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Some Results on Skolem Difference Mean Graphs

Research Article

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from the set$\{1, 2, 3, \ldots, p+q\}$ in such a way that the edge $e= uv$ is labeled with $\frac{|f(u)-f(v)|}{2}$ if $|f(u) - f(v)|$ is even and $\frac{|f(u)-f(v)|+1}{2}$ if $|f(u) - f(v)|$ is odd and the resulting labels of the edges are distinct and are from $\{1, 2, 3, \ldots, q\}$. A graph that admits skolem difference mean labeling is called a skolem difference mean graph. In this paper, the author studied some results on skolem difference mean graphs.

MSC: 05C69.

Keywords: Skolem difference mean labeling, skolem difference mean graph.

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and a simple one. Let G= (V, E) be a graph with p vertices and q edges. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. Terms and notations not defined here are used in the sense of Harary [1].

A graph labeling is an assignment of integers to the vertices or edges or both vertices and edges subject to certain conditions. If the domain of the mapping is the set of vertices $(edges/both)$, then the labeling is called a vertex$(edge/total)$ labeling. There are several types of graph labeling and a detailed survey is found in [2].

The concept of skolem difference mean labeling was introduced by Murugan and Subramanian in the year 2011 in [3]. In this paper, the authors studied some results on skolem difference mean graphs. Following definitions are necessary for the present study.

Definition 1.1. *Let the graphsG_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their union $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. Clearly $G_1 \cup G_2$ has $p_1 + p_2$ vertices and $q_1 + q_2$ edges.*

Definition 1.2. *A graph $G = (V, E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x \in V$ with distinct elements f(x) from the set$\{1, 2, 3, \ldots, p+q\}$ in such a way that the edge e= uv is labeled with $\frac{|f(u)-f(v)|}{2}$ if $|f(u) - f(v)|$ is even and $\frac{|f(u)-f(v)|+1}{2}$ if $|f(u) - f(v)|$ is odd and the resulting labels of*

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the edges are distinct and are from $\{1, 2, 3, \ldots, q\}$. A graph that admits skolem difference mean labeling is called a skolem difference mean graph.

Example 1.3. *The skolem difference mean labeling of the cycle C_3 is given in Figure 1.*

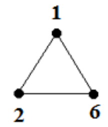

Figure 1.

Previous Results 1.4 ([4]).

(i) The path P_n is skolem difference mean for all $n > 1$.

(ii) The star $K_{1,n}$ is skolem difference mean for all $n \geq 1$.

(iii) The cycle C_n is skolem difference mean for all $n \geq 3$.

2. Some Results on Skolem Difference Mean Graphs

In this section, we find a necessary and sufficient condition for the complete graphK_n, the complete bipartite graph$K_{m,n}$ and wheel graph W_n to be skolem difference mean. Also a necessary condition for the graph G $= (V, E)$ to be skolem difference mean, the total number of possible skolem difference mean labeling of a graph G $= (V, E)$ and some results on skolem difference mean graphs are derived.

Theorem 2.1. *The complete graph K_n is skolem difference mean if and only if $n \leq 3$.*

Proof. when n = 1, there is nothing to prove. When n= 2, the graph is K_2 which is obviously skolem difference mean. When n= 3,the graph is $K_3 \equiv C_{3.}$ By Previous Result 1.4 (3), it is also skolem difference mean.
Conversely assume that $n > 3$. Let V$(K_n) = \{v_i; 1 \leq i \leq n\}$ and E$(K_n) = \{v_1 v_{i+1}; 1 \leq i \leq n-1\} \cup \{v_2 v_{i+2}; 1 \leq i \leq n-2\} \cup \{v_3 v_{i+3}; 1 \leq i \leq n-3\} \cup \ldots \cup \{v_{n-1} v_n\}$. The graph K_n has n vertices and nC_2 edges. Suppose K_n is skolem difference mean. Let $f : V(K_n) \to \{1, 2, 3, \ldots, n + nC_2\}$ be defined as follows. Let $e = uv$ be labeled with nC_2. Then either $\frac{|f(u)-f(v)|}{2} = nC_2$ or $\frac{|f(u)-f(v)|+1}{2} = nC_2$. Without loss of generality let us assume $f(u) > f(v)$

$$\begin{aligned} f(u) &= 2(nC_2) + f(v) \\ &= n^2 - n + f(v) \\ &\geq n^2 - n + 1 \\ &> \frac{n^2 + n}{2} \end{aligned}$$

This is a contradiction or

$$\begin{aligned} f(u) &= 2(nC_2) - 1 + f(v) \\ &= n^2 - n - 1 + f(v) \\ &\geq n^2 - n > \frac{n^2 + n}{2} \end{aligned}$$

This is also a contradiction. Hence the complete graph K_n is skolem difference mean if and only if n≤ 3. ◻

Theorem 2.2. *The complete bipartite graph$K_{m,n}$ is skolem difference mean if and only if $m, n \leq 2$.*

Proof. We prove the first part under three cases

Case (i): when m= $n = 1$, the graph is $K_{1,1} \equiv K_2$ which is skolem difference mean.

