International Journal of Mathematics And its Applications
Volume 3, Issue 4–A (2015), 75–94.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Coincidence Point Theorem Under Mizoguchi-Takahashi Contraction on Ordered Metric Spaces With Application

Research Article

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Abstract: We construct coincidence point result for g -non-decreasing mappings satisfying Mizoguchi-Takahashi contraction on ordered metric spaces. By using our result, we formulate a coupled coincidence point result for generalized compatible pair of mappings $F, G : X^2 \to X$. We give an example and an application to integral equation to show the usefulness of the obtained results. Our results generalize, modify, improve and sharpen several well-known results.

MSC: 47H10, 54H25.

Keywords: Coincidence point, coupled coincidence point, generalized Mizoguchi-Takahashi contraction, ordered metric space, O-compatible, generalized compatibility, g-non-decreasing mapping, mixed monotone mapping.

1. Introduction

In [10] Guo and Lakshmikantham defined the notion of coupled fixed point for single-valued mappings. Thenafter, Bhaskar and Lakshmikantham [2] introduced mixed monotone property and with the help of it presented some coupled fixed point theorems on ordered metric spaces. After sometime, Lakshmikantham and Ciric [17] defined the notion of mixed g-monotone property by extending the notion of mixed monotone property and by applying it, they established some coupled fixed/coincidence point results, which generalized the results of Bhaskar and Lakshmikantham [2]. Subsequently, Choudhury and Kundu [4] modify the results of Lakshmikantham and Ciric [17], by defining the notion of compatibility in coupled coincidence point context. These studies applicable to initial value problems defined by differential or integral equations.

Hussain et al. [13] defined a new concept of generalized compatibility of a pair of mappings $F, G : X^2 \to X$ and presented some coupled coincidence point results. Following this paper, Erhan et al. [8], announced that the results obtained in Hussain et al. [13] can be easily derived from the coincidence point results in the literature.

On the other hand, Samet et al. [22] thrown light on the fact that most of the coupled fixed point theorems on ordered metric spaces can be derived from well-known fixed point theorems.

Recently Ciric et al. [3] proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi condition in the setting of ordered metric spaces. Main results of Ciric et al. [3] extended and

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generalized the results of Bhaskar and Lakshmikantham [2], Du [7] and Harjani et al. [11]. Our basic references are [3, 5–8, 11, 13–16, 20–22].

The main objective of this article is to construct coincidence point result for g-non-decreasing mappings satisfying Mizoguchi-Takahashi contraction on ordered metric spaces. By using our result, we formulate a coupled coincidence point result of generalized compatible pair of mappings F, $G : X^2 \to X$. We give an example and an application to integral equation to show the usability of the obtained results. The established results generalize, modify, improve and sharpen the results of Bhaskar and Lakshmikantham [2], Ciric et al. [3], Du [7], and Harjani et al. [11].

2. Preliminaries

In the sequel, we denote by X a non-empty set. Given a natural number $n \in$ for $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). Let $g : X \to X$ be any mapping, if $x \in X$, we denote $g(x)$ by gx.

Definition 2.1 ([10]). *Let $F : X^2 \to X$ be a given mapping. An element $(x,\, y) \in X^2$ is called a coupled fixed point of F if*

$$F(x, y) = x \text{ and } F(y, x) = y. \tag{1}$$

Definition 2.2 ([2]). *Let $(X,\, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that F has the mixed monotone property if for all $x,\, y \in X$, we have*

$$x_1, x_2 \in X, x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y), \tag{2}$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2). \tag{3}$$

Definition 2.3 ([17]). *Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\, y) \in X^2$ is called a coupled coincidence point of the mappings F and g if*

$$F(x, y) = gx \text{ and } F(y, x) = gy. \tag{4}$$

Definition 2.4 ([17]). *Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\, y) \in X^2$ is called a common coupled fixed point of the mappings F and g if*

$$x = F(x, y) = gx \text{ and } y = F(y, x) = gy. \tag{5}$$

Definition 2.5 ([17]). *The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if*

$$gF(x, y) = F(gx, gy), \text{ for all } (x, y) \in X^2. \tag{6}$$

Definition 2.6 ([17]). *Let $(X,\, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the mixed g-monotone property if for all $x,\, y \in X$, we have*

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \implies F(x_1, y) \preceq F(x_2, y), \tag{7}$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \implies F(x, y_1) \succeq F(x, y_2). \tag{8}$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Definition 2.7 ([4]). *The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if*

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) &= 0, \\ \lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) &= 0, \end{aligned} \tag{9}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n) &= \lim_{n\to\infty} gx_n = x, \\ \lim_{n\to\infty} F(y_n, x_n) &= \lim_{n\to\infty} gy_n = y, \text{ for some } x, y \in X. \end{aligned} \tag{10}$$

Definition 2.8 ([13]). *Suppose that $F,\ G : X^2 \to X$ are two mappings. F is said to be G-increasing with respect to $\preceq$ if for all $x,\ y,\ u,\ v \in X$, with $G(x,\ y) \preceq G(u,\ v)$ we have $F(x,\ y) \preceq F(u,\ v)$.*

Definition 2.9 ([13]). *Let $F,\ G : X^2 \to X$ be two mappings. We say that the pair $\{F,\ G\}$ is commuting if*

$$F(G(x, y), G(y, x)) = G(F(x, y), F(y, x)), \text{ for all } x, y \in X. \tag{11}$$

