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Stability of a Quadratic Functional Equation Originating From Sum of the Medians of a Triangle in Fuzzy Ternary Banach Algebras: Direct and Fixed Point Methods

Research Article

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Abstract: In this paper, we obtain the solution in vector space and the generalized Ulam-Hyers stability of the ternary quadratic homomorphisms and ternary quadratic derivations between fuzzy ternary Banach algebras associated to the quadratic functional equation

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right)=\frac{3}{4}\left(f(x-y)+f(y-z)+f(z-x)\right)$$

originating from sum of the medians of a triangle by using direct and fixed point methods. An application of this functional equation is also studied.

MSC: 39B52, 32B72, 32B82

Keywords: Fuzzy ternary Banach algebra, Quadratic functional equation, Ulam - Hyers stability, Fixed point method.

1. Introduction and Preliminaries

A classical question in the theory of functional equations is the following "When is it true that a function which approximately satisfies a functional equation ϵ must be close to an exact solution ϵ? If the problem accepts a solution, we say that the equation ϵ is stable".

In 1940, Ulam [44] at the University of Wiscosin, he proposed the following stability problem:

Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exists $\delta(\epsilon)>0$ such that if $h:G_1\to G_2$ satisfies the inequality

$$d\left(h(x*y),h(x)\diamond h(y)\right)<\delta \qquad x,y\in G_1,$$

then there is a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$.

In the next year, Hyers [18] gave a affirmative answer to this question for additive groups under the assumption that groups are Banach spaces. In 1950, T. Aoki [3] first generalized the Hyers theorem for unbounded Cauchy difference. In generalizing the definition of Hyers, T. Aoki proved the following result, when $f:X\to Y$ is a mapping and X and Y are normed spaces.

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Theorem 1.1. *Let $f(x)$ from X to Y be an approximately linear transformation, when there exists $K \geq 0$ and $0 \leq p < 1$ such that $\|f(x+y) - f(x) - f(y) \leq K(\|x\|^p + \|y\|^p)\|$ for any x and y in X. Let $f(x)$ and $\phi(x)$ be transformations from X to Y. These are called near when there exists $K \geq 0$and $0 \leq p < 1$ such that $\|f(x) - \phi(x)\| \leq K\|x\|^p$ for any x in X.*

The above result was rediscovered by Th. M. Rassias [38] in 1978 and proved the generalization of Hyers theorem for additive mappings as a special case in the form of following:

Theorem 1.2. *Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on R, and let there exist $\epsilon \geq 0$ and $p \in [0, 1)$ such that $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E$. Then there exists a unique linear mapping $T : E \to F$ such that $\|f(x) - T(x)\| \leq \epsilon \frac{\|x\|^p}{1 - 2^{(p-1)}}$ for all $x \in E$.*

In 1982 J.M. Rassias [35], followed the innovative approach of Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ with $p + q \neq 1$.

In 1990, during the 27th International Symposium on Functional Equations, Th.M.Rassias asked a question whether the Theorem 1.2 can also be proved for value of $p \geq 1$. In 1991, Gajda [27] provided an partial solution to Th.M. Rassiass question for $p > 1$. He established the following result:

Theorem 1.3. *Let X and Y be two (real) normed linear spaces and assume that Y is complete. Let $f : X \to Y$ be a mapping for which there exist two constants $\epsilon \in [0, \infty)$ and $p \in R - \{1\}$ such that $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \|x\|^p$ for all $x \in X$, where $\delta = \frac{2\epsilon}{2-2^p}$ for $p < 1$ and $\delta = \frac{2\epsilon}{2^p-2}$ for $p > 1$, Moreover, for each $x \in X$, the transformation $t \to f(tx)$ is continuous, then the mapping T is linear.*

However, Gajda [14] and Th.M.Rassias and P.Semrl [40] independently showed that a similar result can not be obtained for $p = 1$. They presented the following:

Remark 1.4. *Theorem 1.2 holds for all $p \in R - \{1\}$. Gajda [14] in 1991 gave an example to show that the Theorem 1.2 fails if $p = 1$. Gajda [14] succeeded in constructing an example of a bounded continuous function $g : R \to R$ satisfying $|g(x+y) - g(x) - g(y)| \leq |x| + |y|$ for all $x, y \in R$, with $\lim_{x \to 0} \frac{g(x)}{x} = \infty$.*

In 1994, P. Gavruta [15] provided a further generalization of Th.M. Rassias [38] theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et. al., [41] by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach.

In 2009, C. Park and Th. M. Rassias [33] proved Hyers-Ulam stability of homomorphisms in Banach algebras for the mapping $f : A \to B$ where A and B are Complex Banach algebras which satisfies the functional equation $\mu f(x+y) = f(\mu x) + f(\mu y)$ for all $\mu \in T^1 = \{v \in C : |v| = 1\}$ for all $x, y \in A$ and C linear mapping (i.e. A C- linear mapping $H : A \to B$ is called a homomorphism in Banach algebra if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$). They also obtained the Hyers-Ulam-Rassias stability of derivations on Banach algebra for the Cauchy functional equation. M.S. Moslehian and Th.M. Rassias [32] proved that the Hyers-Ulam-Rassias stability holds for Non-Archimedean normed spaces. They consider that G is an additive group and X is a complete Non-Archimedean space. For more details about stability of functional equations, one can refer to [42].

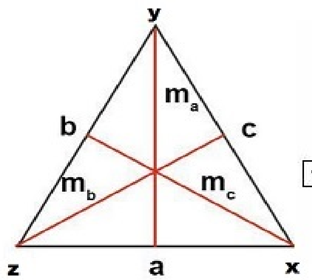
During the last seven decades, the stability problems of various functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces, orthogonal spaces and many spaces have been broadly investigated by number of mathematicians (see [4–7, 11, 13, 19–21, 31, 36–39]).

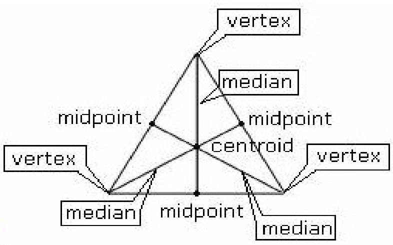
The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is said to be quadratic functional equation because the quadratic function $f(x)=ax^2$ is a solution of the functional equation (1).