Case (ii): when $m = 1$ and $n = 2$ (proof is similar for $m = 2$ and $n = 1$), the graph is $K_{1,2}$ which is also skolem difference mean (By Previous Result 1.4 (2))

Case (iii): when $m = n = 2$, the graph is $K_{2,2}$. The skolem difference mean labeling of $K_{2,2}$ is given in Figure 2.

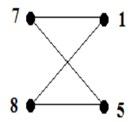

Figure 2.

Conversely suppose that m, $n>$ 2.Let G be the graph $K_{m,n}$. Let $V(G) = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{u_i v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Then the graph G has $m + n$ vertices and mn edges. Suppose that the graph G is skolem difference mean. Let $f : V(G) \to \{1, 2, 3, \ldots, m + n + mn\}$ be defined as follows. Now $1 \leq f(u_i), f(v_j) \leq m + n + mn$ for all u_i and v_j. Let e= $u_i v_j$ be labeled with mn. Without loss of generality, let $f(u_i) > f(v_j)$.

Case (i): Suppose $\frac{|f(u_i)-f(v_j)|}{2} = mn$. Then $f(u_i) = 2mn + f(v_j) > 2mn + 1$. If $2mn + 1 \leq m + n + mn$. Then $mn + 1 \leq m + n$ for all $m, n \geq 2$ which is a contradiction.

Case (ii): Suppose $\frac{|f(u_i)-f(v_j)|+1}{2} = mn$. Then $f(u_i) = 2mn - 1 + f(v_j) \geq 2mn$. If $2mn \leq m+n+mn$. Then $mn \leq m+n$ for all $m, n > 2$ which is also a contradiction. Hence the complete bipartite graph$K_{m,n}$ is skolem difference mean if and only if $m, n \leq 2$. □

Theorem 2.3. *Wheel graph $W_n, n > 2$ is not skolem difference mean.*

Proof. Let G be the wheel graph $W_n, n > 2$. Let $V(G) = \{v, v_i; 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_j v_{j+1}; 1 \leq i \leq n, 1 \leq j \leq n-1\} \cup \{v_n v_1\}$. Then the graph G has $n+1$ vertices and 2n edges. Let $f : V(G) \to \{1, 2, 3, \ldots, 3n+1\}$ be defined as follows. Suppose G is skolem difference mean. Let $f(v) = a$ and $f(v_i) = b$ for some i. Suppose $a > b$. Then $1 \leq b < a \leq 3n + 1$. Now two cases arise.

Case (i): Suppose $\frac{|f(v)-f(v_i)|}{2} = 2n \Rightarrow \frac{a-b}{2} = 2n \Rightarrow a = 4n + b \geq 4n + 1 > 3n + 1$ which is a contradiction.

Case (ii): Suppose $\frac{|f(v)-f(v_i)|+1}{2} = 2n \Rightarrow \frac{a-b+1}{2} = 2n \Rightarrow a = 4n+b-1 \geq 4n > 3n+1$ which is also a contradiction. Hence the wheel graph $W_n, n > 2$ is not skolem difference mean. □

Theorem 2.4. *A necessary condition for the graph $G = (V, E)$ with p vertices and q edges to be skolem difference mean is that $p \geq q$.*

Proof. Suppose that the graph $G = (V, E)$ with p vertices and q edges is skolem difference mean. Suppose $p < q$. Let $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be a skolem difference mean labeling of G. Then the induced edge labels are $\{1, 2, 3, \ldots, q\}$. Let $u, v \in V(G)$ where u and v are adjacent. Let $f(u) = a$ and $f(v) = b$, $a > b$ (Proof is similar if $b > a$). Then $1 \leq a, b \leq p + q$. Now two cases arise.