Definition 2.10 ([13]). *Suppose that $F,\ G : X^2 \to X$ are two mappings. An element $(x,\ y) \in X^2$ is called a coupled coincidence point of mappings F and G if*

$$F(x, y) = G(x, y) \text{ and } F(y, x) = G(y, x). \tag{12}$$

Definition 2.11 ([13]). *Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is g-increasing with respect to $\preceq$ if for any $x,\ y \in X$,*

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y), \tag{13}$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x, y_1) \preceq F(x, y_2). \tag{14}$$

Definition 2.12 ([13]). *Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ be a mapping. We say that F is increasing with respect to $\preceq$ if for any $x,\ y \in X$,*

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y), \tag{15}$$

and

$$y_1 \preceq y_2 \text{ implies } F(x, y_1) \preceq F(x, y_2). \tag{16}$$

Definition 2.13 ([13]). *Let $F,\ G : X^2 \to X$ are two mappings. We say that the pair $\{F,\ G\}$ is generalized compatible if*

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n, y_n), G(y_n,\ x_n)), G(F(x_n, y_n), F(y_n, x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n, x_n), G(x_n,\ y_n)), G(F(y_n, x_n), F(x_n, y_n))) &= 0, \end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} G(x_n, y_n) &= \lim_{n\to\infty} F(x_n, y_n) = x \in X, \\ \lim_{n\to\infty} G(y_n, x_n) &= \lim_{n\to\infty} F(y_n, x_n) = y \in X. \end{aligned} \tag{17}$$

Obviously, a commuting pair is generalized compatible but not conversely in general.

Definition 2.14 ([1, 9]). *A coincidence point of two mappings $T,\ g : X \to X$ is a point $x \in X$ such that $Tx = gx$.*

Definition 2.15 ([8]). *An ordered metric space $(X,\ d,\ \preceq)$ is a metric space $(X,\ d)$ provided with a partial order $\preceq$.*

Definition 2.16 ([2, 13]). *An ordered metric space $(X,\ d,\ \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all n, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all n. $(X,\ d,\ \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.*

Definition 2.17 ([8]). *Let$(X,\ \preceq)$ be a partially ordered set and let $T,\ g : X \to X$ be two mappings. We say that T is $(g, \preceq)$-non-decreasing if $Tx \preceq Ty$ for all $x,\ y \in X$ such that $gx \preceq gy$. If g is the identity mapping on X, we say that T is $\preceq$-non-decreasing.*

Remark 2.18 ([8]). *If T is $(g,\ \preceq)$ -non-decreasing and $gx = gy$, then $Tx = Ty$. It follows that*

$$gx = gy \Rightarrow \left\{ \begin{array}{c} gx \preceq gy, \\ gy \preceq gx \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} Tx \preceq Ty, \\ Ty \preceq Tx \end{array} \right\} \Rightarrow Tx = Ty. \tag{18}$$

Definition 2.19 ([20]). *Let $(X,\ \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order:*

$$(u, v) \sqsubseteq (x, y) \Leftrightarrow x \succeq u \ \text{ and } \ y \preceq v, \ \text{ for all } (u, v), (x, y) \in X^2. \tag{19}$$

Definition 2.20 ([4, 12, 19, 20]). *. Let $(X,\ d,\ \preceq)$ be an ordered metric space. Two mappings $T,\ g : X \to X$ are said to be O-compatible if*

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0, \tag{20}$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X. \tag{21}$$

3. Main results

Lemma 3.1. *Let $(X,\ d)$ be a metric space. Suppose $Y = X^2$ and define $\delta : Y \times Y \to [0,\ +\infty)$ by*

$$\delta((x, y), (u, v)) = \max \{d(x, u), d(y, v)\}, \ \text{ for all } (x, y), (u, v) \in Y. \tag{22}$$

Then δ is metric on Y and $(X,\ d)$ is complete if and only if $(Y,\ \delta)$ is complete.

Let Φ denote the set of all functions $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying

(i_φ) φ is non-decreasing,

(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,

(iii_φ) $\limsup_{t\to 0+} \frac{t}{\varphi(t)} < \infty$.

Let Ψ denote the set of all functions $\psi : [0,\ +\infty) \to [0,\ 1)$ which satisfies $\lim_{r\to t+} \psi(r) < 1$ for all $t \geq 0$.

Theorem 3.2. *Let $(X,\ d,\ \preceq)$ be an ordered metric space and let $T,\ g : X \to X$ be two mappings such that the following properties are fulfilled:*

(i) $T(X) \subseteq g(X)$,

(ii) T *is* $(g, \preceq)$*-non-decreasing,*

(iii) there exists $x_0 \in X$ *such that* $gx_0 \preceq Tx_0$,

(iv) there exist $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\varphi\left(d(Tx, Ty)\right) \leq \psi\left(\varphi\left(d(gx, gy)\right)\right)\varphi\left(d(gx, gy)\right),$$

for all $x,\, y \in X$ *such that* $gx \preceq gy$*. Also assume that, at least, one of the following conditions holds.*

(a) $(X,\, d)$ *is complete,* T *and* g *are continuous and the pair* $(T,\, g)$ *is O-compatible,*

(b) $(X,\, d)$ *is complete,* T *and* g *are continuous and commuting,*

(c) $(g(X),\, d)$ *is complete and* $(X,\, d,\, \preceq)$ *is non-decreasing-regular,*

(d) $(X,\, d)$ *is complete,* $g(X)$ *is closed and* $(X,\, d,\, \preceq)$ *is non-decreasing-regular,*

(e) $(X,\, d)$ *is complete,* g *is continuous and monotone-non-decreasing, the pair* $(T,\, g)$ *is O-compatible and* $(X,\, d,\, \preceq)$ *is non-decreasing-regular.*

Then T *and* g *have, at least, a coincidence point.*

Proof. We divide the proof into four steps.

