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Some Fixed Point Theorems under Generalized Expansion Principle with Control Function

Research Article

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Abstract: We prove common fixed point theorems for semi and weak compatible mapping satisfying a generalized expansion principle by using a control function. Our theorems generalize recent results existing in the literature.

MSC: 47H10; 54H25.

Keywords: Generalized expansion principle, Weak compatible, Semi compatible, Commute map, Control function.

1. Introduction

Generalizing Banach contraction principle in various ways has become a recent research interest and has been studied by many authors. For example, One may refer [2, 3, 7, 10, 12] and [14]. [1] has proved a generalization for weakly contractive mapping in Hilbert space which was proved by [10] in the setup of complete metric space.

On the other hand, [7] and [9] proved fixed point theorem for a self mapping by altering distances between the point and using a control function, whereas [12] extended the concept for weakly commuting pairs of self mapping and proved common fixed point theorem in a complete metric space by using the control function.

More recently, [3] have obtained a fixed point result by generalizing the concept of control function and the weakly contractive mapping. [4] proved a common fixed point theorem for commuting mapping generalizing the Banach's contraction principle. [13] introduced, "Weakly commuting mapping" which was generalized by [5] as, "Compatible mapping" [8] coined the notion of, "R-weakly commuting mapping", whereas [6] defined a term called, "weakly compatible mapping".

In this paper we prove some fixed point theorems using generalized expansion principal with control function and generalize the work of [11].

2. Definition and Preliminaries

Definition 2.1. *Two self mappings T and F of a metric space (X, D) are said to be weak compatible, if $TFx = FTx$ whenever $Fx = Tx$ for all $x \in X$.*

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Definition 2.2 ([7, 9]). *A control function ϕ is defined as $\phi : R^+ \to R^+$ which is continuous at zero, monotonically increasing and $\phi(t) = 0$ if and only if $t = 0$.*

Definition 2.3 ([2]). *A self mapping T of metric space (X, D) is said to be weakly contractive with respect to a self mapping $f : X \to X$, for each $x, y \in X$, $d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy))$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$.*

If $F = I$, the identity mapping, then the Definition 2.3 reduces to the definition of weakly contractive mapping given by [1] and [10]. Combining the generalization of Banach contraction principle given by [7] and the generalization given by [3] and [10] obtained the following result.

Theorem 2.4 ([3]). *Let (X, D) be a complete metric space and $T : X \to X$ be a self map mapping satisfying $\varphi(d(Tx, Ty)) \leq \varphi(d(fx, fy)) - \phi(d(fx, fy))$, where $\phi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone decreasing functions with $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.*

Here we see a following lemma which helps us to prove main result.

Lemma 2.5. *Let (X, D) be a complete metric space and $T : X \to X$ or $F : X \to X$ be continuous self map satisfying $\varphi d(Tx, Ty) \geq \varphi d(Fx, Fy)) + \phi d(Tx, Ty)$, where $\phi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone increasing functions with $\varphi(t) = 0 = \phi(t) \Leftrightarrow t = 0$. If (F, T) is semi compatible then T, F have unique common fixed point.*

3. Main Results

Theorem 3.1. *Let $TandF$ be self mapping of metric space (X, D) with*

(a) $T(X) \subset F(X)$

(b) $\varphi[d(Tx, Ty)] \geq \varphi[d(Tx, Fx)) + d(Tx, Fy)]\phi[d(Tx, Ty)]$

(c) Either T or F is continuous function.

(d) (T, F) is semi compatible and weak compatible.

If φ and ϕ are monotonic increasing function such that $\phi, \varphi : [0, \infty) \to [0, \infty)$ and $\varphi(t) = 0 = \phi(t) \Leftrightarrow t = 0$ then z is unique common fixed point of F and T.

Proof. Let $x_0 \in X$ is an arbitrary point. Since $T(X) \subset F(X)$. Then $x_1 \in X$ such that $Tx_1 = Fx_0$. Inductively we can define a sequence $Tx_{n+1} = Fx_n$.
Using (b) with $x = x_n$, $y = x_{n+1}$

$$\begin{aligned}\phi[d(Tx_n, Tx_{n+1})] &\geq \phi[d(Tx_n, Fx_n)) + d(Tx_n, Fx_{n+1})] + \varphi[d(Tx_n, Tx_{n+1}) \\ &\geq \phi[d(Tx_n, Tx_{n+1})) + d(Tx_n, Tx_{n+2})] + \varphi[d(Tx_n, Tx_{n+1})]\end{aligned}$$

By triangle inequality we have $[d(Tx_{n+1}, Tx_{n+2})] \leq [d(Tx_{n+1}, Tx_n)) + d(Tx_n, Tx_{n+2})]$. Then

$$\begin{aligned}\phi[d(Tx_n, Tx_{n+1})] &\geq \phi[d(Tx_{n+1}, Tx_{n+2})] + \varphi d(Tx_n, Tx_{n+1})] \qquad (1) \\ \phi[d(Tx_n, Tx_{n+1})] &\geq \phi[d(Tx_{n+1}, Tx_{n+2})].\end{aligned}$$