In geometry, a median of a triangle is a line segment joining a vertex to the midpoint of the opposing side. Every triangle has exactly three medians: one running from each vertex to the opposite side. In the case of isosceles and equilateral triangles, a median bisects any angle at a vertex whose two adjacent sides are equal in length.





In a triangle with the sides a, b and c the median drawn to the side c is

$$m_c^2=\frac{1}{2}\left(a^2+b^2\right)-\frac{1}{4}c^2. \tag{2}$$

For a triangle with the vertices $x, y, z \in R^2$ and if we take

$$a=z-x, b=z-y, c=x-y,$$

and the length of a median m_c from z to the midpoint of x and y is

$$m_c=\frac{x+y}{2}-z.$$

In functional equation the length of the median from z is given by

$$f\left(\frac{x+y}{2}-z\right)=\frac{1}{2}\left(f(z-x)+f(z-y)\right)-\frac{1}{4}f(x-y).$$

In a triangle with the sides a, b and c the lengths of the medians m_a ,m_b and m_c, drawn to the sides a, b and c respectively satisfy to the identity

$$m_a^2+m_b^2+m_c^2=\frac{3a^2+3b^2+3c^2}{4}. \tag{3}$$

In functional equation the sum of the medians of a triangle is of the form

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right)=\frac{3}{4}\left(f(x-y)+f(y-z)+f(z-x)\right) \tag{4}$$

having solution $f(x)=ax^2$.

Now, we give some definitions which helps to investigate the stability results in fuzzy ternary banach algebras.

Definition 1.5 ([29]). *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

$(F1)$ $N(x, c) = 0$ *for* $c \leq 0$;

$(F2)$ $x = 0$ *if and only if* $N(x, c) = 1$ *for all* $c > 0$;

$(F3)$ $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

$(F4)$ $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;

$(F5)$ $N(x, \cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty}N(x, t) = 1$;

$(F6)$ *for* $x \neq 0, N(x, \cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X, t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 1.6 ([29]). *Let $(X, || \cdot ||)$ be a normed linear space and $\beta > 0$. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\beta\|x\|}, & t>0, \quad x\in X, \\ 0, & t\leq 0, \quad x\in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.7 ([29]). *Let (X, N) be a fuzzy normed vector space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.*

Definition 1.8. *p[29]] A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 1.9 ([29]). *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Ternary algebraic operations were considered in the nineteenth century by several mathematicians such as Cayley [10] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in [22]. The comments on physical applications of ternary structures can be found in [1, 25, 26, 43, 45].

Definition 1.10. *Let X be a ternary algebra and (X, N) be a fuzzy normed space.*

(1) *The fuzzy normed space (X, N) is called a ternary fuzzy normed algebra if*

$$N([xyz], stu) \geq N(x, s) N(y, t) N(z, u)$$

for all $x, y, z \in X$ and $s, t, u > 0$;

(2) *A complete ternary fuzzy normed algebra is called a ternary fuzzy Banach algebra.*

Example 1.11. *Let $(X, || \cdot ||)$ be a ternary normed (Banach) algebra. Let*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0, \quad x\in X, \\ 0, & t\leq 0, \quad x\in X. \end{cases}$$

Then $N(x, t)$ is a fuzzy norm on X and (X, N) is a ternary fuzzy normed (Banach) algebra.

Definition 1.12. *Let (X,N) and (Y,N') be two ternary fuzzy normed algebras.*

(1) *A $\mathbb{C}$-linear mapping $H:(X,N)\to(Y,N')$ is called a ternary quadratic homomorphism if*

$$H([xyz]) = [H(x)H(y)H(z)]$$

for all $x,y,z\in X$;

(2) *A $\mathbb{C}$-linear mapping $D:(X,N)\to(X,N)$ is called a ternary quadratic derivation if*

$$D([xyz]) = \left[D(x)y^2z^2\right] + \left[x^2D(y)z^2\right] + \left[x^2y^2D(z)\right]$$

for all $x,y,z\in X$.

For more details about fuzzy normed spaces and fuzzy normed algebras, one can refer to [9, 12, 16, 17, 23, 24, 29, 30, 34]. In this paper, we obtain the solution in vector spaces and the generalized Ulam-Hyers stability of the ternary quadratic homomorphisms and ternary quadratic derivations between fuzzy ternary Banach algebras associated to the quadratic functional equation (4) originating from sum of the medians of a triangle by using direct and fixed point methods. An application of this functional equation is also studied.

2. General Solution of the Functional Equation (4)

In this section, the authors investigate the general solution of quadratic functional equation (4). Throughout this section let us consider X and Y be real vector spaces.

Theorem 2.1. *Let X and Y be real vector spaces. If the mapping $f:X\to Y$ satisfies the functional equation (1) for all $x,y\in X$ then $f:X\to Y$ satisfying the functional equation (4) for all $x,y,z\in X$.*

Proof. Setting $x=y=0$ in (1), we get $f(0)=0$. Let $x=0$ in (1), we obtain $f(-x)=f(x)$ for all $x\in X$. Therefore f is an even function. Replacing y by x and $2x$ respectively in (1), we get $f(2x)=2^2f(x)$ and $f(3x)=3^2f(x)$ for all $x\in X$. In general for any positive integer n, we have $f(nx)=n^2f(x)$ for all $x\in X$.
Replacing (x,y) by $(x-z,y-z)$ in (1) and using evenness, we arrive

$$f\left(\frac{x+y}{2}-z\right) = \frac{1}{2}\left(f(z-x)+f(z-y)\right) - \frac{1}{4}f(x-y) \tag{5}$$

for all $x,y,z\in X$. Replacing (x,y,z) by (z,y,x) in (5), we get

$$f\left(\frac{z+y}{2}-x\right) = \frac{1}{2}\left(f(x-z)+f(x-y)\right) - \frac{1}{4}f(z-y) \tag{6}$$

for all $x,y,z\in X$. Replacing (x,y,z) by (x,z,y) in (5), we get

$$f\left(\frac{x+z}{2}-y\right) = \frac{1}{2}\left(f(y-x)+f(y-z)\right) - \frac{1}{4}f(x-z) \tag{7}$$

for all $x,y,z\in X$. Adding (5),(6) and (7) and using evenness, we derive (4) for all $x,y,z\in X$. □

Hereafter throughout this paper, we assume that X is a ternary fuzzy normed algebra and Y is a ternary fuzzy Banach algebra. For the convenience, we define a mapping $F:X\to Y$ by

$$\begin{aligned} F(x,y,z) =& f\left(\frac{x+y}{2}-z\right) + f\left(\frac{y+z}{2}-x\right) + f\left(\frac{z+x}{2}-y\right) \\ &- \frac{3}{4}\left(f(x-y)+f(y-z)+f(z-x)\right) \end{aligned}$$

for all $x,y,z\in X$.