Step 1. We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is $\preceq$-non-decreasing and $gx_{n+1} = Tx_n$, for all $n \geq 0$. Starting from $x_0 \in X$ given in *(iii)* and since $Tx_0 \in T(X) \subseteq g(X)$, therefore there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g,\, \preceq)$-non-decreasing, $Tx_0 \preceq Tx_1$. Now $Tx_1 \in T(X) \subseteq g(X)$, so there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g,\, \preceq)$ -non-decreasing, $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{gx_n\}$ is $\preceq$ -non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{23}$$

Step 2. We claim that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Now, by contractive condition *(iv)* and (i_φ), we have

$$\begin{aligned} & \varphi\left(d(gx_n, gx_{n+1})\right) \\ = \ & \varphi\left(d(Tx_{n-1}, Tx_n)\right) \\ \leq \ & \psi\left(\varphi\left(d(gx_{n-1}, gx_n)\right)\right)\varphi\left(d(gx_{n-1}, gx_n)\right), \end{aligned}$$

which, by the fact that $\psi < 1$, implies

$$\varphi\left(d(gx_n, gx_{n+1})\right) \leq \varphi\left(d(gx_{n-1}, gx_n)\right). \tag{24}$$

Now (24) shows that the sequence $\{\varphi\left(d(gx_n, gx_{n+1})\right)\}$ is non-increasing. Therefore, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \varphi\left(d(gx_n, gx_{n+1})\right) = \delta. \tag{25}$$

Since $\psi \in \Psi$, we have $\lim_{r\to\delta+} \psi(r) < 1$ and $\psi(\delta) < 1$. Then there exists $\alpha \in [0,\, 1)$ and $\varepsilon > 0$ such that $\psi(r) \leq \alpha$ for all $r \in [\delta,\, \delta + \varepsilon)$. From (25), we can take $n_0 \geq 0$ such that $\delta \leq \varphi\left(d(gx_n, gx_{n+1})\right) \leq \delta + \varepsilon$ for all $n \geq n_0$. Then from contractive condition *(iv)* and (i_φ), for all $n \geq n_0$, we have

$$\begin{aligned} & \varphi\left(d(gx_n, gx_{n+1})\right) \\ = \ & \varphi\left(d(Tx_{n-1}, Tx_n)\right) \\ \leq \ & \psi\left(\varphi\left(d(gx_{n-1}, gx_n)\right)\right)\varphi\left(d(gx_{n-1}, gx_n)\right) \\ \leq \ & \alpha\varphi\left(d(gx_{n-1}, gx_n)\right). \end{aligned}$$

Thus, we have

$$\varphi\left(d(gx_n, gx_{n+1})\right) \leq \alpha\varphi\left(d(gx_{n-1}, gx_n)\right), \text{ for all } n \geq n_0. \tag{26}$$

Letting $n \to \infty$ in the above inequality and using (25), we obtain that $\delta \leq \alpha\delta$. Since $\alpha \in [0, 1)$, therefore $\delta = 0$. Thus

$$\lim_{n\to\infty} \varphi\left(d(gx_n, gx_{n+1})\right) = 0. \tag{27}$$

Since $\{\varphi\left(d(gx_n, gx_{n+1})\right)\}$ is a non-increasing sequence and φ is non-decreasing, then $\{d(gx_n, gx_{n+1})\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\theta \geq 0$ such that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \theta. \tag{28}$$

Since φ is non-decreasing, we have

$$\varphi\left(d(gx_n, gx_{n+1})\right) \geq \varphi(\theta).$$

Letting $n \to \infty$ in this inequality, by using (27), we get $0 \geq \varphi[\theta]$ which, by (ii_φ), implies that $\theta = 0$. Thus, by (28), we get

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{29}$$

Suppose that $d(gx_n, gx_{n+1}) = 0$, for some $n \geq 0$. Then, we have $gx_n = gx_{n+1} = Tx_n$, that is, x_n is a coincidence point of F and g. Now, suppose that $d(gx_n, gx_{n+1}) \neq 0$, for all $n \geq 0$. Denote

$$a_n = \varphi\left(d(gx_n, gx_{n+1})\right), \text{ for all } n \geq 0.$$

From (26), we have

$$a_n \leq \alpha a_{n-1}, \text{ for all } n \geq n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} \alpha^{n-n_0} a_{n_0} < \infty. \tag{30}$$

On the other hand, by (iii_φ), we have

$$\limsup_{n\to\infty} \frac{d(gx_n, gx_{n+1})}{\varphi\left(d(gx_n, gx_{n+1})\right)} < \infty. \tag{31}$$

Thus, by (30) and (31), we have $\sum d(gx_n, gx_{n+1})\} < \infty$. It means that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X.

Step 3. We claim that T and g have a coincidence point distinguishing between cases $(a) - (e)$.

Suppose now that (a) holds, that is, (X, d) is complete, T and g are continuous and the pair (T, g) is O-compatible. Since (X, d) is complete, therefore there exists $z \in X$ such that $\{gx_n\} \to z$. Now $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As T and g are continuous, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Since the pair (T, g) is O-compatible, we have that $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. In such a case, we conclude that

$$d(gz, Tz) = \lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0,$$

that is, z is a coincidence point of T and g.