Since ϕ is an increasing function then we have $d(Tx_n, Tx_{n+1}) \geq d(Tx_{n+1}, Tx_{n+2})$. Therefore the sequence $d(Tx_n, Tx_{n+1})$ will be decreasing. Let $r \geq 0$ such that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = r. \tag{2}$$

Hence on taking $\lim n \to \infty$ we have by (1) $\phi(r) \geq \phi(r) + \varphi(r)$. It is only possible when$r = 0$. Then by (2)

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0 \tag{3}$$

Now we shall show that $\{Tx_n\}$ is Cauchy sequence. Let we assume contrary. Then there exist $\epsilon > 0$ such that for $m, n \to \infty$ and for $m_i < n_i < m_{i+1}$,

$$d(Tx_{m_i}, Tx_{n_i}) \geq \epsilon \text{ and} \tag{4}$$

$$d(Tx_{m_i}, Tx_{n_{i-1}}) < \epsilon$$

Then it follows that

$$\epsilon \leq [d(Tx_{m_i}, Tx_{n_i})] \leq d(Tx_{m_i}, Tx_{n_{i-1}}) + d(Tx_{n_{i-1}}, Tx_{n_i}) < \varepsilon + d(Tx_{n_{i-1}}, Tx_{n_i}) d(Tx_{m_i}, Tx_{n_i}) < \epsilon + d(Tx_{n_{i-1}}, Tx_{n_i})$$

$$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) < \epsilon + \lim_{i\to\infty} d(Tx_{n_{i-1}}, Tx_{n_i})$$

By (3)

$$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) < \epsilon \tag{5}$$

By (4) and (5)

$$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) = \epsilon \tag{6}$$

Now by using (b) with $x = x_{m_i}$, $y = x_{n_i}$

$$\phi[d(Tx_{m_i}, Tx_{n_i})] \geq \phi[d(Tx_{m_i}, Fx_{m_i}) + d(Tx_{m_i}, Fx_{n_i})] + \varphi d(Tx_{m_i}, Tx_{n_i})$$

$$\phi[d(Tx_{m_i}, Tx_{n_i})] \geq \phi[d(Tx_{m_i}, Tx_{m_{i+1}}) + d(Tx_{m_i}, Tx_{n_{i+1}})] + \varphi d(Tx_{m_i}, Tx_{n_i})$$

$\lim_{i\to\infty}$ and by (3) & (6) we have $\phi(\varepsilon) \geq \phi(\varepsilon) + \varphi(\varepsilon)$. It is only possible when $\varepsilon = 0$. Which is contradiction to our assumption that $\varepsilon > 0$. Therefore for all $m, n \to \infty$ we have $d(Tx_{n_i}, Tx_{m_i}) < \varepsilon$. Therefore $\{Tx_n\}$ is a Cauchy sequence. Since (X, D) is complete metric space, then it will be converge at some point $z \in X$, or $\lim_{n\to\infty} Tx_n = z$. So all of its subsequence also converge to z or $\lim_{n\to\infty} Tx_{n+1} = z$, $\lim_{n\to\infty} Fx_n = z$.

Case (1): When T is continuous map

Since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} TTx_n = Tz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Tz$. Since pair (T, F) is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Tz$. Now using (b) with $x = Tx_n$, $y = x_n$

$$\phi[d(TTx_n, Tx_n)] \geq \phi[d(TTx_n, FTx_n)) + d(TTx_n, Fx_n)] + \varphi[d(TTx_n, Tx_n)$$

Now limiting $\lim_{n\to\infty}$ we have

$$\phi[d(Tz, z)] \geq \phi[d(Tz, Tz) + d(Tz, z)] + \varphi d(Tz, z)]$$

$$\phi[d(Tz, z)] \geq \phi[d(Tz, z)] + \varphi d(Tz, z)$$

$$\varphi[d(Tz, z)] \leq 0$$

It is only possible when, $Tz = z$. Since $T(X) \subset F(X)$. Then let $u \in X$ such that $Tz = Fu = z$. Now by using (b) with $x = u$, $y = x_n$

$$\phi[d(Tu, Tx_n)] \geq \phi[d(Tu, Fu) + d(Tu, Fx_n)] + \varphi d(Tu, Tx_n)]$$

Taking $\lim n \to \infty$

$$\phi[d(Tu, z)] \geq \phi[d(Tu, z) + d(Tu, z)] + \varphi d(Tu, z)$$

$$\phi[d(Tu, z)] \geq \phi[2d(Tu, z)] + \varphi d(Tu, z)$$

Since ϕ and φ are increasing function therefore obtained inequality is only possible when $d(Tu, z) = 0 \Rightarrow Tu = z$ or $Tz = Fu = z$. Since (F, T) is weak compatible then $FTu = TFu = z$ or $Tz = Fz = z$. Therefore z is common fixed point of F and T.