3. Stability Results: Direct Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (4).

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed and let $\alpha : X^3 \to [0, \infty)$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2^2}\right)^j < 1$*

$$N'\left(\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right), r\right) \geq N'\left(d^{nj}\alpha(x, y, z), r\right) \tag{8}$$

for all $x, y, z \in X$ and all $r > 0$ and

$$\lim_{n\to\infty} N'\left(\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right), r\right) = 1 \tag{9}$$

for all $x, y, z \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the following inequalities

$$N(F(x, y, z), r) \geq N'(\alpha(x, y, z), r) \tag{10}$$

and

$$N(f([xyz]) - [f(x)f(y)f(z)], r) \geq N'(\alpha(x, y, z), r) \tag{11}$$

for all $x, y, z \in X$ and all $r > 0$. Then there exists a unique ternary quadratic homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), r) \geq N'\left(\alpha(x, x, -x), r|2^2 - d|\right) \tag{12}$$

for all $x \in X$ and all $r > 0$. The mapping $H(x)$ is defined by

$$H(x) = N - \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{2nj}} \tag{13}$$

for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x, y, z) by $(x, x, -x)$ in (10), we get

$$N\left(f(2x) - 2^2 f(x), r\right) \geq N'(\alpha(x, x, -x), r) \tag{14}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^n x$ in (14), we obtain

$$N\left(\frac{f(2^{n+1}x)}{2^2} - f(2^n x), \frac{r}{2^2}\right) \geq N'(\alpha(2^n x, 2^n x, -2^n x), r) \tag{15}$$

for all $x \in X$ and all $r > 0$. Using (8) and $(F3)$ in (15), we arrive

$$N\left(\frac{f(2^{n+1}x)}{2^2} - f(2^n x), \frac{r}{2^2}\right) \geq N'\left(\alpha(x, x, -x), \frac{r}{d^n}\right) \tag{16}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (16), that

$$N\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{r}{2^{2(n+1)}}\right) \geq N'\left(\alpha(x, x, -x), \frac{r}{d^n}\right) \tag{17}$$

holds for all $x \in X$ and all $r > 0$. Replacing r by $d^n r$ in (17), we get

$$N\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{d^n\ r}{2^{2(n+1)}}\right) \geq N'(\alpha(x, x, -x), r) \tag{18}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^{2n}} - f(x) = \sum_{i=0}^{n-1} \left[\frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}} \right] \tag{19}$$

for all $x \in X$. From equations (18) and (19), we have

$$\begin{aligned} N\left(\frac{f(2^n x)}{2^{2n}} - f(x), \sum_{i=0}^{n-1} \frac{d^i\ r}{2^{2(i+1)}} \right) &\geq min \bigcup_{i=0}^{n-1} \left\{ \frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}}, \frac{d^i\ r}{2^{2(i+1)}} \right\} \\ &\geq min \bigcup_{i=0}^{n-1} \left\{ N'\left(\alpha(x,x,-x), r\right) \right\} \geq N'\left(\alpha(x,x,-x), r\right) \end{aligned} \tag{20}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (20) and using (8), $(F3)$, we obtain

$$N\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=0}^{n-1} \frac{d^i\ r}{2^{2(m+i+1)}} \right) \geq N'\left(\alpha(x,x,-x), \frac{r}{d^m} \right) \tag{21}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $d^m r$ in (21), we get

$$N\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=m}^{m+n-1} \frac{d^i\ r}{2^{2(i+1)}} \right) \geq N'\left(\alpha(x,x,-x), r\right) \tag{22}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Using $(F3)$ in (22), we obtain

$$N\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, r \right) \geq N'\left(\alpha(x,x,-x), \frac{r}{\sum_{i=m}^{m+n-1} \frac{d^i}{2^{2(i+1)}}} \right) \tag{23}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < n^2$ and $\sum_{i=0}^{n} \left(\frac{d}{n^2}\right)^i < \infty$, the cauchy criterion for convergence and $(F5)$ implies that $\left\{ \frac{f(2^n x)}{2^{2n}} \right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy ternary Banach space, this sequence converges to some point $H(x) \in Y$. So one can we define the mapping $H : X \to Y$ by

$$H(x) = N - \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$. Letting $m = 0$ in (23), we get

$$N\left(\frac{f(2^n x)}{2^{2n}} - f(x), r \right) \geq N'\left(\alpha(x,x,-x), \frac{r}{\sum_{i=0}^{n-1} \frac{d^i}{2^{2(i+1)}}} \right) \tag{24}$$

for all $x \in X$ and all $r > 0$. Letting $n \to \infty$ in (24) and using $(F6)$, we arrive

$$N\left(f(x) - H(x), r\right) \geq N'\left(\alpha(x,x,-x), r(2^2 - d)\right)$$

for all $x \in X$ and all $r > 0$. Now, we need to prove H satisfies the (4), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ in (10), respectively , we obtain

$$N\left(\frac{1}{2^{2n}} Df\left(2^n x, 2^n y, 2^n z\right), r \right) \geq N'\left(\alpha\left(2^n x, 2^n y, 2^n z\right), 2^{2n} r \right) \tag{25}$$