Suppose now that (b) holds, that is, (X, d) is complete, T and g are continuous and commuting. Clearly (b) implies (a).

Suppose now that (c) holds, that is, $(g(X),\, d)$ is complete and $(X,\, d,\, \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X),\, d)$, so there exist $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$. In this case $\{gx_n\} \to gz$. Indeed, as $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$ -non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv) and (i_φ),

$$\varphi\left(d(gx_{n+1}, Tz)\right) = \varphi\left(d(Tx_n,\ Tz)\right) \leq \psi\left(\varphi\left(d(gx_n, gz)\right)\right)\varphi\left(d(gx_n, gz)\right),$$

which, by the fact that $\psi < 1$, implies

$$\varphi\left(d(gx_{n+1}, Tz)\right) \leq \varphi\left(d(gx_n, gz)\right).$$

Since φ is non-decreasing, we have

$$d(gx_{n+1}, Tz) \leq d(gx_n, gz).$$

Letting $n \to \infty$ in the above inequality, using $\lim_{n\to\infty} gx_n = gz$, we get $d(gz,\, Tz) = 0$, that is, z is a coincidence point of T and g.

Suppose now that (d) holds, that is, $(X,\, d)$ is complete, $g(X)$ is closed and $(X,\, d,\, \preceq)$ is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, $(g(X),\, d)$ is complete and $(X,\, d,\, \preceq)$ is non-decreasing-regular. Thus (c) is applicable.

Suppose now that (e) holds, that is, $(X,\, d)$ is complete, g is continuous and monotone-non-decreasing, the pair $(T,\, g)$ is O-compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular. As $(X,\, d)$ is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. Since $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As T and g are continuous, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Since the pair $(T,\, g)$ is O-compatible, we get that $\lim_{n\to\infty} d(gTx_n,\, Tgx_n) = 0$. In such a case, we conclude that

$$d(gz, Tz) = \lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0,$$

that is, z is a coincidence point of T and g. Indeed, as $(X,\, d,\, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$ -non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv) and (i_φ),

$$\varphi\left(d(gx_{n+1}, Tz)\right) = \varphi\left(d(Tx_n,\ Tz)\right) \leq \psi\left(\varphi\left(d(gx_n, gz)\right)\right)\varphi\left(d(gx_n, gz)\right),$$

which, by the fact that $\psi < 1$, implies

$$\varphi\left(d(gx_{n+1}, Tz)\right) \leq \varphi\left(d(gx_n, gz)\right).$$

Since φ is non-decreasing, we have

$$d(gx_{n+1}, Tz) \leq d(gx_n, gz).$$

Letting $n \to \infty$ in the above inequality, using $\lim_{n\to\infty} gx_n = gz$, we get $d(gz,\, Tz) = 0$, that is, z is a coincidence point of T and g.

Next, we obtain the coupled version of Theorem 3.2. Given the ordered metric space $(X,\, d,\, \preceq)$, let us consider the ordered metric space $(X^2,\, \delta,\, \sqsubseteq)$, where δ was defined in Lemma 3.1 and $\sqsubseteq$ was introduced in (19). We define the mappings T_F, $T_G : X^2 \to X^2$, for all $(x,\, y) \in X^2$, by,

$$T_F(x, y) = (F(x, y), F(y, x)) \ \text{ and } T_G(x, y) = (G(x, y), G(y, x)).$$

Under these conditions, the following properties hold: $\square$

Lemma 3.3. *Let $(X, d, \preceq)$ be an ordered metric space and let $F, G : X^2 \to X$ be two mappings. Then*

(1) (X, d) is complete if and only if (X^2, δ) is complete.

(2) If $(X, d, \preceq)$ is regular, then $(X^2, \delta, \sqsubseteq)$ is also regular.

(3) If F is d-continuous, then T_F is δ-continuous.

(4) If F is G-increasing with respect to $\preceq$, then T_F is $(T_G, \sqsubseteq)$-non-decreasing.

(5) If there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \preceq F(x_0, y_0)$ and $G(y_0, x_0) \succeq F(y_0, x_0)$, then there exists a point $(x_0, y_0) \in X^2$ such that $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.

(6) For any $x, y \in X$, there exist $u, v \in X$ such that $F(x, y) = G(u, v)$ and $F(y, x) = G(v, u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(7) There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\begin{aligned} & \varphi\left(d(F(x, y), F(u, v))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right], \end{aligned} \tag{32}$$

for all $x, y, u, v \in X$, where $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, then

$$\begin{aligned} & \varphi\left(\delta(T_F(x, y), T_F(u, v))\right) \\ \leq\ & \psi\left(\varphi\left(\delta(T_G(x, y), T_G(u, v))\right)\right)\varphi\left(\delta(T_G(x, y), T_G(u, v))\right), \end{aligned}$$

for all $(x, y), (u, v) \in X^2$, where $T_G(x, y) \sqsubseteq T_G(u, v)$.

(8) If the pair $\{F, G\}$ is generalized compatible, then the mappings T_F and T_G are O-compatible in $(X^2, \delta, \sqsubseteq)$.

(9) A point $(x, y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of T_F and T_G.

Proof. Item (1) follows from Lemma 3.1 and items (2), (3), (5), (6) and (9) are obvious.

