [6].

Definition 1.2. *Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f : V(G) \to \{1, 2, \ldots\ldots p\}$ is called a prime labeling if for each edge $e = uv$, $\gcd\{f(u), f(v)\} = 1$. A graph which admits prime labeling is called a prime graph.*

The notion of a prime labeling was originated by Entringer and was discussed in a paper by Tout et al. [9]. Many researchers have studied prime graphs. For e.g. Fu and Huang [5] have proved that P_n and $K_{1,n}$ are prime graphs. Lee et al. [7] have proved that W_n is a prime graph if and only if n is even. Deretsky et al. [4] have proved that cycle C_n is a prime graph.

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Prime labeling of some classes of graph were discussed by S.K.Vaidya and Udayan M Prajapati in [11]. Prime labeling in the context of some graph operation was discussed by S.Meena and K.Vaithiligam [8].

Definition 1.3. *A graph G is said to be a strongly prime graph if for any vertex, v of G there exists a prime labeling , f satisfying , $f(v) = 1$.*

The concept of strongly prime graph was introduced by Samir K.Vaidya and Udayan M Prajapati [10] and they proved that the graphs C_n,P_n,$K_{1,n}$ and W_n for every even integer $n \geq 4$ are strongly prime graphs.

Definition 1.4. *Triangular snake T_n is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n-1$, that is every edge of path is replaced by a triangle C_3.*

Definition 1.5. *A quadrilateral snake Q_n is obtained from a path $\{u_1, u_2, \ldots, u_n\}$ by joining u_i and u_{i+1} to two vertices v_i and w_i, $1 \leq i \leq n-1$ respectively and then joining v_i and w_i.*

Definition 1.6. *The product $P_2 \times P_n$ is called a ladder and it is denoted by L_n.*

Definition 1.7. *The corona of two graphs G_1 and G_2 is the graph $G = G_1 \odot G_2$ formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the i'th copy of G_2.*

Definition 1.8. *The prism $D_n, n \geq 3$ is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. We denote a graph obtained by attaching P_2 at each vertex of outer cycle of D_n by $(D_n; P_2)$.*

Definition 1.9. *The helm H_n is a graph obtained from a wheel by attaching a pendant edge at each vertex of then n-cycle.*

Definition 1.10. *The gear graph G_n is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges.*

Definition 1.11 (Bertrand's Postulate)**.** *For every positive integer $n > 1$ there is a prime p such that $n < p < 2n$.*

The present work is aimed to discuss some new families of strongly prime graphs.

2. Strongly Prime Graphs

Theorem 2.1. *The graph $G \odot K_1$ is a strongly prime graph where $G = T_n$ for all integer $n \geq 2$.*

Proof. Let $\{u_1, u_2, ...u_n\}$ be a path of length n. Let v_i, $1 \leq i \leq n-1$ be the new vertex joined to u_i and u_{i+1}. The resulting graph is called T_n and let x_i be the vertex which is joined to u_i, $1 \leq i \leq n$, let y_i be the vertex which is joined to v_i, $1 \leq i \leq n-1$. The resulting graph is G_1 (i.e.) $G \odot K_1$ where $G = T_n$ graph.

Now the vertex set of $V(G_1) = \{u_1, u_2, ...u_n, v_1, v_2, ...v_{n-1}, x_1, x_2, ...x_n, y_1, y_2, ...y_{n-1}\}$ and the edge set $E(G_1) = \{u_iu_{i+1}, u_iv_i / 1 \leq i \leq n-1\} \cup \{u_ix_i / 1 \leq i \leq n\} \cup \{v_iu_{i+1}, v_iy_i / 1 \leq i \leq n-1\}$. Here $|V(G_1)| = 4n-2$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): If $v = u_j$ for some $j \in \{1, 2, ...n\}$ then the function $f : V(G) \to \{1, 2, ...4n-2\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j - 1 & if\ i = 1, 2, ...j-1; \\ 4i - 4j + 1 & if\ i = j, j+1, ...n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 1 & if\ i = 1, 2, ...j-1; \\ 4i - 4j + 3 & if\ i = j, j+1, ...n-1; \end{cases}$$

$$f(x_i) = \begin{cases} 4n + 4i - 4j & if \ i = 1, 2, ...j - 1; \\ 4i - 4j + 2 & if \ i = j, j + 1, ...n; \end{cases}$$

$$f(y_i) = \begin{cases} 4n + 4i - 4j + 2 & if \ i = 1, 2, ...j - 1; \\ 4i - 4j + 4 & if \ i = j, j + 1, ...n - 1; \end{cases}$$

is a prime labeling for G_1 with $f(v) = f(u_j) = 1$. Thus f is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in G_1.