Case (2): When F is continuous map

Since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Fz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} FFx_n = Fz$. Since pair (T, F) is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Fz$. Now using (b) with, $x = Fx_n$, $y = x_n$

$$\phi[d(TFx_n, Tx_n)] \geq \phi[d(TFx_n, FFx_n)) + d(TFx_n, Fx_n)] + \varphi[d(TFx_n, Tx_n)$$

Now limiting $n \to \infty$ we have

$$\phi[d(Fz, z)] \geq \phi[d(Fz, Fz) + d(Fz, z)] + \varphi d(Fz, z)$$

$$\phi[d(Fz, z)] \geq \phi[d(Fz, z)] + \varphi d(Fz, z)$$

$$\varphi d(Fz, z)] \leq 0$$

It is only possible when $Fz = z$. Again using (b) with $x = z$, $y = x_n$

$$\phi[d(Tz, Tx_n)] \geq \phi[d(Tz, Fz) + d(Tz, Fx_n)] + \varphi d(Tz, Tx_n)$$

Limiting $n \to \infty$

$$\phi[d(Tz, z)] \geq \phi[d(Tz, z) + d(Tz, z)] + \varphi d(Tz, z)$$

$$\phi[d(Tz, z)] \geq \phi[2d(Tz, z)] + \varphi d(Tz, z)$$

Since ϕ and φ are increasing function therefore obtained inequality is only possible when $d(Tz, z) = 0 \Rightarrow Tz = z$ or $Fz = Tz = z$.

Uniqueness: Let w be another fixed point of F and T, then $Fw = Tw = w$. By using (b) with $x = z$, $y = w$ we have

$$\phi[d(Tz, Tw)] \geq \phi[d(Tz, Fz) + d(Tz, Fw)] + \varphi d(Tz, Tw)$$

$$\phi[d(z, w)] \geq \phi[d(z, z) + d(z, w)] + \varphi d(z, w)$$

$$\phi[d(z, w)] \geq \phi[d(z, w)] + \varphi d(z, w)$$

$$\varphi[d(z, w)] \leq 0 \Rightarrow z = w$$

Hence z is a unique common fixed point of F and T. This completes the proof. □

Corollary 3.2. *Let T,F and S be self mapping of metric space (X, D) with*

(a) $T(X) \subset F(X)$, $S(X) \subset F(X)$,

(b) $\phi[d(Tx, Sy)] \geq \phi[d(Tx, Fx) + d(Tx, Fy)] + \varphi[d(Tx, Sy)]$

(c) Either T or F is continuous function.

(d) (T, F) is semi compatible and weak compatible.

(e) $TS = ST$, $FS = SF$.

If ϕ and φ are monotonic increasing function such that $\phi, \varphi : [0, \infty) \to [0, \infty)$ and $\varphi(t) = 0 = \phi(t) \Leftrightarrow t = 0$ then z is unique common fixed point of F and T.

Theorem 3.3. *Let T and F be self mapping of metric space (X, D) with*

(a) $T(X) \subset F(X)$

(b) $\phi[d(Tx, Ty)] \geq \phi \min[d(Fx, Fy), d(Ty, Fy) + d(Fx, Ty)] + \varphi[d(Tx, Ty]$

(c) Either T or F is continuous function.

(d) (T, F) is semi compatible and commute.

If ϕ and φ are monotonic increasing function such that $\phi, \varphi : [0, \infty) \to [0, \infty)$ and $\varphi(t) = 0 = \phi(t) \Leftrightarrow t = 0$, if T^2 is an identity map then z is unique common fixed point of F and T.

Proof. Let $x_0 \in X$ is an arbitrary point. Since $T(X) \subset F(X)$. Then $x_1 \in X$ such that $Tx_1 = Fx_0$. Inductively we can define a sequence $Tx_{n+1} = Fx_n$. Using (b) with $x = x_n$, $y = x_{n+1}$ we have

$$\begin{aligned}
&\phi[d(Tx_n, Tx_{n+1})] \geq \phi \min[d(Fx_n, Fx_{n+1}), d(Tx_{n+1}, Fx_{n+1}) + d(Fx_n, Tx_{n+1})] + \varphi[d(Tx_n, Tx_{n+1}) \\
&\phi[d(Tx_n, Tx_{n+1})] \geq \phi \min[d(Tx_{n+1}, Tx_{n+2}), d(Tx_{n+1}, Tx_{n+2}) + d(Tx_{n+1}, Tx_{n+1})] + \varphi[d(Tx_n, Tx_{n+1})] \\
&\phi[d(Tx_n, Tx_{n+1})] \geq \phi[d(Tx_{n+1}, Tx_{n+2})] + \varphi d(Tx_n, Tx_{n+2})] \qquad (7) \\
&\phi[d(Tx_n, Tx_{n+1})] \geq \phi[d(Tx_{n+1}, Tx_{n+2})]
\end{aligned}$$