(4) Assume that F is G-increasing with respect to $\preceq$ and let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Then $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Since F is G-increasing with respect to $\preceq$, we deduce that $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. Therefore $T_F(x, y) \sqsubseteq T_F(u, v)$, which means that T_F is $(T_G, \sqsubseteq)$-non-decreasing.

(7) Suppose that there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\begin{aligned} & \varphi\left(d(F(x, y), F(u, v))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right], \end{aligned}$$

for all $x, y, u, v \in X$, where $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$ and let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Therefore $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Using (32), we have

$$\begin{aligned} & \varphi\left(d(F(x, y), F(u, v))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right]. \end{aligned} \tag{33}$$

Furthermore since $G(y, x) \succeq G(v, u)$ and $G(x, y) \preceq G(u, v)$, therefore the contractive condition (32) also guarantees that

$$\begin{aligned} & \varphi\left(d(F(y, x), F(v, u))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right]\right) \\ & \times\varphi\left[\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right]. \end{aligned} \tag{34}$$

Combining (33) and (34), we get

$$\begin{aligned}
& \max\{\varphi(d(F(x,\ y),\ F(u,\ v))),\ \varphi(d(F(y,\ x),\ F(v,\ u)))\} \\
\leq\ & \psi(\varphi[\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}]) \\
& \times\varphi[\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}].
\end{aligned}$$

Since φ is non-decreasing, therefore

$$\begin{aligned}
& \varphi(\max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\}) \\
\leq\ & \psi(\varphi[\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}]) \\
& \times\varphi[\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}].
\end{aligned} \tag{35}$$

Thus, it follows from (35) that

$$\begin{aligned}
& \varphi(\delta(T_F(x,\ y),\ T_F(u,\ v))) \\
=\ & \varphi(\delta((F(x,\ y),\ F(y,\ x)),\ (F(u,\ v),\ F(v,\ u)))) \\
=\ & \varphi(\max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\}) \\
\leq\ & \psi(\varphi[\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}]) \\
& \times\varphi[\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}] \\
\leq\ & \psi(\varphi(\delta(T_G(x,\ y),\ T_G(u,\ v))))\,\varphi(\delta(T_G(x,\ y),\ T_G(u,\ v))).
\end{aligned}$$

(8) Let $\{(x_n,\ y_n)\} \subseteq X^2$ be any sequence such that $T_F(x_n,\ y_n) \xrightarrow{\delta} (x,\ y)$ and $T_G(x_n,\ y_n) \xrightarrow{\delta} (x,\ y)$ (notice that we do not need to suppose that $\{T_G(x_n,\ y_n)\}$ is $\sqsubseteq$-monotone). Therefore,

$$\begin{aligned}
& (F(x_n,\ y_n),\ F(y_n,\ x_n)) \xrightarrow{\delta} (x,\ y) \\
\Rightarrow\ & F(x_n,\ y_n) \xrightarrow{d} x \text{ and } F(y_n,\ x_n) \xrightarrow{d} y,
\end{aligned}$$

and

$$\begin{aligned}
& (G(x_n,\ y_n),\ G(y_n,\ x_n)) \xrightarrow{\delta} (x,\ y) \\
\Rightarrow\ & G(x_n,\ y_n) \xrightarrow{d} x \text{ and } G(y_n,\ x_n) \xrightarrow{d} y.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} G(x_n,\ y_n) = x \in X, \\
\lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} G(y_n,\ x_n) = y \in X.
\end{aligned}$$

Since the pair $\{F,\ G\}$ is generalized compatible, we deduce that

$$\begin{aligned}
\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\
\lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0.
\end{aligned}$$

In particular,

$$
\begin{aligned}
& \lim_{n\to\infty} \delta(T_G T_F(x_n,\ y_n),\ T_F T_G(x_n,\ y_n)) \\
= & \lim_{n\to\infty} \delta(T_G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ T_F(G(x_n,\ y_n),\ G(y_n,\ x_n))) \\
= & \lim_{n\to\infty} \delta \begin{pmatrix} (G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))), \\ (F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{pmatrix} \\
= & \lim_{n\to\infty} \max \begin{Bmatrix} d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ F(G(x_n,\ y_n),\ G(y_n,\ x_n))), \\ d(G(F(y_n,\ x_n),\ F(x_n,\ y_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{Bmatrix} \\
= & \ 0.
\end{aligned}
$$

Hence, the mappings T_F and T_G are O-compatible in $(X^2,\ \delta,\ \sqsubseteq)$. □

Theorem 3.4. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ satisfying (32) and for any $x,\ y \in X$, there exist $u,\ v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{36}$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Proof. Applying Theorem 3.2 for the mappings $T = T_F$ and $g = T_G$ in the ordered metric space $(X^2,\ \delta,\ \sqsubseteq)$ with the help of Lemma 3.3. □

Corollary 3.5. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two commuting mappings satisfying (32) and (36) such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Now, we obtain result without using mixed g-monotone property of F.

Corollary 3.6. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$ and there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$
\begin{aligned}
& \varphi\left(d(F(x,\ y),\ F(u,\ v))\right) \\
\leq & \ \psi\left(\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right]\right)\varphi\left[\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right],
\end{aligned} \tag{37}
$$

for all $x,\ y,\ u,\ v \in X$, where $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$ and the pair $\{F,\ g\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Corollary 3.7. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$, satisfying (37). Suppose that $F(X^2) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$ and the pair $\{F,\ g\}$ is commutng. Also suppose that either*

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Next, we obtain result without mixed monotone property of F.