Case (ii): If $v = x_j$ for some $j \in \{1, 2, ...n\}$ then define a labeling f_2 using the labeling f defined in case (i)as follows: $f_2(u_j) = f(x_j)$, $f_2(x_j) = f(u_j)$for $j \in \{1, 2, ...n\}$and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = x_j$ in G_1.

Case (iii): If $v = v_j$ for some $j \in \{1, 2, ...n - 1\}$ then define a labeling f_3 using the labeling f_2 defined in case (ii)as follows: $f_3(x_j) = f_2(v_j)$, $f_3(v_j) = f_2(x_j)$ for $j \in \{1, 2, ...n - 1\}$and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling f_3 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1.

Case (iv): If $v = y_j$ for some $j \in \{1, 2, ...n - 1\}$ then define a labeling f_4 using the labeling f_2 defined in case (ii)as follows: $f_4(x_j) = f_2(y_j)$,$f_4(y_j) = f_2(x_j)$,$f_4(u_j) = f_2(v_j)$,$f_4(v_j) = f_2(u_j)$for $j \in \{1, 2, ...n - 1\}$ and $f_4(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling f_4 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = y_j$ in G_1.

[In this case if $f_4(u_{j-1})$ is a multiple of 3 then interchange $f_4(u_{j-1})$ and $f_4(x_{j-1})$. Similarly $f_4(v_{j-1})$ is a multiple of 3 then interchange $f_4(v_{j-1})$ and $f_4(y_{j-1})$]

Thus from all the cases described above G_1 is a strongly prime graph.

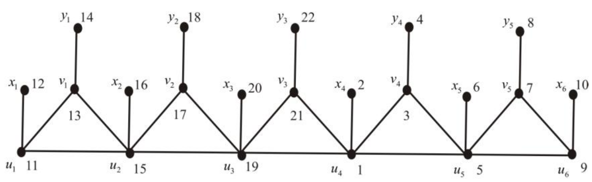

Figure 1. **A prime labeling of $G \odot K_1$ where $G = T_n$ having u_4 as label 1**

□

Theorem 2.2. *The graph $G \odot K_1$ is a strongly prime graph where $G = Q_n$ for all integer $n \geq 2$.*

Proof. Let $\{u_1, u_2, ...u_n\}$ be a path. Let v_i and w_i be two vertices joined to u_i and u_{i+1} respectively and then join v_i and w_i, $1 \leq i \leq n - 1$. The resulting graph is called as quadrilateral snake Q_n. Let $x_i$be the new vertex joined to u_i, $1 \leq i \leq n$, Let y_i be the new vertex joined to v_i, $1 \leq i \leq n - 1$ and let z_i be the new vertex joined to w_i, $1 \leq i \leq n - 1$. The resulting graph is G_1 (i.e.) $G \odot K_1$ where $G = Q_n$ graph.

Now the vertex set $V(G_1) = \{u_1, u_2, ...u_n, v_1, v_2, ...v_{n-1}, w_1, w_2, ...w_{n-1}, x_1, x_2, ...x_n, y_1, y_2, ...y_{n-1}, z_1, z_2, ...z_{n-1}, \}$. The edge set $E(G_1) = \{u_iu_{i+1}/1 \leq i \leq n - 1\} \cup \{u_ix_i/1 \leq i \leq n\} \cup \{u_iv_i, v_iy_i, v_iw_i, w_iz_i/1 \leq i \leq n - 1\} \cup \{w_iu_{i+1}/1 \leq i \leq$

$n-1\}$.Here$|V(G_1)| = 6n-4$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): Let $v = u_j$ for some $j \in \{1, 2, ...n\}$ then the function $f : V(G) \to \{1, 2, ...6n-4\}$ defined by

$$f(u_i) = \begin{cases} 6n + 6i - 6j - 1 & if\ i = 1, 2, ...j-1; \\ 6i - 6j + 1 & if\ i = j, j+1, ...n; \end{cases}$$

$$f(v_i) = \begin{cases} 6n + 6i - 6j - 3 & if\ i = 1, 2, ...j-1; \\ 6i - 6j + 3 & if\ i = j, j+1, ...n-1; \end{cases}$$

$$f(w_i) = \begin{cases} 6n + 6i - 6j + 1 & if\ i = 1, 2, ...j-1; \\ 6i - 6j + 5 & if\ i = j, j+1, ...n-1; \end{cases}$$

$$f(x_i) = \begin{cases} 6n + 6i - 6j & if\ i = 1, 2, ...j-1; \\ 6i - 6j + 2 & if\ i = j, j+1, ...n; \end{cases}$$

$$f(y_i) = \begin{cases} 6n + 6i - 6j - 2 & if\ i = 1, 2, ...j-1; \\ 6i - 6j + 4 & if\ i = j, j+1, ...n-1; \end{cases}$$

$$f(z_i) = \begin{cases} 6n + 6i - 6j + 2 & if\ i = 1, 2, ...j-1 \\ 6i - 6j + 6 & if\ i = j, j+1, ...n-1 \end{cases}$$

is a prime labeling for G_1 with $f(v) = f(u_j) = 1$. Thus $f$is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in G_1.