for all $r > 0$ and all $x, y, z \in X$. Now,

$$\begin{aligned}
&N\left(H\left(\frac{x+y}{2}-z\right)+H\left(\frac{y+z}{2}-x\right)+H\left(\frac{z+x}{2}-y\right)-\frac{3}{4}\left(H(x-y)+H(y-z)+H(z-y)\right),r\right)\\
&\geq min\left\{N\left(H\left(\frac{x+y}{2}-z\right)-\frac{1}{2^{2n}}f\left(\frac{2^n(x+y)}{2}-2^nz\right),\frac{r}{7}\right),N\left(H\left(\frac{y+z}{2}-x\right)-\frac{1}{2^{2n}}f\left(\frac{2^n(y+z)}{2}-2^nx\right),\frac{r}{7}\right),\right.\\
&\quad N\left(H\left(\frac{z+x}{2}-y\right)-\frac{1}{2^{2n}}f\left(\frac{2^n(z+x)}{2}-2^ny\right),\frac{r}{7}\right),N\left(-\frac{3}{4}H(x-y)+\frac{3}{(4)2^{2n}}f\left(2^nx-2^ny\right),\frac{r}{7}\right),\\
&\quad N\left(-\frac{3}{4}H(y-z)+\frac{3}{(4)2^{2n}}f\left(2^ny-2^nz\right),\frac{r}{7}\right),N\left(-\frac{3}{4}H(z-x)+\frac{3}{(4)2^{2n}}f\left(2^nz-2^nx\right),\frac{r}{7}\right),\\
&\quad N\left(\frac{1}{2^{2n}}\left(f\left(\frac{2^n(x+y)}{2}-z\right)+f\left(\frac{2^n(y+z)}{2}-x\right)+f\left(\frac{2^n(z+x)}{2}-y\right)\right.\right.\\
&\quad\left.\left.\left.-\frac{3}{4}\left(f\left(2^nx-2^ny\right)+f\left(2^ny-2^nz\right)+f\left(2^nz-2^ny\right)\right)\right),\frac{r}{7}\right)\right\}
\end{aligned} \tag{26}$$

for all $x, y, z \in X$ and all $r > 0$. Using (25) and $(F5)$ in (26), we arrive

$$\begin{aligned}
&N\left(H\left(\frac{x+y}{2}-z\right)+H\left(\frac{y+z}{2}-x\right)+H\left(\frac{z+x}{2}-y\right)-\frac{3}{4}\left(H(x-y)+H(y-z)+H(z-y)\right),r\right)\\
&\geq min\left\{1,1,1,1,1,1,N'\left(\alpha\left(2^nx,2^ny,2^nz\right),2^{2n}r\right)\right\}\\
&\geq N'\left(\alpha\left(2^nx,2^ny,2^nz\right),2^{2n}r\right)
\end{aligned} \tag{27}$$

for all $x, y, z \in X$ and all $r > 0$. Letting $k \to \infty$ in (27) and using (9), we see that

$$N\left(H\left(\frac{x+y}{2}-z\right)+H\left(\frac{y+z}{2}-x\right)+H\left(\frac{z+x}{2}-y\right)-\frac{3}{4}\left(H(x-y)+H(y-z)+H(z-y)\right),r\right)=1 \tag{28}$$

for all $x, y, z \in X$ and all $r > 0$. Using $(F2)$ in the above inequality gives

$$H\left(\frac{x+y}{2}-z\right)+H\left(\frac{y+z}{2}-x\right)+H\left(\frac{z+x}{2}-y\right)=\frac{3}{4}\left(H(x-y)+H(y-z)+H(z-y)\right)$$

for all $x, y, z \in X$. Hence H satisfies the quadratic functional equation (4). This shows that H is quadratic. So it follows that

$$\begin{aligned}
N\left(H\left([xyz]\right)-\left[H(x)H(y)H(z)\right],r\right)&=N\left(\frac{1}{2^{6n}}\left(f\left(2^{3n}[xyz]\right)-\left[f(2^nx)f(2^ny)f(2^nz)\right]\right),\frac{r}{2^{6n}}\right)\\
&\geq N'\left(\alpha\left(2^nx,2^ny,2^nz\right),r\right)
\end{aligned} \tag{29}$$

for all $x, y, z \in X$ and all $r > 0$. Letting $n \to \infty$ in (29) and using (9), we gain

$$N\left(H\left([xyz]\right)-\left[H(x)H(y)H(z)\right],r\right)=1$$

for all $x, y, z \in X$ and all $r > 0$. Hence we have $H([xyz]) = [H(x)H(y)H(z)]$ for all $x, y, z \in X$. Therefore , H is a ternary quadratic homomorphism. In order to prove $H(x)$ is unique, let $H'(x)$ be another quadratic functional equation satisfying (4) and (12). Hence,

$$\begin{aligned}
N(H(x)-H'(x),r)&=N\left(\frac{H(2^nx)}{2^{2n}}-\frac{H'(2^nx)}{2^{2n}},r\right)\\
&\geq min\left\{N\left(\frac{H(2^nx)}{2^{2n}}-\frac{f(2^nx)}{2^{2n}},\frac{r}{2}\right),N\left(\frac{f(2^nx)}{2^{2n}}-\frac{H'(2^nx)}{2^{2n}},\frac{r}{2}\right)\right\}\\
&\geq N'\left(\alpha(2^nx,2^nx,-2^nx),\frac{r\left(2^2-d\right)}{2}\right)\\
&\geq N'\left(\alpha(x,x,-x),\frac{r\left(2^2-d\right)}{2d^n}\right)
\end{aligned}$$

for all $x \in X$ and all $r > 0$. Since

$$\lim_{n\to\infty} \frac{r\left(2^2-d\right)}{2d^n} = \infty,$$

we obtain

$$\lim_{n\to\infty} N'\left(\alpha(x,x,-x), \frac{r\left(2^2-d\right)}{2d^n}\right) = 1$$

for all $x \in X$ and all $r > 0$. Thus

$$N(H(x) - H'(x), r) = 1$$

for all $x \in X$ and all $r > 0$. Hence, we have $H(x) = H'(x)$. Therefore $H(x)$ is unique. Thus the mapping $H : X \to Y$ is a unique ternary quadratic homomorphism.

For $j = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

From Theorem 3.1, we obtain the following corollary concerning the generalized Ulam-Hyers stability for the functional equation (4).