Corollary 3.8. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F : X^2 \to X$ be an increasing mapping with respect to $\preceq$ and there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} & \varphi\left(d(F(x,\ y),\ F(u,\ v))\right) \\ \leq\ & \psi\left(\varphi\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right]\right)\varphi\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right], \end{aligned} \tag{38}$$

for all $x,\ y,\ u,\ v \in X$, where $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point.

If we put $\psi(t) = 1 - \frac{\widetilde{\psi}(t)}{t}$ for all $t \geq 0$ in Theorem 3.4, then we get the following result:

Corollary 3.9. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that there exist $\varphi \in \Phi$ and $\widetilde{\psi} \in \Psi$ such that

$$\begin{aligned} & \varphi\left(H(F(x,\ y),\ F(u,\ v))\right) \\ \leq\ & \varphi\left[\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\}\right] \\ & -\widetilde{\psi}\left(\varphi\left[\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\}\right]\right), \end{aligned} \tag{39}$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ satisfying (36). *Also suppose that either*

(*a*) *F is continuous or*

(*b*) *$(X,\ d,\ \preceq)$ is regular.*

Then F and G have a coupled coincidence point.

Corollary 3.10. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two commuting mappings satisfying* (36) *and* (39) *such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property, and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Also suppose that either

(*a*) *F is continuous or*

(*b*) *$(X,\ d,\ \preceq)$ is regular.*

Then F and G have a coupled coincidence point.

If we put $\varphi(t) = 2t$ for all $t \geq 0$ in Theorem 3.4, then we get the following result:

Corollary 3.11. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property, and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that there exist $\psi \in \Psi$ such that

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \psi\left(2\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\}\right) \\ & \times\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\}, \end{aligned} \tag{40}$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ satisfying (36). *Also suppose that either*

(*a*) *F is continuous or*

(*b*) *$(X,\ d,\ \preceq)$ is regular.*

Then F and G have a coupled coincidence point.

Corollary 3.12. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two commuting mappings satisfying* (36) *and* (40) *such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u).$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

If we put $\psi(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 3.11, then we get the following result:

Corollary 3.13. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property satisfying*

$$H(F(x,\ y),\ F(u,\ v)) \leq k \max \left\{ \begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array} \right\}, \tag{41}$$

for all $x,\ y,\ u,\ v \in X$ and $0 < k < 1$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ satisfying (36). *Suppose there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Corollary 3.14. *Let $(X,\ d,\ \preceq)$ be a complete ordered metric space. Assume $F,\ G : X^2 \to X$ be two commuting mappings satisfying* (36) *and* (41) *such that F is G-increasing with respect to $\preceq$, G is continuous with mixed monotone property, and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Example 3.15. *Suppose that $X = [0,\ 1]$, furnished with the usual metric $d : X^2 \to [0,\ +\infty)$ with the natural ordering of real numbers. Let $F,\ G : X^2 \to X$ be defined as*

$$F(x,\ y) = \begin{cases} \frac{x^2-y^2}{4}, & \textit{if } x \geq y, \\ 0, & \textit{if } x < y, \end{cases}$$

and

$$G(x,\ y) = \begin{cases} x^2 - y^2, & \textit{if } x \geq y, \\ 0, & \textit{if } x < y. \end{cases}$$

Define $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ by

$$\varphi(t) = \begin{cases} \ln(t+1), & \textit{for } t \neq 1, \\ \frac{3}{4}, & \textit{for } t = 1, \end{cases}$$

and $\psi : [0,\ +\infty) \to [0,\ 1)$ defined by

$$\psi(t) = \frac{\varphi(t)}{t}, \ \textit{for all } t \geq 0.$$

First, we shall show that F is G-increasing. Let $(x,\ y),\ (u,\ v) \in X \times X$ with $G(x,\ y) \leq G(u,\ v)$. We consider the following cases:

Case 1: If $x < y$, then $F(x,\ y) = 0 \leq F(u,\ v)$.

Case 2: If $x \geq y$ and $u \geq v$, then $G(x,\ y) \leq G(u,\ v) \Rightarrow x^2 - y^2 \leq u^2 - v^2 \Rightarrow \frac{x^2-y^2}{4} \leq \frac{u^2-v^2}{4} \Rightarrow F(x,\ y) \leq F(u,\ v)$. But if $u < v$, then $G(x,\ y) \leq G(u,\ v) \Rightarrow 0 \leq x^2 - y^2 \leq 0 \Rightarrow x^2 = y^2 \Rightarrow F(x,\ y) = 0 \leq F(u,\ v)$.

Thus F is G-increasing. Now, we prove that for any $x,\ y \in X$, there exist $u,\ v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u).$$

Let $(x,\ y),\ (u,\ v) \in X \times X$ be fixed. We consider the following cases:

Case 1: If $x = y$, then we have $F(x,\ y) = 0 = G(x,\ y)$ and $F(y,\ x) = 0 = G(y,\ x)$.

Case 2: If $x > y$, then we have $F(x,\ y) = \frac{x^2-y^2}{4} = G(\frac{x}{2},\ \frac{y}{2})$ and $F(y,\ x) = 0 = G(\frac{y}{2},\ \frac{x}{2})$.

Case 3: If $x < y$, then we have $F(x,\ y) = 0 = G(\frac{x}{2},\ \frac{y}{2})$ and $F(y,\ x) = \frac{y^2-x^2}{4} = G(\frac{y}{2},\ \frac{x}{2})$.