Case (ii): Let $v = x_j$ for some $j \in \{1, 2, ...n\}$ then define a labeling f_2 using the labeling f defined in case (i) as follows: $f_2(u_j) = f(x_j)$,$f_2(x_j) = f(u_j)$ for $j \in \{1, 2, ...n\}$ and $f_2(v) = f(v)$for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = x_j$ in G_1.

Case (iii): Let $v = v_j$ for some $j \in \{1, 2, ...n-1\}$ then define a labeling f_3 using the labeling f_2 defined in case (ii) as follows: $f_3(x_j) = f_2(v_j)$, $f_3(v_j) = f_2(x_j)$for $j \in \{1, 2, ...n-1\}$ and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling f_3 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1.

Case (iv): Let $v = w_j$ for some $j \in \{1, 2, ...n-1\}$ then define a labeling f_4 using the labeling f_3 defined in case (iii) as follows: $f_4(w_j) = f_3(v_j)$, $f_4(v_j) = f_3(w_j)$for $j \in \{1, 2, ...n-1\}$ and $f_4(v) = f_3(v)$ for all the remaining vertices. Then the resulting labeling f_4 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = w_j$ in G_1.

Case (v): Let $v = z_j$ for some $j \in \{1, 2, ...n-1\}$ then define a labeling f_5 using the labeling f_4 defined in case (iv) as follows: $f_5(z_j) = f_4(w_j)$, $f_5(w_j) = f_4(z_j)$ for $j \in \{1, 2, ...n-1\}$ and $f_5(v) = f_4(v)$for all the remaining vertices. Then the resulting labeling f_5 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = z_j$ in G_1.

Case (vi): Let $v = y_j$ for some $j \in \{1, 2, ...n-1\}$ then define a labeling f_6 using the labeling f_2 defined in case (ii) as follows: $f_6(u_j) = f_2(v_j)$,$f_6(v_j) = f_2(u_j)$,$f_6(x_j) = f_2(y_j)$,$f_6(y_j) = f_2(x_j)$ for $j \in \{1, 2, ...n-1\}$ and $f_6(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling f_6 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = y_j$ in G_1. Thus from all the cases described above G_1 is a strongly prime graph.

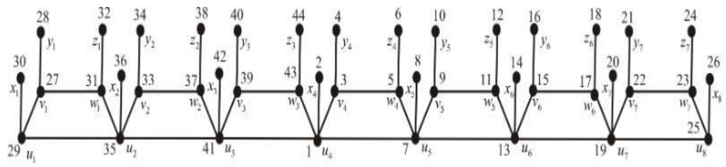

Figure 2. **A prime labeling of $G \odot K_1$ where $G = Q_n$ having u_4 as label 1**

□

Theorem 2.3. *The graph $G \odot K_1$ is a strongly prime graph where $G = L_n$ for all integer $n \geq 2$.*

Proof. Let G be the Ladder graph with vertices $\{u_1, u_2, ...u_n, v_1, v_2, ...v_n\}$. Let $u_i^{'}$ be the new vertex joined to u_i, $1 \leq i \leq n$ and $v_i^{'}$ be the new vertex joined to v_i, $1 \leq i \leq n$ in G. The resulting graph is G_1 (i.e.) $G \odot K_1$ where $G = L_n$ graph. Now the vertex set $V(G_1) = \{u_1, u_2, ...u_n, v_1, v_2, ...v_n, u_1^{'}, u_2^{'}, ...u_n^{'}, v_1^{'}, v_2^{'}, ...v_n^{'}\}$.
The edge set $E(G_1) = \{v_iv_{i+1}, u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i, u_iu_i^{'}, v_iv_i^{'}/1 \leq i \leq n\}$. Here$|V(G_1)| = 4n$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:
Case (i): If $v = u_j$ for some $j \in \{1, 2, ...n\}$then the function $f : V(G_1) \rightarrow \{1, 2, ...4n\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j + 1 & if\ i = 1, 2, ...j - 1; \\ 4i - 4j + 1 & if\ i = j, j + 1, j + 2, ...n; \end{cases}$$

$$f(u_i^{'}) = \begin{cases} 4n + 4i - 4j + 2 & if\ i = 1, 2, ...j - 1; \\ 4i - 4j + 2 & if\ i = j, j + 1, j + 2, ...n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 3 & if\ i = 1, 2, ...j - 2; \\ 4i - 4j + 3 & if\ i = j, j + 1, j + 2, ...n; \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 4n + 4i - 4j + 4 & if\ i = 1, 2, ...j - 2; \\ 4i - 4j + 4 & if\ i = j, j + 1, j + 2, ...n; \end{cases}$$

$$f(v_{j-1}) = \begin{cases} 4n\ \ if\ \ 4n - 1\ \ is\, multiple\, of\ 3; \\ 4n - 1 \qquad otherwise; \end{cases}$$

$$f(v_{j-1}^{'}) = \begin{cases} 4n - 1\ \ if\ 4n - 1\ is\, multiple\, of\ 3; \\ 4n \qquad otherwise; \end{cases}$$

is a prime labeling for G_1 with $f(v) = f(u_j) = 1$. Thus f is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in G_1 graph.