Since ϕ is an increasing function therefore $d(Tx_n, Tx_{n+1}) \geq d(Tx_{n+1}, Tx_{n+2})$. Therefore the sequence $d(Tx_n, Tx_{n+1})$ will be decreasing. Let $r \geq 0$ such that

$$\lim_{n \to \infty} d(Tx_n, Tx_{n+1}) = r. \qquad (8)$$

Hence on taking $\lim n \to \infty$ we have by (7) $\phi(r) \geq \phi(r) + \varphi(r)$. It is only possible when $r = 0$. Then by (8)

$$\lim_{n \to \infty} d(Tx_n, Tx_{n+1}) = 0 \qquad (9)$$

Now we shall show that $\{Tx_n\}$ is Cauchy sequence. Let we assume contrary. Then there exist $\epsilon > 0$ such that for $m, n \to \infty$ and for $m_i < n_i < m_{i+1}$

$$d(Tx_{m_i}, Tx_{n_i}) \geq \epsilon \qquad (10)$$

$$d(Tx_{m_i}, Tx_{n_{i-1}}) < \epsilon$$

Then it follows that

$$\epsilon \leq [d(Tx_{m_i}, Tx_{n_i})] \leq d(Tx_{m_i}, Tx_{n_{i-1}}) + d(Tx_{n_{i-1}}, Tx_{n_i}) < \varepsilon + d(Tx_{n_{i-1}}, Tx_{n_i}) d(Tx_{m_i}, Tx_{n_i}) < \epsilon + d(Tx_{n_{i-1}}, Tx_{n_i})$$

$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) < \epsilon + \lim_{i\to\infty} d(Tx_{n_{i-1}}, Tx_{n_i})$. By (9)

$$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) < \epsilon \tag{11}$$

By (10) and (11)

$$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) = \epsilon \tag{12}$$

by using (b) with $x = x_{m_i}$, $y = x_{n_i}$

$$\phi[d(Tx_{m_i}, Tx_{n_i})] \geq \phi \min[d(Fx_{m_i}, Fx_{n_i}), \{d(Tx_{n_i}, Fx_{n_i}) + d(Fx_{m_i}, Tx_{n_i})\}] + \varphi d(Tx_{m_i}, Tx_{n_i})$$
$$\phi[d(Tx_{m_i}, Tx_{n_i})] \geq \phi \min[d(Tx_{m_{i+1}}, Tx_{n_{i+1}}), \{d(Tx_{n_i}, Tx_{n_{i+1}})] + d(Tx_{m_{i+i}}, Tx_{n_i})\} + \varphi d(Tx_{m_i}, Tx_{n_i})$$

$\lim_{n\to\infty}$ and By (9) & (12) we have

$$\phi(\varepsilon) \geq \phi \min \varepsilon, \varepsilon + \varphi(\varepsilon)$$
$$\phi(\varepsilon) \geq \phi(\varepsilon) + \varphi(\varepsilon).$$

It is only possible when $\varepsilon = 0$. Which is contradiction to our assumption that $\varepsilon > 0$. Therefore for all $m, n \to \infty$ we have $d(Tx_{n_i}, Tx_{m_i}) < \varepsilon$. Therefore $\{Tx_n\}$ is a Cauchy sequence. Since (X, D) is complete metric space, then it will be converge at some point $z \in X$, or $\lim_{n\to\infty} Tx_n = z$ So all of its subsequence also converge to z or $\lim_{n\to\infty} Tx_{n+1} = z$, $\lim_{n\to\infty} Fx_n = z$.

Case (1): When T is continuous map

Since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} TTx_n = Tz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Tz$. Since pair $(T, F)$is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Tz$. Now using (b) with $x = Tx_n$, $y = x_n$

$$\phi[d(TTx_n, Tx_n)] \geq \phi \min[d(FTx_n, Fx_n), d(Tx_n, Fx_n) + d(FTx_n, Tx_n)] + \varphi[d(TTx_n, Tx_n)$$

limiting $n \to \infty$ we have

$$\phi[d(Tz, z)] \geq \phi \min[d(Tz, z), \{d(z, z) + d(Tz, z)\} + \varphi d(Tz, z)]$$
$$\phi[d(Tz, z)] \geq \phi \min[d(Tz, z), d(Tz, z)] + \varphi d(Tz, z)$$
$$\phi[d(Tz, z)] \geq \phi d(Tz, z) + \varphi d(Tz, z) \text{ it is only possible when, } d(Tz, z) = 0 \;\; Tz = z.$$

Again by using (b) with $x = z$, $y = x_{n+1}$

$$\phi[d(Tz, Tx_{n+1})] \geq \phi \min[d(Fz, Fx_{n+1}), \{d(Tx_{n+1}, Fx_{n+1}) + d(Fz, Tx_{n+1})\}] + \varphi[d(Tz, Tx_{n+1})$$

Liming $n \to \infty$

$$\phi[d(z, z)] \geq \phi \min[d(Fz, z), \{d(z, z) + d(Fz, z)\}] + \varphi d(z, z) 0 \geq \phi d(Fz, z) \Rightarrow Fz = z$$

Therefore $Tz = Fz = z$. z is common fixed point of F and T.