Corollary 3.2. *Suppose that a function $F : X \to Y$ satisfies the inequality*

$$N\left(F(x,y,z), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), & s \neq 2; \\ N'\left(\epsilon\left(||x||^s ||y||^s ||z||^s\right), r\right), & s \neq \frac{2}{3}; \\ N'\left(\epsilon\left\{||x||^s ||y||^s ||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}, r\right), & s \neq \frac{2}{3}; \end{cases} \tag{30}$$

for all $x, y, z \in X$ and all $r > 0$ and

$$N\left(H\left([xyz]\right) - \left[H(x)H(y)H(z)\right], r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right) \\ N'\left(\epsilon\left(||x||^s ||y||^s ||z||^s\right), r\right) \\ N'\left(\epsilon\left\{||x||^s ||y||^s ||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}, r\right) \end{cases} \tag{31}$$

for all $x, y, z \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique ternary quadratic homomorphism $H : X \to Y$ such that

$$N\left(f(x) - H(x), r\right) \geq \begin{cases} N'\left(\epsilon, |3|\, r\right), \\ N'\left(3\epsilon||x||^s, r\left|2^2 - 2^s\right|\right), \\ N'\left(\epsilon||x||^{3s}, r\left|2^2 - 2^{3s}\right|\right), \\ N'\left(4\epsilon||x||^{3s}, r\left|2^2 - 2^{3s}\right|\right) \end{cases} \tag{32}$$

for all $x \in X$ and all $r > 0$.

Theorem 3.3. *Let $j = \pm 1$. Let $\alpha : X^3 \to [0,\infty)$ be a mapping such that for some d with $0 < \left(\frac{d}{2^2}\right)^j < 1$*

$$N'\left(\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right), r\right) \geq N'\left(d^{nj}\alpha\left(x,y,z\right), r\right) \tag{33}$$

for all $x, y, z \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{n\to\infty} N'\left(\alpha\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right), r\right) = 1 \tag{34}$$

for all x, y, z and all $r > 0$. Suppose that a function $f : X \to X$ satisfies the inequalities

$$N\left(F(x,y,z), r\right) \geq N'\left(\alpha(x,y,z), r\right) \tag{35}$$

and

$$N\left(f\left([xyz]\right) - \left[f(x)y^2z^2\right] - \left[x^2f(y)z^2\right] - \left[x^2y^2f(z)\right], r\right) \geq N'\left(\alpha(x,y,z), r\right) \tag{36}$$

for all $x, y, z \in X$ and all $r > 0$. Then there exists a unique ternary quadratic derivation $D : X \to X$ such that

$$N\left(f(x) - D(x), r\right) \geq N'\left(\alpha(x,x,-x), r|2^2 - d|\right) \tag{37}$$

for all $x \in X$ and all $r > 0$. The mapping $D(x)$ is defined by

$$D(x) = N - \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{2nj}} \tag{38}$$

for all $x \in X$.

Proof. By the same reasoning as that in the proof of the Theorem 3.1, there exist a unique quadratic mapping $D : X \to X$ satisfying (37). The mapping $D : X \to X$ ginven by $D(x) = N - \lim_{n\to\infty} \dfrac{f(2^{nj}x)}{2^{2nj}}$ for all $x \in X$. It follows from (35) that

$$\begin{aligned}
&N\left(D\left([xyz]\right) - \left[D(x)y^2z^2\right] - \left[x^2D(y)z^2\right] - \left[x^2y^2D(z)\right], r\right)\\
&= N\left(\frac{1}{2^{6n}}\left(f\left(2^{3n}[xyz]\right) - \left[f(2^nx)2^{2n}y^22^{2n}z^2\right] - \left[2^{2n}x^2f(2^ny)2^{2n}z^2\right] - \left[2^{2n}x^22^{2n}y^2f(2^nz)\right]\right), \frac{r}{2^{6n}}\right)\\
&\geq N'\left(\alpha\left(2^nx, 2^ny, 2^nz\right), r\right)
\end{aligned} \tag{39}$$

for all $r > 0$ and all $x, y, z \in X$. Letting $n \to \infty$ in (39) and using (34), we reach

$$N\left(D\left([xyz]\right) - \left[D(x)y^2z^2\right] - \left[x^2D(y)z^2\right] - \left[x^2y^2D(z)\right], r\right) = 1$$

for all $x, y, z \in X$ and $r > 0$. Hence, we have $D\left([xyz]\right) = \left[D(x)y^2z^2\right] + \left[x^2D(y)z^2\right] + \left[x^2y^2D(z)\right]$ for all $x, y, z \in X$. Therefore $D : X \to X$ is a ternary quadratic derivation satisfying (37). The rest of the proof is similar to that of Theorem 3.1. □

From Theorem 3.3, we obtain the following corollary concerning the generalized Ulam-Hyers stability for the functional equation (4).

Corollary 3.4. *Suppose that a function $F : X \to X$ satisfies the inequality*

$$N\left(F(x,y,z), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), & s \neq 2; \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right), r\right), & s \neq \frac{2}{3}; \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}, r\right), & s \neq \frac{2}{3}; \end{cases} \tag{40}$$

for all $x, y, z \in X$ and all $r > 0$ and

$$N\left(D\left([xyz]\right) - \left[D(x)y^2z^2\right] - \left[x^2D(y)z^2\right] - \left[x^2y^2D(z)\right], r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right) \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right), r\right) \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}, r\right) \end{cases} \tag{41}$$

for all $x, y, z \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique ternary quadratic derivation $D : X \to X$ such that

$$N\left(f(x) - D(x), r\right) \geq \begin{cases} N'\left(\epsilon, |3|\, r\right), \\ N'\left(3\epsilon||x||^s, r\left|2^2 - 2^s\right|\right), \\ N'\left(\epsilon||x||^{3s}, r\left|2^2 - 2^{3s}\right|\right), \\ N'\left(4\epsilon||x||^{3s}, r\left|2^2 - 2^{3s}\right|\right) \end{cases} \tag{42}$$

for all $x \in X$ and all $r > 0$.

4. Stability Results: Fixed Point Method

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (4) in fuzzy ternary banach algebra by fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1. *[28](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}$ $d(y, Ty)$ for all $y \in Y$.