Case (ii): If $v = u_j^{'}$ for some $j \in \{1, 2, ...n\}$then define a labeling f_2 using the labeling f defined in case (i) as follows: $f_2(u_j) = f(u_j^{'})$,$f_2(u_j^{'}) = f(u_j)$ for$j \in \{1, 2, ...n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling

f_2 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j^{'}$ in G_1.

Case (iii): If $v = v_j$ for some $j \in \{1, 2, ...n\}$ then define a labeling f_3 using the labeling f defined in case (i) as follows: $f_3(u_i) = f(v_i)$, $f_3(v_i) = f(u_i)$, $f_3(u_i^{'}) = f(v_i^{'})$,$f_3(v_i^{'}) = f(u_i^{'})$ for $1 \leq i \leq n$in G_1. Then the resulting labeling f_3 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1 graph.

Case (iv): If $v = v_j^{'}$ for some $j \in \{1, 2, ...n\}$ then define a labeling f_4 using the labeling f_3 defined in case (iii) as follows: $f_4(v_j) = f_3(v_j^{'})$,$f_4(v_j^{'}) = f_3(v_j)$ for $j \in \{1, 2, ...n\}$ and $f_4(v) = f_3(v)$for all the remaining vertices. Then the resulting labeling f_4 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j^{'}$ in G_1. Thus from all the cases described above gives G_1 graph is a strongly prime graph.

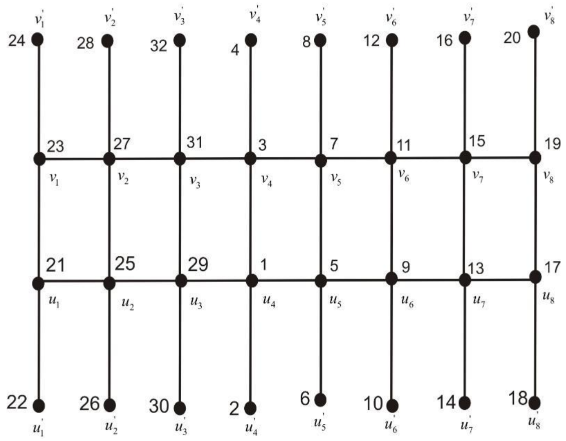

Figure 3. **A prime labeling of $G \odot K_1$ where $G = L_n$ having u_4 as label 1**

□

Theorem 2.4. *The graph obtained by attaching P_2 at each vertex of outer cycle of prismD_n by $(D_n; P_2)$for all integer $n \geq 3$, is a strongly prime graph.*

Proof. Let u_i and v_i be the vertices of the inner and outer cycle of $(D_n; P_2)$ respectively in which u_i and v_i are adjacent, $1 \leq i \leq n$. Let w_i be the pendant vertex which is joined with v_i, $1 \leq i \leq n$. The vertex set $V(D_n; P_2) = \{u_1, u_2, ...u_n, v_1, v_2, ...v_{n-1}, w_1, w_2, ...w_n\}$. The edge set $E(D_n; P_2) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_nu_1, v_nv_1\} \cup \{u_iv_i, v_iw_i/1 \leq i \leq n\}$.

Here $|V(D_n; P_2)| = 3n$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): If v is the vertex of the inner cycle. Let $v = u_j$ for some $j \in \{1, 2, ...n\}$ then the function $f : V(D_n; P_2) \rightarrow \{1, 2, ...3n\}$ defined by

$$f(u_i) = \begin{cases} 3n + 3i - 3j + 1 & if\ i = 1, 2, ...j - 1; \\ 3i - 3j + 1 & if\ i = j, j + 1, j + 2, ...n; \end{cases}$$

$$f(v_i) = \begin{cases} 3n + 3i - 3j + 2 & if\ i = 1, 2, ...j - 2; \\ 3i - 3j + 2 & if\ i = j, j + 1, j + 2, ...n; \end{cases}$$

$$f(w_i) = \begin{cases} 3n+3i-3j+3 & if \;\; i=1,2,...j-2; \\ 3i-3j+3 & if \;\; i=j,j+1,...n; \end{cases}$$

$$f(v_{j-1}) = \begin{cases} 3n-1 & if \; 3n \; is \; even; \\ 3n & otherwise; \end{cases}$$

$$f(w_{j-1}) = \begin{cases} 3n & if \; 3n \; is \; even; \\ 3n-1 & otherwise; \end{cases}$$

is a prime labeling for $(D_n; P_2)$ with $f(v) = f(u_j) = 1$. Thus f is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of the inner cycle $v = u_j$ in $(D_n; P_2)$ graph.