Case (2) - When F is continuous map

since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Fz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} FFx_n = Fz$. Since pair (T, F) is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Fz$. Now using (b) with, $x = Fx_n$, $y = x_n$

$$\phi[d(TFx_n, Tx_n)] \geq \phi \min[d(FFx_n, Fx_n), \{d(Tx_n, Fx_n) + d(FFx_n, Tx_n)\}] + \varphi[d(TFx_n, Tx_n)$$

Now limiting $n \to \infty$

$$\phi[d(Fz, z)] \geq \phi \min[d(Fz, z), \{d(z, z) + d(Fz, z)\}] + \varphi d(Fz, z)$$
$$\phi[d(Fz, z)] \geq \phi \min[d(Fz, z), d(Fz, z)] + \varphi d(Fz, z)$$
$$\phi[d(Fz, z)] \geq \phi[d(Fz, z)] + \varphi d(Fz, z)$$

It is only possible when, $d(Fz, z) = 0 \Rightarrow Fz = z$. By using (b) with $x = Tz$, $y = x_n$

$$\phi[d(T^2z, Tx_n)] \geq \phi \min[d(FTz, Fx_n), \{d(Tx_n, Fx_n) + d(FTz, Tx_n)\}] + \varphi d(T^2z, Tx_n),$$

since $T^2 = I$ and pair(F,T) is commute also limit $n \to \infty$

$$\phi[d(z, z)] \geq \phi \min[d(TFz, z), \{d(z, z) + d(TFz, z)\}] + \varphi d(z, z)$$
$$0 \geq \phi \min[d(Tz, z), d(Tz, z)]$$
$$0 \geq \phi d(Tz, z)$$

Which is possible when $d(Tz, z) = 0 \Rightarrow Tz = z$. Therefore $Fz = Tz = z$. z is common fixed point of T and F. Uniqueness can be proved easily. □

Theorem 3.4. *Let T,F,S and A be self mapping of metric space (X, D) with*

(a) $T(X) \subset F(X), S(X) \subset A(X)$,

(b) $\phi[d(Tx, Sy)] \geq \phi[d(Tx, Fx)) + d(Tx, Ay)] + \varphi[d(Tx, Sy]$

(c) Either T or F is continuous function.

(d) (T, F) *is semi compatible and weak compatible.*

(e) $TS = ST, FS = SF$.

If ϕ and φ are monotonic increasing function such that $\phi, \varphi : [0, \infty) \to [0, \infty)$ and $\varphi(t) = 0 = \phi(t) \Leftrightarrow t = 0$ then z is unique common fixed point of F and T.

Proof. Let $x_0 \in X$ is an arbitrary point. Since $T(X) \subset F(X), S(X) \subset A(X)$. Then there exist $x_1, x_2 \in X$ such that $Tx_1 = Fx_0$ and $sx_2 = Ax_1$. Inductively we can define a sequence $Tx_{n+1} = Fx_n = y_n$ and $Sx_{n+2} = Ax_{n+1} = y_{n+1}$. Using (b) with $x = x_n$, $y = x_{n+1}$

$$\phi[d(Tx_n, Sx_{n+1})] \geq \phi[d(Tx_n, Fx_n)) + d(Tx_n, Ax_{n+1})] + \varphi[d(Tx_n, Sx_{n+1})$$
$$\phi[d(Tx_n, Tx_{n+1})] \geq \phi[d(Tx_n, Tx_{n+1})) + d(Tx_n, Tx_{n+2})] + \varphi[d(Tx_n, Tx_{n+1})]$$

By triangle inequality we have

$$[d(Tx_{n+1}, Tx_{n+2})] \leq [d(Tx_{n+1}, Tx_n)) + d(Tx_n, Tx_{n+2})]$$

Then

$$\phi[d(Tx_n, Tx_{n+1})] \geq \phi[d(Tx_{n+1}, Tx_{n+2})] + \varphi d(Tx_n, Tx_{n+2})] \quad (13)$$
$$\phi[d(Tx_n, Tx_{n+1})] \geq \phi[d(Tx_{n+1}, Tx_{n+2})]$$

Since ϕ is an increasing function then we have $d(Tx_n, Tx_{n+1}) \geq d(Tx_{n+1}, Tx_{n+2})$. Therefore the sequence $d(Tx_n, Tx_{n+1})$ will be decreasing. Let $r \geq 0$ such that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = r. \quad (14)$$

Hence on taking $\lim_{n\to\infty}$ we have by (13) $\phi(r) \geq \phi(r) + \varphi(r)$. It is only possible when $r = 0$. Then by (14)

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0 \quad (15)$$

From Theorem 3.1 it can be easily shown that $\{Tx_n\}$ is Cauchy sequence. Since (X, D) is complete metric space, then it will be converge at some point $z \in X$, or $\lim\ Tx_n = z$ So all of its subsequence also converge to z. Or $\lim_{n\to\infty} Tx_{n+1} = z$, $\lim_{n\to\infty} Fx_n = z$, $\lim_{n\to\infty} Sx_{n+2} = z$ and $\lim_{n\to\infty} Ax_{n+1} = z$.