For to prove the stability result we define the following:

δ_i is a constant such that

$$\delta_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

Theorem 4.2. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} N'\left(\alpha\left(\delta_i^n x, \delta_i^n y, \delta_i^n z\right), \delta_i^{2n} r\right) = 1, \quad \forall\ x, y, z \in X, r > 0 \tag{43}$$

and satisfying the functional inequality

$$N\left(F(x,y,z),r\right) \geq N'\left(\alpha(x,y,z),r\right) \tag{44}$$

for all $x,y,z \in X$ *and all* $r>0$ *and*

$$N\left(f\left([xyz]\right)-\left[f(x)f(y)f(z)\right],r\right) \geq N'\left(\alpha(x,y,z),r\right) \tag{45}$$

for all $x,y,z \in X$ *and all* $r>0$. *If there exists* $L=L(i)$ *such that the function*

$$x \to \beta(x) = \alpha\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x),r\right) = N'\left(\beta(x),r\right), \ \forall \ x \in X, r>0. \tag{46}$$

Then there exists unique ternary quadratic homomorphism $H : X \to Y$ *satisfying the functional equation (4) and*

$$N\left(f(x)-H(x),r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right), \ \forall \ x \in X, r>0. \tag{47}$$

Proof. Let d be a general metric on Ω, such that

$$d(g,h) = inf\left\{K \in (0,\infty)|N\left(g(x)-h(x),r\right) \geq N'\left(K\beta(x),r\right), x \in X, r>0\right\}.$$

It is easy to see that (Ω,d) is complete. Define $T:\Omega \to \Omega$ by $Tg(x)=\frac{1}{\delta_i^2}g(\delta_i x)$, for all $x \in X$. For $g,h \in \Omega$, we have $d(g,h) \leq K$

$$\begin{aligned}
&\Rightarrow && N\left(g(x)-h(x),r\right) \geq N'\left(K\beta(x),r\right)\\
&\Rightarrow && N\left(\frac{g(\delta_i x)}{\delta_i^2}-\frac{h(\delta_i x)}{\delta_i^2},r\right) \geq N'\left(\frac{K}{\delta_i^2}\beta(\delta_i x),r\right)\\
&\Rightarrow && N\left(Tg(x)-Th(x),r\right) \geq N'\left(KL\beta(x),r\right)\\
&\Rightarrow && d\left(Tg(x),Th(x)\right) \leq KL\\
&\Rightarrow && d\left(Tg,Th\right) \leq Ld(g,h)
\end{aligned} \tag{48}$$

for all $g,h \in \Omega$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (x,y,z) by $(x,x,-x)$ in (44), we get

$$N\left(f(2x)-2^2f(x),r\right) \geq N'\left(\alpha(x,x,-x),r\right). \tag{49}$$

for all $x \in X, r>0$. Using (F3) in (49), we arrive

$$N\left(\frac{f(2x)}{2^2}-f(x),r\right) \geq N'\left(\alpha(x,x,-x),2^2r\right) \tag{50}$$

for all $x \in X, r>0$, with the help of (46) when $i=0$, it follows from (50), we get

$$\begin{aligned}
&\Rightarrow && N\left(\frac{f(2x)}{2^2x}-f(x),r\right) \geq N'\left(L\beta(x),r\right)\\
&\Rightarrow && d(Tf,f) \leq L = L^1 = L^{1-i}.
\end{aligned} \tag{51}$$

Replacing x by $\frac{x}{2}$ in (49), we obtain

$$N\left(f(x)-2^2f\left(\frac{x}{2}\right),r\right)\geq N'\left(\alpha\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right),r\right) \tag{52}$$

for all $x\in X, r>0$, with the help of (46) when $i=1$, it follows from (52), we get

$$\begin{aligned} &\Rightarrow \quad N\left(f(x)-2^2f\left(\frac{x}{2}\right),r\right)\geq N'\left(\beta(x),r\right)\\ &\Rightarrow \quad d(f,Tf)\leq 1=L^0=L^{1-i}. \end{aligned} \tag{53}$$

Then from (51) and (53), we can conclude

$$d(f,Tf)\leq L^{1-i}<\infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point H of T in Ω such that

$$H(x)=N-\lim_{n\to\infty}\frac{f(2^nx)}{2^{2n}}, \qquad \forall x\in X, r>0. \tag{54}$$

To prove $H:X\to Y$ is quadratic. Replacing (x,y,z) by $(\delta_i x,\delta_i y,\delta_i z)$ in (44), we arrive

$$N\left(\frac{1}{\delta_i^{2n}}F(\delta_i x,\delta_i y,\delta_i z),r\right)\geq N'\left(\alpha(\delta_i x,\delta_i y,\delta_i z),\delta_i^{2n}r\right) \tag{55}$$

for all $r>0$ and all $x,y,z\in X$.

By proceeding the same procedure as in the Theorem 3.1, we can prove the ternary quadratic homomorphism $H:X\to Y$ satisfies the functional equation (4).

By fixed point alternative, since H is unique fixed point of T in the set

$$\Delta=\{f\in\Omega | d(f,H)<\infty\},$$

therefore H is a unique function such that

$$N\left(f(x)-H(x),r\right)\geq N'\left(K\beta(x),r\right) \tag{56}$$

for all $x\in X, r>0$ and $K>0$. Again using the fixed point alternative, we obtain

$$\begin{aligned} &\qquad d(f,H)\leq \frac{1}{1-L}d(f,Tf)\\ &\Rightarrow \quad d(f,H)\leq \frac{L^{1-i}}{1-L}\\ &\Rightarrow \quad N\left(f(x)-H(x),r\right)\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right), \end{aligned} \tag{57}$$

for all $x\in X$ and $r>0$. This completes the proof of the theorem. □

From Theorem 4.2, we obtain the following corollary concerning the stability for the functional equation (4).