Case (ii): If v is any pendent vertex. Let $v = w_j$ for some $j \in \{1, 2, ...n\}$, then define a labeling f_2 using the labeling f defined in case (i) as follows: $f_2(u_j) = f(w_j)$,$f_2(w_j) = f(u_j)$ for$j \in \{1, 2, ...n\}$ and $f_2(v) = f(v)$for all the remaining vertices. Then the resulting labeling f_2 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of the pendent vertex $v = w_j$ in $(D_n; P_2)$ graph.

Case (iii): If v is the vertex of the outer cycle. Let $v = v_j$ for some $j \in \{1, 2, ...n\}$. then define a labeling f_3 using the labeling f_2 defined in case (ii) as follows: $f_2(v_j) = f(w_j)$,$f_2(w_j) = f(v_j)$ for$j \in \{1, 2, ...n\}$ and $f_3(v) = f_2(v)$and $f_3(v) = f_2(v)$ for all other remaining vertices. Then the resulting labeling f_3 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of outer cycle $v = v_j$ in $(D_n; P_2)$ graph. Thus from all the cases described above gives $(D_n; P_2)$ graph is a strongly prime graph.

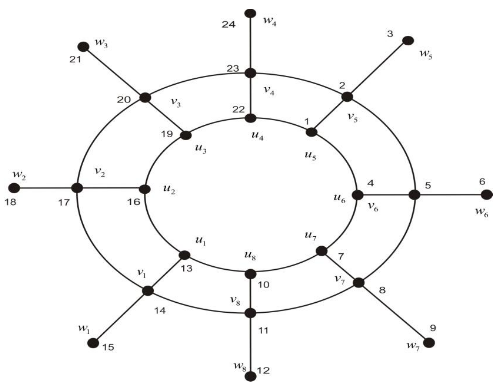

Figure 4. **A prime labeling of $(D_n; P_2)$ having u_5 as label 1**

□

Theorem 2.5. *The Helm H_n is a strongly prime graph.*

Proof. Let v_0 be the apex vertex $v_1, v_2, ...v_n$ be the consecutive rim vertices of H_n and $v_1^{'}, v_2^{'}, ...v_n^{'}$ be the pendent vertices of H_n. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): If $v$is the apex vertex $v = v_0$ then the function $f : V(H_n) \to \{1, 2, ...2n + 1\}$ defined as

$$f(v_0) = 1,$$
$$f(v_1) = 2,$$
$$f(v_1^{'}) = 3,$$

$f(v_i) = 2i + 1$ if $2 \leq i \leq n$, $f(v_i^{'}) = 2i$ if $2 \leq i \leq n$, then clearly f is an injection. For an arbitrary edge $e = ab$ of H_n we claim that $(f(a), f(b)) = 1$

Subcase (i): If $e = v_0 v_i$ for some $i \in \{2, 3, ...n\}$ then $\gcd(f(v_0), f(v_i)) = \gcd(1, f(v_i)) = 1$.

Subcase (ii): If $e = v_i v_{i+1}$ for some $i \in \{1, 2, ...n-1\}$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(2i + 1, 2i + 3) = 1$ as $2i+1$, $2i+3$ are consecutive odd positive integers. If $e = v_1 v_2$ then $\gcd(f(v_1), f(v_2)) = \gcd(2, 5) = 1$ and if $e = v_n v_1$ then $\gcd(f(v_n), f(v_1)) = \gcd(2n + 1, 2) = 1$ as $2n + 1$ is an odd integer.

Subcase (iii): If $e = v_i v_i^{'}$ for some $i \in \{2, 3, ...n\}$ then $\gcd(f(v_i), f(v_i^{'})) = \gcd(2i + 1, 2i) = 1$ as $2i + 1$, $2i$ are consecutive positive integers and if $e = v_1 v_1^{'}$ then $\gcd(f(v_1), f(v_1^{'})) = \gcd(2, 3) = 1$as 2 and 3 are consecutive positive integers.

Case (ii): If $v = v_j$ for some $j \in \{1, 2, ...n\}$, v is one of the rim vertices then we may assume that $v = v_1$ then define a labeling f_2 using the labeling f defined in case (i) as follows: $f_2(v_0) = f(v_1)$, $f_2(v_1) = f(v_0)$ and $f_2(v) = f(v)$for all other remaining vertices. Clearly f is an injection. For an arbitrary edge $e = ab$ of G we claim that $\gcd(f(a), f(b)) = 1$. To prove our claim the following cases are to be considered.