Case (1) - When T is continuous map

Since $\lim_{n\to\infty} Tx_n = z$, therefor $\lim_{n\to\infty} TTx_n = Tz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Tz$. Since pair (T, F) is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Tz$. Now using (b) with $x = Tx_n, y = x_n$

$$\phi[d(TTx_n, Sx_n)] \geq \phi[d(TTx_n, FTx_n)) + d(TTx_n, Ax_n)] + \varphi[d(TTx_n, Sx_n)$$

Now limiting $\lim n \to \infty$ we have

$$\phi[d(Tz, z)] \geq \phi[d(Tz, Tz) + d(Tz, z)] + \varphi d(Tz, z)]$$
$$\phi[d(Tz, z)] \geq \phi[d(Tz, z)] + \varphi d(Tz, z)$$
$$\varphi[d(Tz, z)] \leq 0.$$

It is only possible when, $Tz = z$. Since $T(X) \subset F(X)$. Then let $u \in X$ such that $Tz = Fu = z$. Now by using (b) with $x = u$, $y = x_n$

$$\phi[d(Tu, Sx_n)] \geq \phi[d(Tu, Fu) + d(Tu, Ax_n)] + \varphi d(Tu, Sx_n)]$$

Taking $\lim n \to \infty$

$$\phi[d(Tu, z)] \geq \phi[d(Tu, z) + d(Tu, z)] + \varphi d(Tu, z)$$
$$\phi[d(Tu, z)] \geq \phi[2d(Tu, z)] + \varphi d(Tu, z)$$

Since ϕ and φ are increasing function therefore obtained inequality is only possible when $d(Tu, z) = 0 \Rightarrow Tu = z$ or $Tz = Fu = z$. Since (F, T) is weak compatible then $FTu = TFu \Rightarrow Fz = Tz$ or $Fz = Tz = z$. Now by using (b) with $x = Sz$ and $y = x_n$ we have

$$\phi[d(TSz, Sx_n)] \geq \phi[d(TSz, FSz) + d(TSz, Ax_n)] + \varphi d(TSz, Sx_n)]$$

Since $TS = ST$ and $FS = SF$ and limiting $n \to \infty$ we have

$$\phi[d(STz, z)] \geq \phi[d(STz, SFz) + d(STz, z)] + \varphi d(STz, z)$$
$$\phi[d(Sz, z)] \geq \phi[d(Sz, Sz) + d(Sz, z)] + \varphi d(Sz, z)$$
$$\phi[d(Sz, z)] \geq \phi[d(Sz, z)] + \varphi d(Sz, z)$$
$$\varphi[d(Sz, z)] \leq 0 \Rightarrow Sz = z$$

By using (b) with $x = x_n$, $y = z$

$$\phi[d(Tx_n, Sz)] \geq \phi[d(Tx_n, Fx_n) + d(Tx_n, Az)] + \varphi d(Tx_n, Sz)]$$

Now liming $n \to \infty$

$$\phi[d(z, z)] \geq \phi[d(z, z) + d(z, Az)] + \varphi d(z, z)$$
$$\phi[d(z, Az)] \leq 0 \Rightarrow Az = z \text{ or } Sz = Tz = Fz = Az = z.$$

Therefore z is common fixed point T, F, S and A.

Case (2) - When F is continuous map

since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Fz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} FFx_n = Fz$ Since pair (T, F) is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Fz$ Now using (b) with, $x = Fx_n$, $y = x_n$

$$\phi[d(TFx_n, Sx_n)] \geq \phi[d(TFx_n, FFx_n)) + d(TFx_n, Ax_n)] + \varphi[d(TFx_n, Sx_n)$$

Now limiting $n \to \infty$ we have

$$\phi[d(Fz, z)] \geq \phi[d(Fz, Fz) + d(Fz, z)] + \varphi d(Fz, z)$$
$$\phi[d(Fz, z)] \geq \phi[d(Fz, z)] + \varphi d(Fz, z)$$
$$\varphi d(Fz, z)] \leq 0$$

It is only possible when$d(Fz, z) \Rightarrow Fz = z$. Again using (b) with $x = z$, $y = x_n$

$$\phi[d(Tz, Sx_n)] \geq \phi[d(Tz, Fz) + d(Tz, Ax_n)] + \varphi d(Tz, Sx_n)$$

Limiting $n \to \infty$

$$\phi[d(Tz, z)] \geq \phi[d(Tz, z) + d(Tz, z)] + \varphi d(Tz, z)$$
$$\phi[d(Tz, z)] \geq \phi[2d(Tz, z)] + \varphi d(Tz, z)$$

Since ϕ and φ are increasing function therefore obtained inequality is only possible when $d(Tz, z) = 0 \Rightarrow Tz = z$ Or $Fz = Tz = z$. Now by using (b) with $x = Sz$ and $y = x_n$ we have

$$\phi[d(TSz, Sx_n)] \geq \phi[d(TSz, FSz) + d(TSz, Ax_n)] + \varphi d(TSz, Sx_n)]$$

Since $TS = ST$ and $FS = SF$ and limiting $n \to \infty$ we have

$$\begin{aligned}
\phi[d(STz, z)] &\geq \phi[d(STz, SFz) + d(STz, z)] + \varphi d(STz, z) \\
\phi[d(Sz, z)] &\geq \phi[d(Sz, Sz) + d(Sz, z)] + \varphi d(Sz, z) \\
\phi[d(Sz, z)] &\geq \phi[d(Sz, z)] + \varphi d(Sz, z) \\
\varphi[d(Sz, z)] &\leq 0 \Rightarrow Sz = z
\end{aligned}$$