Now we prove that G is continuous and has the mixed monotone property. Clearly G is continuous. Let $(x,\ y) \in X \times X$ be fixed. Suppose that $x_1,\ x_2 \in X$ are such that $x_1 < x_2$.

Case 1: If $x_1 < y$, then we have $G(x_1,\ y) = 0 \leq G(x_2,\ y)$.

Case 2: If $x_1 \geq y$, then we have $G(x_1,\ y) = x_1^2 - y^2 \leq x_2^2 - y^2 = G(x_2,\ y)$.

Similarly, we can show that if $y_1,\ y_2 \in X$ are such that $y_1 < y_2$, then $G(x,\ y_1) \geq G(x,\ y_2)$.

Now, we prove that the pair $\{F,\ G\}$ satisfies the generalized compatibility hypothesis.

Let (x_n) and (y_n) be two sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = t_1, \\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = t_2, \end{aligned}$$

then we must have $t_1 = t_2 = 0$ and it is easy to see that

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0. \end{aligned}$$

Now we prove that there exist two elements $x_0,\ y_0 \in X$ with

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

Since we have $G(0,\ \frac{1}{2}) = 0 = F(0, \frac{1}{2})$ and $G(\frac{1}{2},\ 0) = \frac{1}{4} \geq \frac{1}{16} = F(\frac{1}{2},\ 0)$. Next, we shall show that the mappings F and G satisfy the condition (32). Let $x,\ y,\ u,\ v \in X$ such that $G(x,\ y) \leq G(u,\ v)$ and $G(y,\ x) \geq G(v,\ u)$. Then

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
= & \left| \frac{x^2 - y^2}{4} - \frac{u^2 - v^2}{4} \right| \\
\leq & \ \ln(\left|(x^2 - y^2) - (u^2 - v^2)\right| + 1) \\
\leq & \ \ln(|G(x,\ y) - G(u,\ v)| + 1) \\
\leq & \ \ln(d(G(x,\ y),\ G(u,\ v)) + 1) \\
\leq & \ \ln\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\} + 1\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
& \varphi\left(d(F(x,\ y),\ F(u,\ v))\right) \\
= & \ \ln\left[d(F(x,\ y),\ F(u,\ v)) + 1\right] \\
\leq & \ \ln\left[\ln\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\} + 1\right] + 1\right] \\
\leq & \ \frac{\ln\left[\ln\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\} + 1\right] + 1\right]}{\ln\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\} + 1\right]} \\
& \times \ln\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\} + 1\right] \\
\leq & \ \psi\left(\varphi\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\}\right]\right) \\
& \times \varphi\left[\max\left\{\begin{array}{c} d(G(x,\ y),\ G(u,\ v)), \\ d(G(y,\ x),\ G(v,\ u)) \end{array}\right\}\right].
\end{aligned}$$

Thus the contractive condition (32) is satisfied for all $x,\ y,\ u,\ v \in X$ and $z = (0,\ 0)$ is a coupled coincidence point of F and G.

4. Application to Integral Equations

As an application of the results established in previous section of our paper, we study the existence of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))\left[f(q,\ x(q)) + g(q,\ x(q))\right] dq + h(p), \tag{41}$$

for all $p \in I = [a,\ b]$.

Let Θ denote the set of all functions $\theta : [0,\ +\infty) \to [0,\ +\infty)$ satisfying

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) = \ln(p+1)$.

Assumption 4.1. *We assume that the functions $K_1,\ K_2,\ f,\ g$ fulfill the following conditions:*

(i) $K_1(p,\ q) \geq 0$ and $K_2(p,\ q) \leq 0$ for all $p,\ q \in I$.

(ii) There exist positive numbers $\lambda,\ \mu$ and $\theta \in \Theta$ such that for all $x,\ y \in \mathbb{R}$ with $x \succeq y$, the following conditions hold:

$$0 \ \leq\ f(q,\ x) - f(q,\ y) \leq \lambda\theta(x-y), \tag{42}$$

$$-\mu\theta(x-y) \ \leq\ g(q,\ x) - g(q,\ y) \leq 0, \tag{43}$$

(iii)

$$\max\{\lambda,\ \mu\} \sup_{p\in I} \int_a^b [K_1(p,\ q) - K_2(p,\ q)]dq \leq \frac{1}{2}. \tag{44}$$

Definition 4.2 ([18])**.** *A pair $(\alpha,\ \beta) \in X \times X$ with $X = C(I,\ \mathbb{R})$, where $C(I,\ \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a coupled lower-upper solution of* (42) *if, for all $p \in I$,*

$$\begin{aligned}
\alpha(p) \ &\leq\ \int_a^b K_1(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq \\
&\quad + \int_a^b K_2(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq + h(p), \\
\textit{and } \beta(q) \ &\geq\ \int_a^b K_1(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq \\
&\quad + \int_a^b K_2(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq + h(p).
\end{aligned}$$

Theorem 4.3. *Consider the integral equation* (42) *with $K_1,\ K_2 \in C(I \times I,\ \mathbb{R})$, $f,\ g \in C(I \times \mathbb{R},\ \mathbb{R})$ and $h \in C(I,\ \mathbb{R})$. Suppose that there exists a coupled lower-upper solution $(\alpha,\ \beta)$ of* (42) *and that Assumption 4.1 is satisfied. Then the integral equation* (42) *has a solution in $C(I,\ \mathbb{R})$.*