Subcase (i): If $e = v_0 v_i$ for some $i \in \{2, 3, ...n\}$ then $\gcd(f(v_0), f(v_i)) = \gcd(2, 2i + 1) = 1$ as $2i + 1$ is an odd positive integer and it is not divisible by 2. If $e = v_0 v_1$ then $\gcd(f(v_0), f(v_1)) = \gcd(2, 1) = 1$. When $e = v_i v_{i+1}$ for some $i \in \{2, 3, ...n - 1\}$ are as same as subcase (ii) in case (i). If $e = v_1 v_2$ then $\gcd(f(v_1), f(v_2)) = \gcd(1, 5) = 1$ and if $e = v_n v_1$ then$\gcd(f(v_n), f(v_1)) = \gcd(2n + 1, 1) = 1$.

Subcase (ii): When $e = v_i v_i^{'}$ for some $i \in \{2, 3, ...n\}$are as same as subcase (iii) in case (i). If $e = v_1 v_1^{'}$ then $\gcd(f(v_1), f(v_1^{'})) = \gcd(1, 3) = 1$.

Case (iii): If v is one of the pendent vertices then we assume that $v = v_i^{'}$ for $i = \frac{p-1}{2}$ (or) $\frac{p-3}{2}$, where p is the largest prime less than or equal to $2n + 1$. According to Bertrand's postulate such a prime p exist with $\frac{2n+1}{2} < p < 2n + 1$.

Subcase (i): If $n \neq 3k + 1$where k. Define a function $f : V(H_n) \to \{1, 2, ...2n + 1\}$

$$f(v_i) = \begin{cases} p & if \ \ i = 0; \\ 2i + 1 & if \ \ i \in \{1, 2, 3, ...n\} - \left\{\frac{p-1}{2}\right\}; \\ p - 1 & if \ \ i = \frac{p-1}{2}; \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 2i & if \ \ \{1, 2, 3, ...n\} - \left\{\frac{p-1}{2}\right\}; \\ 1 & if \ \ i = \frac{p-1}{2}; \end{cases}$$

Subcase (ii): If $f\left(V_{\frac{p-1}{2}}\right)$ and $f\left(V_{\frac{p+1}{2}}\right)$ are multiple of 3 then the above case $\gcd\left(\left(\frac{p-1}{2}\right), f\left(\frac{p+1}{2}\right)\right) \neq 1$ then define a function $f : V(H_n) \to \{1, 2, ...2n + 1\}$ as

$$f(v_i) = \begin{cases} p & if \ \ i = 0; \\ 2i + 1 & if \ \ i \in \{1, 2, 3, ...n\} - \left\{\frac{p-1}{2}, \frac{p-3}{2}\right\}; \\ p - 2 & if \ \ i = \frac{p-1}{2}; \\ p - 3 & if \ \ i = \frac{p-3}{2}; \end{cases}$$

$$f(v_i^{'}) = \begin{cases} 2i & if \ \{1,2,3,...n\} - \{\frac{p-3}{2}\}; \\ 1 & if \ i = \frac{p-3}{2}; \end{cases}$$

Case (iv): When $n = 3k+1$ then define the labeling f_2 using labeling $f$defined in subcase (i) **of** case (iii) as follows:$f_2(v_1) = f(v_1^{'})$,$f_2(v_1^{'}) = f(v_1)$ and $f_2(v) = f(v)$ for all other remaining vertices. Clearly f is an injection. For an arbitrary edge $e = ab$ of G we claim that $\gcd(f(a), f(b)) = 1$. To prove our claim the following cases are to be considered.

Subcase (i): If $e = v_0v_i$ for some $i \in \{1,2,3,...n\}$ then $\gcd(f(v_0), f(v_i)) = \gcd(p, f(v_i)) = 1$ as p is co-prime to every integer from $\{1,2,...2n+1\} - \{p\}$.

Subcase (ii): If $e = v_iv_{i+1}$ for some $i \in \{1,2,...n-1\}$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(2i+1, 2i+3) = 1$ as $2i+1,\ 2i+3$ are consecutive odd positive integers. If $e = V_{\frac{p-3}{2}}, V_{\frac{p-1}{2}}$ then $\gcd\left(f\left(V_{\frac{p-3}{2}}, V_{\frac{p-1}{2}}\right)\right) = \gcd(p-1, p-2) = 1$ as $p-1$ and $p-2$ are consecutive positive integers. If $e = V_{\frac{p-1}{2}}, V_{\frac{p+1}{2}}$ then $\gcd\left(f\left(V_{\frac{p-1}{2}}, V_{\frac{p+1}{2}}\right)\right) = \gcd(p-1, p+2) = 1$ as $p-1$ is even and it is differ by 3. Similarly we prove for any arbitrary edge $e = ab$ of H_n have $\gcd(f(a), f(b)) = 1$ subcases (ii) and case (iv). Thus in all the possibilities described above f is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of H_n. That is H_n is strongly prime graph.