By using (b) with $x = x_n$, $y = z$

$$\phi[d(Tx_n, Sz)] \geq \phi[d(Tx_n, Fx_n) + d(Tx_n, Az)] + \varphi d(Tx_n, Sz)]$$

Now liming $n \to \infty$

$$\begin{aligned}
\phi[d(z, z)] &\geq \phi[d(z, z) + d(z, Az)] + \varphi d(z, z) \\
\phi[d(z, Az)] &\leq 0 \Rightarrow Az = z \text{ or } Sz = Tz = Fz = Az = z
\end{aligned}$$

Therefore z is common fixed point of T, F, S and A. Uniqueness can easily proved. □

Theorem 3.5. *Let T and F be self mapping of metric space (X, D) with*

(a) $T(X) \subset F(X)$

(b) $\phi[d(Tx, Ty)] \geq \phi[d(Ty, Fx)) + d(Ty, Fy)] + \varphi[d(Tx, Ty)]$

(c) Either T or F is continuous function.

(d) (T, F) is semi compatible and weak compatible.

If ϕ and φ are monotonic increasing function such that $\phi, \varphi : [0, \infty) \to [0, \infty)$ and $\varphi(t) = 0 = \phi(t) \Rightarrow t = 0$ then z is unique common fixed point of F and T.

Proof. Let $x_0 \in X$ is an arbitrary point. Since $T(X) \subset F(X)$. Then $x_1 \in X$ such that$Tx_1 = Fx_0$. Inductively we can define a sequence $Tx_{n+1} = Fx_n$. Using (b) with $x = x_n$, $y = x_{n+1}$

$$\begin{aligned}
\phi[d(Tx_n, Tx_{n+1})] &\geq \phi[d(Tx_{n+1}, Fx_n)) + d(Tx_{n+1}, Fx_{n+1})] + \varphi[d(Tx_n, Tx_{n+1}) \\
&\geq \phi[d(Tx_{n+1}, Tx_{n+1})) + d(Tx_{n+1}, Tx_{n+2})] + \varphi[d(Tx_n, Tx_{n+1})] \\
\phi[d(Tx_n, Tx_{n+1})] &\geq \phi[d(Tx_{n+1}, Tx_{n+2})] + \varphi d(Tx_n, Tx_{n+1})] \qquad (16) \\
\phi[d(Tx_n, Tx_{n+1})] &\geq \phi[d(Tx_{n+1}, Tx_{n+2})]
\end{aligned}$$

Since ϕ is an increasing function then we have $d(Tx_n, Tx_{n+1}) \geq d(Tx_{n+1}, Tx_{n+2})$. Therefore the sequence $d(Tx_n, Tx_{n+1})$ will be decreasing. Let $r \geq 0$ such that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = r. \qquad (17)$$

Hence on taking $\lim_{n\to\infty}$ we have by (16), $\phi(r) \geq \phi(r) + \varphi(r)$. It is only possible when $r = 0$. Then by (17)

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0 \qquad (18)$$

Now we shall show that $\{Tx_n\}$ is Cauchy sequence. Let we assume contrary. Then there exist $\epsilon > 0$ such that for $m, n \to \infty$ and for $m_i < n_i < m_{i+1}$

$$d(Tx_{m_i}, Tx_{n_i}) \geq \epsilon \tag{19}$$

and $d(Tx_{m_i}, Tx_{n_{i-1}}) < \epsilon$. Then it follows that

$$\begin{aligned}
\epsilon \leq [d(Tx_{m_i}, Tx_{n_i})] &\leq d(Tx_{m_i}, Tx_{n_{i-1}}) + d(Tx_{n_{i-1}}, Tx_{n_i}) < \varepsilon + d(Tx_{n_{i-1}}, Tx_{n_i}) \\
d(Tx_{m_i}, Tx_{n_i}) &< \epsilon + d(Tx_{n_{i-1}}, Tx_{n_i}) \\
\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) &< \epsilon + \lim_{i\to\infty} d(Tx_{n_{i-1}}, Tx_{n_i})
\end{aligned}$$

By (18)

$$\lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) < \epsilon \tag{20}$$

By (19) and (20)

$$\lim_{n\to\infty} d(Tx_{m_i}, Tx_{n_i}) = \epsilon \tag{21}$$

Now by using (b) with $x = x_{m_i}$, $y = x_{n_i}$

$$\begin{aligned}
\phi[d(Tx_{m_i}, Tx_{n_i})] &\geq \phi[d(Tx_{n_i}, Fx_{m_i}) + \phi[d(Tx_{n_i}, Fx_{n_i})]] + \varphi[d(Tx_{m_i}, Tx_{n_i})] \\
\phi[d(Tx_{m_i}, Tx_{n_i})] &\geq \phi[d(Tx_{n_i}, Tx_{m_{i+1}}) + \phi[d(Tx_{n_i}, Tx_{n_{i+1}})]] + \varphi d(Tx_{m_i}, Tx_{n_i})
\end{aligned}$$

$\lim_{i\to\infty}$ and By (18) and (21) we have $\phi(\varepsilon) \geq \phi(\varepsilon) + \varphi(\varepsilon)$.s. It is only possible when $\varepsilon = 0$. Which is contradiction to our assumption that $\varepsilon > 0$. Therefore for all $m, n \to \infty$ we have $d(Tx_{n_i}, Tx_{m_i}) < \varepsilon$. Therefore $\{Tx_n\}$ is a Cauchy sequence. Since $(X, D)$is complete metric space, then it will be converge at some point $z \in X$, or $\lim_{n\to\infty} Tx_n = z$ So all of its subsequence also converge to z. Or $\lim_{n\to\infty} Tx_{n+1} = z$, $\lim_{n\to\infty} Fx_n = z$.