Corollary 4.3. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(F(x,y,z),r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), & s \neq 2; \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right),r\right), & s \neq \frac{2}{3}; \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s+||x||^{3s}+||y||^{3s}+||z||^{3s}\right\},r\right), & s \neq \frac{2}{3}; \end{cases} \tag{58}$$

for all $x, y, z \in X$ and all $r > 0$ and

$$N\left(H\left([xyz]\right)-\left[H(x)H(y)H(z)\right],r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right),r\right), \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s+||x||^{3s}+||y||^{3s}+||z||^{3s}\right\},r\right), \end{cases} \tag{59}$$

for all $x, y, z \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique ternary quadratic homomorphism $H : X \to Y$ such that

$$N\left(f(x)-H(x),r\right) \geq \begin{cases} N'\left(\epsilon, |3|\, r\right), \\ N'\left(3\epsilon||x||^s, r\left|2^2-2^s\right|\right), \\ N'\left(\epsilon||x||^{3s}, r\left|2^2-2^{3s}\right|\right), \\ N'\left(4\epsilon||x||^{3s}, r\left|2^2-2^{3s}\right|\right), \end{cases} \tag{60}$$

for all $x \in X$ and all $r > 0$.

Proof. Setting

$$\alpha(x,y,z) = \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right),r\right), \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right),r\right), \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s+||x||^{3s}+||y||^{3s}+||z||^{3s}\right\},r\right), \end{cases}$$

for all $x, y, z \in X$ and all $r > 0$. Then,

$$N'\left(\alpha(\delta_i^n x, \delta_i^n y, \delta_i^n z), \delta_i^{2n} r\right) = \begin{cases} N'\left(\epsilon, \delta_i^{2n} r\right), \\ N'\left(\epsilon\left(||x||^s+||y||^s+||z||^s\right), \delta_i^{(2-s)n} r\right), \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right), \delta_i^{(2-3s)n} r\right), \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s+||x||^{3s}+||y||^{3s}+||z||^{3s}\right\}, \delta_i^{(2-3s)n} r\right), \end{cases}$$

$$= \begin{cases} \to 1 \text{ as } n \to \infty, \\ \to 1 \text{ as } n \to \infty, \\ \to 1 \text{ as } n \to \infty, \\ \to 1 \text{ as } n \to \infty. \end{cases}$$

Thus, (43) is holds. But we have $\beta(x) = \alpha\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right)$ has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall\ x \in X, r > 0.$$

Hence

$$N'\left(\beta(x), r\right) = N'\left(\alpha\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right), r\right) = \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\frac{3\epsilon}{2^s}||x||^s, r\right), \\ N'\left(\frac{\epsilon}{2^{3s}}||x||^{3s}, r\right), \\ N'\left(\frac{(4)\epsilon}{2^{3s}}||x||^{3s}, r\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = \begin{cases} N'\left(\frac{\epsilon}{\delta_i^2}, r\right), \\ N'\left(\frac{3\epsilon}{2^s\delta_i^2}||\delta_i x||^s, r\right), \\ N'\left(\frac{\epsilon}{2^{3s}\delta_i^2}||\delta_i x||^{3s}, r\right), \\ N'\left(\frac{4\epsilon}{2^{3s}\delta_i^2}||\delta_i x||^{3s}, r\right), \end{cases} = \begin{cases} N'\left(\delta_i^{-2}\beta(x), r\right), \\ N'\left(\delta_i^{s-2}\beta(x), r\right), \\ N'\left(\delta_i^{3s-2}\beta(x), r\right), \\ N'\left(\delta_i^{3s-2}\beta(x), r\right). \end{cases}$$

From (47), we prove the following cases:

Case:1 $L = 2^{-2}$ if $i = 0$

$$N\left(f(x) - H(x), r\right) \geq N'\left(\frac{2^{-2}}{1 - 2^{-2}}\beta(x), r\right) = N'\left(\frac{\epsilon}{(2^2 - 1)}, r\right) = N'\left(\epsilon, 3r\right).$$

Case:2 $L = 2^2$ if $i = 1$

$$N\left(f(x) - H(x), r\right) \geq N'\left(\frac{1}{1 - 2^2}\beta(x), r\right) = N'\left(\frac{\epsilon}{-3}, r\right) = N'\left(\epsilon, |-3|\, r\right).$$

Case:3 $L = 2^{s-2}$ for $s < 2$ if $i = 0$

$$N\left(f(x) - H(x), r\right) \geq N'\left(\frac{2^{s-2}}{1 - 2^{s-2}}\beta(x), r\right) = N'\left(\frac{3\epsilon}{2^2 - 2^s}||x||^s, r\right) = N'\left(3\epsilon||x||^s, (2^2 - 2^s)r\right).$$

Case:4 $L = 2^{2-s}$ for $s > 2$ if $i = 1$

$$N\left(f(x) - H(x), r\right) \geq N'\left(\frac{1}{1 - 2^{2-s}}\beta(x), r\right) = N'\left(\frac{3\epsilon}{2^s - 2^2}||x||^s, r\right) = N'\left(3\epsilon||x||^s, (2^s - 2^2)r\right).$$

Case:5 $L = 2^{3s-2}$ for $s < \frac{2}{3}$ if $i = 0$

$$N\left(f(x) - H(x), r\right) \geq N'\left(\frac{2^{3s-2}}{1 - 2^{3s-2}}\beta(x), r\right) = N'\left(\frac{\epsilon}{2^2 - 2^{3s}}||x||^s, r\right) = N'\left(\epsilon||x||^s, 2^2 - 2^{3s}r\right).$$

Case:6 $L = 2^{2-3s}$ for $s > \frac{2}{3}$ if $i = 1$

$$N\left(f(x) - H(x), r\right) \geq N'\left(\frac{1}{1 - 2^{2-3s}}\beta(x), r\right) = N'\left(\frac{\epsilon}{2^{3s} - 2^2}||x||^s, r\right) = N'\left(\epsilon||x||^s, (2^{3s} - 2^2)r\right).$$

Hence the proof is complete. □

Theorem 4.4. *Let $f : X \to X$ be a mapping for which there exist a function $\alpha : X^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} N'\left(\alpha\left(\delta_i^n x, \delta_i^n y, \delta_i^n z\right), \delta_i^{2n} r\right) = 1, \quad \forall\ x, y, z \in X, r > 0 \tag{61}$$

and satisfying the functional inequality

$$N\left(F(x, y, z), r\right) \geq N'\left(\alpha(x, y, z), r\right) \tag{62}$$

for all $x, y, z \in X$ and all $r > 0$ and

$$N\left(f\left([xyz]\right) - \left[f(x)y^2z^2\right] - \left[x^2 f(y)z^2\right] - \left[x^2y^2 f(z)\right], r\right) \geq N'\left(\alpha(x, y, z), r\right) \tag{63}$$

for all $x, y, z \in X$ and all $r > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x) = \alpha\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall\ x \in X, r > 0 \tag{64}$$