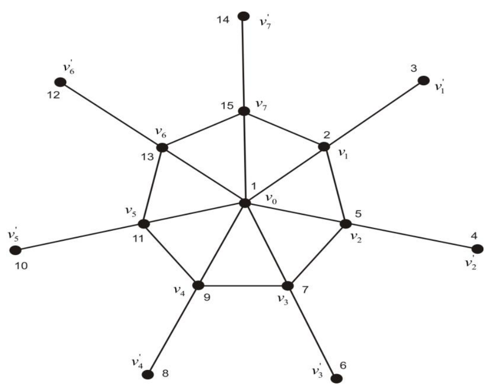

Figure 5. **A prime labeling of Helm graph ofH_7 having the apex vertex (v_0) as label 1**

□

Theorem 2.6. *The Gear graphG_n is a strongly prime graph.*

Proof. Let v_0 be the apex vertex $v_1, v_2, ...v_n, v_1^{'}, v_2^{'}, ...v_n^{'}$ be the consecutive rim vertices. Let v be an arbitrary vertex of G_n that is $v = v_0$. Then the function $f : V(G_n) \to \{1,2,...2n+1\}$ defined as $f(v_i) = 2i+1$ for $i = 0,1,2,...n$, $f(v_i^{'}) = 2i+2$ for $i = 1,2,...n-1$,

$$f(v_n^{'}) = 2.$$

Clearly f is an injection. For an arbitrary edge $e = ab$ of G_n we claim that $\gcd(f(a), f(b)) = 1$. To prove our claim the following cases are to be considered.

Subcase (i): If $e = v_0v_i$ for some $i \in \{1,2,3,...n\}$ then $\gcd(1, 2i+1) = 1$.

Subcase (ii): If $e = v_iv_i^{'}$ for some $i \in \{1,2,3,...n-1\}$ then $\gcd(f(v_i), f(v_i^{'})) = \gcd(2i+1, 2i+2) = 1$ as $2i+1, 2i+2$ are consecutive positive integers. If $e = v_nv_n^{'}$ then $\gcd(f(v_n), f(v_n^{'})) = \gcd(2n+1, 2) = 1$ as $2n+1$ is an odd positive integer and it is not divisible by 2.

Subcase (iii): If $e = v_i^{'}v_{i+1}$ for some $i \in \{1,2,3,...n-1\}$ then $\gcd(f(v_i^{'}), f(v_{i+1})) = \gcd(2i+2, 2i+3) = 1$ as $2i+1$ and $2i+3$ are consecutive positive integers. If $e = v_n^{'}v_1$ then $\gcd(f(v_n^{'}), f(v_1)) = \gcd(2,3) = 1$ as 2 and 3 are consecutive positive integers.

Case (ii): When v is of degree 2. Define a labeling f_2 using the labeling f in case (i)as follows: $f_2(v_n^{'}) = f(v_0)$, $f_2(v_0) = f(v_n^{'})$and $f_2(v) = f(v)$for all other remaining vertices. Then clearly f is an injection. For an arbitrary edge $e = ab$ of G_n we claim that $\gcd(f(a), f(b)) = 1$. To prove our claim the following cases are to be considered.

Subcase (i): If $e = v_0v_i$ for some $i \in \{1, 2, 3, ...n\}$ then $\gcd(f(v_0), f(v_i)) = \gcd(2, 2 + i) = 1$ as $2i + 1$ is an odd positive integer and it is not divisible by 2.

Subcase (ii): If $e = v_iv_i^{'}$ for some $i \in \{1, 2, 3, ...n - 1\}$ are as same as subcase (ii) in case (i). If $e = v_nv_n^{'}$ for some $i \in \{2, 3, ...n\}$ then $\gcd(f(v_n), f(v_n^{'})) = \gcd(2n + 1, 1) = 1$.

Subcase (iii): If $e = v_i^{'}v_{i+1}$ for some $i \in \{1, 2, 3, ...n - 1\}$are as same as subcase (iii) in case (ii). If $e = v_n^{'}v_1$ then $\gcd(f(v_n^{'}), f(v_1)) = \gcd(2n + 1, 1) = 1$.

Case (iii): When v is of degree 3. We may assume that $v = V_{\frac{p-1}{2}}$ where p is the largest prime less then or equal to $2n+1$. According to the Bertrand's postulate such a prime p exist with $\frac{2n+1}{2} < p < 2n + 1$. Now let f_3 be the labeling obtained from f in case (i) by interchanging the label $f(v_0)$ and $f\left(v_{\frac{p-1}{2}}\right)$ and for all the remaining vertices $f_3(v) = f(v)$. Then clearly f is an injection. For an arbitrary edge $e = ab$ of G_n we claim that $\gcd(f(a), f(b)) = 1$. To prove our claim the following cases are to be considered.