Case (1) When T is continuous map

Since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} TTx_n = Tz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} TFx_n = Tz$. Since pair (T, F) is semi compatible map. since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Tz$. Now using (b) with $x = Tx_n$, $y = x_n$

$$\phi[d(TTx_n, Tx_n)] \geq \phi[d(Tx_n, FTx_n) + d(Tx_n, Fx_n) + \varphi[d(TTx_n, Tx_n)$$

Now limiting $n \to \infty$ we have

$$\begin{aligned}
\phi[d(Tz, z)] &\geq \phi[d(z, Tz) + d(z, z)] + \varphi d(Tz, z)] \\
\phi[d(Tz, z)] &\geq \phi d(z, Tz) + \varphi d(Tz, z) \\
\varphi d(Tz, z) &\leq 0
\end{aligned}$$

It is only possible when $d(Tz, z) = 0 \Rightarrow Tz = z$. Since $T(X) \subset F(X)$. Then let $u \in X$ such that $Tz = Fu = z$. Now by using (b) with $x = x_n$, $y = u$

$$\phi[d(Tx_n, Tu)] \geq \phi d(Tu, Fx_n), d(Tu, Fu) + \varphi[d(Tx_n, u).$$

Taking $\lim_{n\to\infty}$

$$\begin{aligned}
\phi[d(z, Tu)] &\geq \phi[d(Tu, z) + d(Tu, z)] + \varphi d(z, Tu) \\
\phi[d(z, Tu) &\geq \phi[2d(Tu, z)] + \varphi d(z, Tu).
\end{aligned}$$

Since ϕ and φ are increasing function therefore obtained inequality is only possible when $d(Tu,z) = 0 \Rightarrow Tu = z$ or $Fu = Tu = z$. Since (F,T) is weak compatible then $FTu = TFu \Rightarrow Fz = Tz$ or $Fz = Tz = z$. Therefore z is common fixed point of F and T.

Case (2) - When F is continuous map

since $\lim_{n\to\infty} Tx_n = z$, therefore $\lim_{n\to\infty} FTx_n = Fz$. Also $\lim_{n\to\infty} Fx_n = z$, therefore $\lim_{n\to\infty} FFx_n = Fz$ Since pair (T,F) is semi compatible map then since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Tx_n = z$, therefore, $\lim_{n\to\infty} TFx_n = Fz$. Now using (b) with, $x = Fx_n$, $y = x_n$

$$\phi[d(TFx_n, Tx_n)] \geq \phi[d(Tx_n, FFx_n)) + d(Tx_n, Fx_n)] + \varphi[d(TFx_n, Tx_n)$$

Now limiting $n \to \infty$ we have

$$\phi[d(Fz,z)] \geq \phi[d(z,Fz) + d(z,z)] + \varphi d(Fz,z)$$
$$\phi[d(Fz,z)] \geq \phi[d(z,Fz)] + \varphi d(Fz,z)$$
$$\varphi d(Fz,z)] \leq 0.$$

It is only possible when $d(Fz,z) = 0 \Rightarrow Fz = z$. Again using (b) with $x = x_n$, $y = z$

$$\phi[d(Tx_n, Tz)] \geq \phi[d(Tz, Fx_n) + d(Tz, Fz)] + \varphi d(Tx_n, Tz)$$

Taking $\lim_{n} \to \infty$

$$\phi[d(z,Tz)] \geq \phi[d(Tz,z) + d(Tz,z)] + \varphi d(z,Tz)$$
$$\phi[d(Tz,z)] \geq \phi[2d(Tz,z)] + \varphi d(z,Tz)$$

Since ϕ and φ are increasing function therefore obtained inequality is only possible when $d(Tz,z) = 0 \Rightarrow Tz = z$ Or $Fz = Tz = z$.

Uniqueness: Let w be another fixed point of F and T. Then $Fw = Tw = w$. By using (b) with $x = z$, $y = w$ we have

$$\phi[d(Tz,Tw)] \geq \phi[d(Tw,Fz) + d(Tw,Fw)] + \varphi d(Tz,Tw)$$
$$\phi[d(z,w)] \geq \phi[d(w,z) + d(w,w)] + \varphi d(z,w)$$
$$\phi[d(z,w)] \geq \phi[d(w,z)] + \varphi d(z,w)$$
$$\varphi d(z,w) \leq 0 \Rightarrow z = w.$$

Hence z is a unique common fixed point of F and T. This complete the proof. □

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