Proof. Consider $X = C(I,\ \mathbb{R})$, the natural partial order relation, that is, for $x,\ y \in C(I,\ \mathbb{R})$,

$$x \preceq y \Longleftrightarrow x(p) \leq y(p),\ \ \forall p \in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p\in I} |x(p) - y(p)|\,.$$

The partial order on X^2 is define as follows: for $(x,\ y),\ (u,\ v) \in X \times X$,

$$(x,\ y) \preceq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p), \text{ for all } p \in I.$$

Define $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ by

$$\varphi(t) = \begin{cases} \ln(t+1), & \text{for } t \neq 1, \\ \frac{3}{4}, & \text{for } t = 1, \end{cases}$$

and $\psi : [0,\, +\infty) \to [0,\, 1)$ defined by

$$\psi(t) = \frac{\varphi(t)}{t}, \text{ for all } t \geq 0.$$

Define the mapping $F : X^2 \to X$, for all $p \in I$, by

$$\begin{aligned} F(x,\ y)(p) \ &= \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\ &+ \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq + h(p). \end{aligned}$$

It is not difficult to prove, like in [13], that F is increasing. Now for $x,\, y,\, u,\, v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ = & \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq \\ & - \int_a^b K_1(p,\ q)\,[f(q,\ u(q)) + g(q,\ v(q))]\,dq \\ & - \int_a^b K_2(p,\ q)\,[f(q,\ v(q)) + g(q,\ u(q))]\,dq \\ = & \int_a^b K_1(p,\ q)\,[(f(q,\ x(q)) - f(q,\ u(q))) - (g(q,\ v(q)) - g(q,\ y(q)))]\,dq \\ & - \int_a^b K_2(p,\ q)\,[(f(q,\ v(q)) - f(q,\ y(q))) - (g(q,\ x(q)) - g(q,\ u(q)))]\,dq. \end{aligned}$$

Thus, by using (43) and (44), we get

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ \leq & \int_a^b K_1(p,\ q)\,[\lambda\theta\,(x(q) - u(q)) + \mu\theta\,(v(q) - y(q))]\,dq \\ & - \int_a^b K_2(p,\ q)\,[\lambda\theta\,(v(q) - y(q)) + \mu\theta\,(x(q) - u(q))]\,dq. \end{aligned} \tag{45}$$

Since the function θ is non-decreasing and $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} \theta\,(x(q) - u(q)) \ &\leq\ \theta\left(\sup_{q\in I}|x(q) - u(q)|\right) = \theta(d(x,\ u)), \\ \theta\,(v(q) - y(q)) \ &\leq\ \theta\left(\sup_{q\in I}|v(q) - y(q)|\right) = \theta(d(y,\ v)). \end{aligned}$$

Hence by (46), in view of the fact that $K_2(p,\ q) \leq 0$, we obtain

$$\begin{aligned}
& |F(x,\ y)(p) - F(u,\ v)(p)| \\
\leq\ & \int_a^b K_1(p,\ q)\left[\lambda\theta(d(x,\ u)) + \mu\theta(d(y,\ v))\right] dq \\
& - \int_a^b K_2(p,\ q)\left[\lambda\theta(d(y,\ v)) + \mu\theta(d(x,\ u))\right] dq, \\
\leq\ & \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(x,\ u)) + \max\{\lambda,\ \mu\}\theta(d(y,\ v))\right] dq \\
& - \int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(y,\ v)) + \max\{\lambda,\ \mu\}\theta(d(x,\ u))\right] dq,
\end{aligned}$$

as all the quantities on the right hand side of (46) are non-negative. Now, taking the supremum with respect to p, by using (45), we get

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b (K_1(p,\ q) - K_2(p,\ q))\, dq.\left[\theta(d(x,\ u)) + \theta(d(y,\ v))\right] \\
\leq\ & \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2}.
\end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2}. \tag{46}$$

Now, since θ is non-decreasing, we have

$$\begin{aligned}
\theta(d(x,\ u)) &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \\
\theta(d(y,\ v)) &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right),
\end{aligned}$$

which implies, by (ii_θ), that

$$\begin{aligned}
& \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2} \\
\leq\ & \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right) \\
\leq\ & \ln\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\} + 1\right].
\end{aligned} \tag{47}$$

Thus by (47) and (48), we have

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \ln\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\} + 1\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
& \varphi\left(d(F(x,\ y),\ F(u,\ v))\right) \\
= \ & \ln\left[d(F(x,\ y),\ F(u,\ v))+1\right] \\
\leq \ & \ln\left[\ln\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}+1\right]+1\right] \\
\leq \ & \frac{\ln\left[\ln\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}+1\right]+1\right]}{\ln\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}+1\right]} \\
& \times \ln\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}+1\right] \\
\leq \ & \psi\left(\varphi\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right]\right) \\
& \times \varphi\left[\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right],
\end{aligned}$$

which is the contractive condition (38) of Corollary 3.8. Now, let $(\alpha,\ \beta) \in X^2$ be a coupled upper-lower solution of (42), then we have $\alpha(p) \leq F(\alpha,\ \beta)(p)$ and $\beta(p) \geq F(\ \beta,\ \alpha)(p)$, for all $p \in I$, which shows that all hypothesis of Corollary 3.8 are satisfied. This proves that F has a coupled fixed point $(x,\ y) \in X^2$ which is the solution in $X = C(I,\ \mathbb{R})$ of the integral equation (42). □

Remark 4.4. *Using the same fact that can be used in [14–16, 20–22] it is possible to formulate tripled, quadruple and in general, multidimensional coincidence point theorems from Theorem 3.2.*

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