Case (i): when $|f(u) - f(v)|$ is odd. Then $\frac{|f(u)-f(v)|+1}{2} = q$

$$\Rightarrow \frac{a-b+1}{2} = q$$

$$\Rightarrow a = 2q + b - 1 \tag{1}$$

Now $p < q$

$$\Rightarrow p + q \leq 2q - 1$$

$$\Rightarrow a \leq p + q \leq 2q - 1$$

$$\Rightarrow 2q + b - 1 \leq 2q - 1 \quad \text{using } (1)$$

$$\Rightarrow b \leq 0. \text{ This is a contradiction.}$$

Case (ii): when $|f(u) - f(v)|$ is even. Then $\frac{|f(u)-f(v)|}{2} = q \Rightarrow a - b = 2q$. As in Case (i), $a = 2q + b \leq 2q - 1 \Rightarrow b \leq -1$. This is also a contradiction. Hence a necessary condition for the graph $G = (V, E)$ with p vertices and q edges to be skolem difference mean is that $p \geq q$. □

Observation 2.5. *Let G be a skolem difference mean graph. Then the following two cannot hold simultaneously.*

(i) $4x + 4$ *and* $2x + 2y + 2$ *are labels of adjacent vertices.*

(ii) $4x + 3$ *and* $2x + 2y + 1$ *are labels of adjacent vertices.*

Theorem 2.6. *The total number of possible skolem difference mean labeling of a graph* $G = (V, E)$ *with p vertices and q edges is atmost* $\prod_{i=1}^{q} 4i - 1 if p = q + 1$ *and* $\prod_{i=1}^{q} 4i - 3 if p = q$.

Proof. Let $G = (V, E)$ be a skolem difference mean graph with p vertices and q edges. Then the set of labeling of the edges $\{1, 2, 3, \ldots, q\}$ is as follows.

Case (i): when $p = q + 1$

Edge label	Choice of labels for the adjacent vertices	Number of edges
q	$(1, p+q)(1, p+q-1), (2, p+q)$	3
$q-1$	$(1, p+q-2), (1, p+q-3), (2, p+q-1), (2, p+q-2), (3, p+q)(3, p+q-1), (4, p+q)$	7
$q-2$	$(1, p+q-4), (1, p+q-5), (2, p+q-3), (2, p+q-4), (3, p+q-2), (3, p+q-3), (4, p+q-1),$ $(4, p+q-2), (5, p+q), (5, p+q-1), (6, p+q)$	11
...	...	...
1	$(1, 2), (1, 3), (2, 3), (2, 4), (3, 4) \ldots (p+q-2, p+q-1), (p+q-2, p+q), (p+q-1, p+q)$	$4q-1$

Hence the number of possible skolem difference mean labeling is almost $3.7.11 \ldots 4q - 1 = \prod_{i=1}^{q} 4i - 1$

Case (ii): when $p = q$

Edge label	Choice of labels for the adjacent vertices	Number of edges
q	$(1, p+q)$	1
$q-1$	$(1, p+q-1), (1, p+q-2), (2, p+q), (2, p+q-1), (3, p+q)$	5
$q-2$	$(1, p+q-3), (1, p+q-4), (2, p+q-2), (2, p+q-3), (3, p+q-1), (3, p+q-2), (4, p+q),$ $(4, p+q-1), (5, p+q)$	9
...	...	...
1	$(1, 2), (1, 3), (2, 3), (2, 4), (3, 4) \ldots (p+q-2, p+q-1), (p+q-2, p+q), (p+q-1, p+q)$	$4q-3$

Hence the number of possible skolem difference mean labeling is almost $1.5.9 \ldots 4q - 3 = \prod_{i=1}^{q} 4i - 3$. □

Theorem 2.7. *Let G be a graph which admits skolem difference mean labeling. Then the graph* $G \cup \bar{K_n}$ *is also skolem difference mean.*

Proof. Let $G = (V, E)$ be a skolem difference mean graph with p vertices and q edges. Let $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be a skolem difference mean labeling of G. Let f^* be the induced edge labeling of f. Then the edge labels are distinct and are $\{1, 2, 3, \ldots, q\}$. Let $V(\bar{K_n}) = \{v_i; 1 \leq i \leq n\}$. Now the graph $G \cup \bar{K_n}$ has $p + n$ vertices and q edges. Let $g : V(G \cup \bar{K_n}) \to \{1, 2, 3, \ldots, p + q + n\}$ be defined as follows.

$$g(V(G)) = f(V(G))$$
$$g(v_i) = p + q + i; 1 \leq i \leq n$$

Let g^* be the induced edge labeling of g. Then the induced edge labels are $\{1, 2, 3, \ldots, q\}$ which are distinct. Hence the graph $G \cup \bar{K_n}$ is also skolem difference mean. □