Subcase (i): If $e = v_0v_i$ for some $i \in \{1, 2, 3, ...n\}$ then $\gcd(f(v_0), f(v_i)) = \gcd(p, f(v_i)) = 1$ as p is co-prime to every integer from $\{1, 2, 3, ...n\} - \{\frac{p-1}{2}\}$.

Subcase (ii): If $e = v_iv_i^{'}$ for some $i \in \{1, 2, 3, ...n - 1\} - \{\frac{p-1}{2}\}$ are as same as subcase (ii) in case (i). If $e = v_{\frac{p-1}{2}}v_{\frac{p-1}{2}}^{'}$ then $\gcd\left(f\left(v_{\frac{p-1}{2}}\right), f\left(v_{\frac{p-1}{2}}^{'}\right)\right) = \left(1, f\left(v_{\frac{p-1}{2}}^{'}\right)\right) = 1$. If $e = v_nv_n^{'}$ then $\gcd(f(v_n), f(v_n^{'})) = \gcd(2n + 1, 2) = 1$ as $2n + 1$ is an odd positive integer and it is not divisible by 2.

Subcase (iii): If $e = v_i^{'}v_{i+1}$ for some $i \in \{1, 2, 3, ...n-1\} - \{\frac{p-3}{2}\}$ are as same as subcase (iii) in case (i). If $e = v_{\frac{p-3}{2}}^{'}v_{(\frac{p-1}{2})}$ then $\gcd\left(f\left(v_{\frac{p-3}{2}}^{'}\right), f\left(v_{\frac{p-1}{2}}\right)\right) = \left(f\left(v_{\frac{p-3}{2}}^{'}\right), 1\right) = 1$. If $e = v_n^{'}v_1$ then $\gcd(f(v_n^{'}), f(v_1)) = \gcd(2, 3) = 1$. Thus in all the possibilities described above f admits prime labeling and also it is possible to assign label 1 to any arbitrary vertex of G_n. Thus G_n is strongly prime graph for all n.

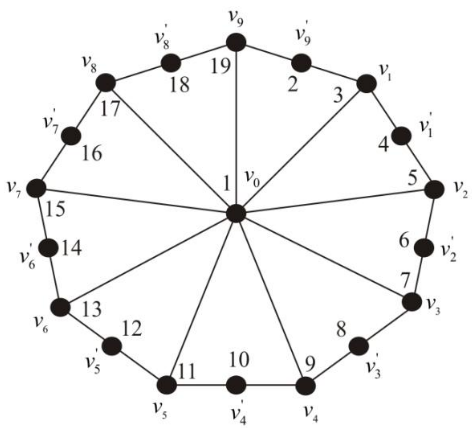

Figure 6. **A prime labeling of Gear graph of G_9 having the apex vertex(v_0) as label 1**

□

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References

[1] L.W.Beineke and S.M.Hegde, *Strongly multiplicative graphs*, Discussiones Mathematicae Graph Theory, 21(2001), 63-75.

[2] G.S.Bloom and S.W.Golomb, *Applications of numbered undirected graphs*, Proceedings of IEEE, 65(4)(1977), 562-570.

[3] J.A.Bondy and U.S.R.Murthy, *Graph Theory and Applications*, North-Holland, Newyork, (1976).

[4] T.Deretsky, S. M.Lee and J.Mitchem, On Vertex Prime Labelings of Graphs," In: J. Alvi, G. Chartrand, O. Oel-lerman, A. Schwenk, Eds., *Graph Theory, Combinatorics and Applications: Proceedings of the 6th International Conference Theory and Applications of Graphs*, Wiley, New York, (1991), 359-369.

[5] H.L.Fu and K.C.Huang, *On Prime labellings*, Discrete Mathematics, 127(1-3)(1994), 181-186.

[6] J.A.Gallian, *A dynamic survey of graph labeling*, The Electronics Journal of Combinatorics, 16(2009), DS6.

[7] S.M.Lee, I.Wui and J.Yeh, *On the Amalgamation of Prime graphs*, Bulletin of the Malaysian Mathematical Science Society(Second Series), 11(1988), 59-67.

[8] S.Meena and K.Vaithilingam, *Prime labeling for some helm related graphs*, International Journal of Innovative Research in Science, Engineering and Technology, 2(4)(2013).

[9] A.Tout, A.N.Dabboucy and K.Howalla, *Prime labeling of graphs*, National Academy Science letters, 11(1982), 365-368.

[10] S.K.Vaidya and Udayan M.Prajapati, *Some new results on prime graph*, Open Journal of Discrete Mathematics, (2012), 99-104.

[11] S.K.Vaidya and Udayan M.Prajapati, *Some results on prime and k-Prime Labeling*, Journal of Mathematics Research, 3(1)(2011), 66-75.