Then there exists unique ternary quadratic derivation $D : X \to X$ satisfying the functional equation (4) and

$$N\left(f(x) - D(x), r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), \ \forall\ x \in X, r > 0. \tag{65}$$

Proof. By the same reasoning as that in the proof of the Theorem 4.2, there exist a unique quadratic mapping $D : X \to X$ satisfying (65). The mapping $D : X \to X$ ginven by $D(x) = N - \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{2nj}}$ for all $x \in X$. It follows from (62) that

$$\begin{aligned}
&N\left(D\left([xyz]\right) - \left[D(x)y^2z^2\right] - \left[x^2D(y)z^2\right] - \left[x^2y^2D(z)\right], r\right)\\
&= N\left(\frac{1}{2^{6n}}\left(f\left(2^{3n}[xyz]\right) - \left[f(2^n x)2^{2n}y^2 2^{2n}z^2\right] - \left[2^{2n}x^2 f(2^n y)2^{2n}z^2\right] - \left[2^{2n}x^2 2^{2n}y^2 f(2^n z)\right]\right), \frac{r}{2^{6n}}\right)\\
&\geq N'\left(\alpha\left(2^n x, 2^n y, 2^n z\right), r\right)
\end{aligned} \tag{66}$$

for all $r > 0$ and all $x, y, z \in X$. Letting $n \to \infty$ in (66) and using (61), we reach

$$N\left(D\left([xyz]\right) - \left[D(x)y^2z^2\right] - \left[x^2D(y)z^2\right] - \left[x^2y^2D(z)\right], r\right) = 1$$

for all $x, y, z \in X$ and $r > 0$. Hence, we have $D\left([xyz]\right) = \left[D(x)y^2z^2\right] + \left[x^2D(y)z^2\right] + \left[x^2y^2D(z)\right]$ for all $x, y, z \in X$. Therefore $D : X \to X$ is a ternary quadratic derivation satisfying (65). The rest of the proof is similar to that of Theorem 4.2. □

From Theorem 4.4, we obtain the following corollary concerning the generalized Ulam-Hyers stability for the functional equation (4) and the corollary proof is similar to that of Corollary 4.3.

Corollary 4.5. *Suppose that a function $F : X \to X$ satisfies the inequality*

$$N\left(F(x, y, z), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), & s \neq 2; \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right), r\right), & s \neq \frac{2}{3}; \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}, r\right), & s \neq \frac{2}{3}; \end{cases} \tag{67}$$

for all $x, y, z \in X$ *and all* $r > 0$ *and*

$$N\left(D\left([xyz]\right) - \left[D(x)y^2z^2\right] - \left[x^2D(y)z^2\right] - \left[x^2y^2D(z)\right], r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left(||x||^s + ||y||^s + ||z||^s\right), r\right), \\ N'\left(\epsilon\left(||x||^s||y||^s||z||^s\right), r\right), \\ N'\left(\epsilon\left\{||x||^s||y||^s||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}, r\right), \end{cases} \tag{68}$$

for all $x, y, z \in X$ *and all* $r > 0$*, where* ϵ, s *are constants with* $\epsilon > 0$*. Then there exists a unique ternary quadratic derivation* $D : X \to X$ *such that*

$$N\left(f(x) - D(x), r\right) \geq \begin{cases} N'\left(\epsilon, |3|\, r\right), \\ N'\left(3\epsilon||x||^s, r\left|2^2 - 2^s\right|\right), \\ N'\left(\epsilon||x||^{3s}, r\left|2^2 - 2^{3s}\right|\right), \\ N'\left(4\epsilon||x||^{3s}, r\left|2^2 - 2^{3s}\right|\right), \end{cases} \tag{69}$$

for all $x \in X$ *and all* $r > 0$.

5. Application of the Functional Equation(4)

Consider the quadratic functional equation (4), that is

$$f\left(\frac{x+y}{2} - z\right) + f\left(\frac{y+z}{2} - x\right) + f\left(\frac{z+x}{2} - y\right) = \frac{3}{4}\left(f(x-y) + f(y-z) + f(z-x)\right).$$

This functional equation can be used to find the sum of the length of the median in a triangle. Since $f(x) = x^2$ is the solution of the functional equation, the above equation is written as follows

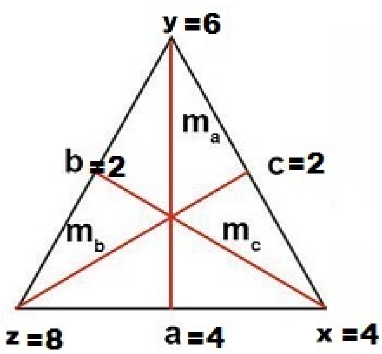
$$\left(\frac{x+y}{2} - z\right)^2 + \left(\frac{y+z}{2} - x\right)^2 + \left(\frac{z+x}{2} - y\right)^2 = \frac{3}{4}\left((x-y)^2 + (y-z)^2 + (z-x)^2\right). \tag{70}$$

Hence the above quadratic identity can be written as

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right). \tag{71}$$

The above identity shows that "three times the sum of the squares of the sides of a triangle is equal to four times the sum of squares of the medians of that triangle".

Example 5.1. *Find the sum of the medians of a following triangle.*



Solution. Using (70), we get

$$L.H.S\ of\ (71)\ is\ m_a^2+m_b^2+m_c^2=\left(\frac{x+y}{2}-z\right)^2+\left(\frac{y+z}{2}-x\right)^2+\left(\frac{z+x}{2}-y\right)^2$$
$$=\left(\frac{4+6}{2}-8\right)^2+\left(\frac{6+8}{2}-4\right)^2+\left(\frac{8+4}{2}-6\right)^2=18.$$

$$R.H.S\ of\ (71)\ is\ \frac{3}{4}\left(a^2+b^2+c^2\right)=\frac{3}{4}\left((z-x)^2+(y-z)^2+(x-y)^2\right)$$
$$=\frac{3}{4}\left(4^2+2^2+2^2\right)=18.$$

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