Theorem 2.8. *Let the path $G_1 = (V_1, E_1)$ with p_1 vertices and q_1 edges and the star $G_2 = (V_2, E_2)$ with p_2 vertices and q_2 edges have skolem difference mean labeling f and g respectively. Let u be the end vertex of G_1 and v be the central vertex of G_2 such that $f(u) = 1$ and $f(v) = 1$. Then the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also skolem difference mean.*

Proof. Let $V(G_1) = \{u, u_i; 1 \leq i \leq p_1 - 1\}$ and $V(G_2) = \{v, v_i; 1 \leq i \leq p_2 - 1\}$. Then $(G_1)_f * (G_2)_g$ has $p_1 + p_2 - 1$ vertices and $q_1 + q_2$ edges. Let $h : V[(G_1)_f * (G_2)_g] \to \{1, 2, 3, \ldots, p_1 + p_2 + q_1 + q_2 - 1\}$ be defined as follows.

$$h(u_i) = f(u_i); 1 \leq i \leq p_1 - 1$$
$$h(u) = f(u)$$
$$h(v_i) = p_1 + q_1 - 1 + g(v_i); 1 \leq i \leq p_2 - 1$$

Then the induced edge labels of G_1 are $1, 2, 3, \ldots, q_1$ and that of G_2 are $q_1 + 1, q_1 + 2, q_1 + 3, \ldots, q_1 + q_2$. Hence the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also skolem difference mean. □

Theorem 2.9. *Let $G = (V_1, E_1)$ be a tree with p_1 vertices and q_1 edges. Let $H = (V_2, E_2)$ be another tree with p_2 vertices and q_2 edges. Let G and H be skolem difference mean with skolem difference mean labeling f and g respectively. Then $G \cup H$ is also skolem difference mean.*

Proof. Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be any two skolem difference mean trees. Let $V(G) = \{v_i; 1 \leq i \leq p_1\}$ and $V(H) = \{u_i; 1 \leq i \leq p_2\}$. Let $f : V(G) \to \{1, 2, 3, \ldots, p_1 + q_1\}$ and $g : V(H) \to \{1, 2, 3, \ldots, p_2 + q_2\}$ be the skolem difference mean labeling of G and H respectively. Let f^* and g^* be the induced edge labeling of f and g respectively. Then the induced edge labels of G and H are $\{1, 2, 3, \ldots, q_1\}$ and $\{1, 2, 3, \ldots, q_2\}$ respectively. Let $h : V(G \cup H) \to \{1, 2, 3, \ldots p_1 + p_2 + q_1 + q_2\}$ be defined as follows.

$$h(v_i) = f(v_i); 1 \leq i \leq p_1.$$

Let $g(u_{i1}) = \max\{g(u_i); 1 \leq i \leq p_2\}$. Define $h(u_{i1}) = g(u_{i1}) + p_1 + q_1$ and $h(u) = g(u) + 1$ for all $u \in N[u_{i1}]$. Let $G_1 = G - N[u_{i1}]$. Let $g(u_{i2}) = \max\{g(u_i); u_i \in V(G_1)\}$. Define $h(u_{i2}) = g(u_{i2}) + p_1 + q_1$ and $h(u) = g(u) + 1$ for all $u \in N_G[u_{i2}]$. Let $G_2 = G_1 - N_G[u_{i2}]$. Proceeding in this way at least a finite number of stages, we get a graph G_t such that $V(G_t) = \{\}$. Suppose $h(u_{ij}) = h(u)$ for some u_{ij} and u.
$\Leftrightarrow g(u_{ij}) + p_1 + q_1 = g(u) + 1 \Leftrightarrow g(u_{ij}) - g(u) = 1 - p_1 - q_1$ which is impossible. Suppose $h(u_{ij}) = h(v_i)$ for some u_{ij} and v_i.
$\Leftrightarrow g(u_{ij}) + p_1 + q_1 = f(v_i) \Leftrightarrow g(u_{ij}) - f(v_i) = -p_1 - q_1$ which is also impossible. Hence $G \cup H$ is also skolem difference mean. □

3. Conclusion

In this paper, the author derived a necessary and sufficient condition for the complete graph K_n, the complete bipartite graph $K_{m,n}$ and the wheel graph W_n to be skolem difference mean. He derived a necessary condition for the graph $G = (V, E)$ to be skolem difference mean, the total number of possible skolem difference mean labeling of a graph $G = (V, E)$ and some results on skolem difference mean graphs.

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