International Journal of Mathematics And its Applications
Volume 3, Issue 4–B (2015), 9–16.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Analysis of Dual-Matrix Approach for Solving the Transportation Problems

Research Article

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Abstract: A transportation decision problem is considered. The existence of solution and characterization of solution space are established. The dual behavior of transportation problem is studied. The step by step procedure is described and an algorithm is proposed to solve the transportation problem. The whole procedure is illustrated step by step through a numerical example.

Keywords: Linear programming models, Transportation problem, Dual- Matrix, Dual simplex.

1. Introduction

Transportation problem is the special form of linear programming model. There are numerous papers in the area of transportation problem. Important among of these for solving transportation problem are Arsham H. and A.B. Kahn [5], Balinski M.L and R.E. Gomory [6], Charnes A. and W.W. Cooper [1], Ford, L. R. and D.R. Fulkerson [7], Shafaat and S.K. Goyal [9], Ping J.I. and K.F CHU [2]. Ping and CHU [2] have developed dual-matrix approach to solve transportation problem. They have suggested that if the original transportation problem is converted into corresponding dual transportation problem then a better starting basic feasible solution can be obtained. They further mentioned that time and space can be saved while going through the initial simplex table to the optimal simplex table. It is difficult to observe the existence of solution and characterization of solution space mathematically, in their paper. Further, converging behavior of the solution is also missing.

In the present paper we have developed mathematical background (existence, characterization and convergence) of the algorithm given by [2].

2. The Problem and its Dual

The transportation problem can be expressed as a linear programming model as follows:

LP(1) Maximize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

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Subject to

$$\begin{aligned} &\sum_{j=1}^{n} x_{ij} \leq a_i \quad (i = i, \ldots, m) \\ &\sum_{i=1}^{m} x_{ij} \geq b_i \quad (j = i, \ldots, n) \\ &\quad x_{ij} \geq 0 \quad (i = 1, \ldots, m;\ j = 1, dots, n) \end{aligned} \tag{1}$$

Here a_i and b_j are assumed to be positive, a_i and b_j are called supplies and demands. The cost c_{ij} are all nonnegative.

Dual of the L.P(1) is as follows:

LP(2) Maximize $\psi = \sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i$

Subject to

$$\begin{aligned} v_j - u_i \leq c_{ij} \quad (i = 1, \ldots, m;\ j = 1, dots, n) \\ u_i, v_j \geq 0 \quad (i = 1, \ldots, m;\ j = 1, dots, n) \end{aligned} \tag{2}$$

3. Mathematical Analysis of the Solution and Characterization of the Solution Space

Our approach to find the good solution is the dual approach. So from L.P(2) $u_i = 0, v_j = c_{rj} = \min_i c_{ij}$ gives the basic solution. ie., minimum cost variable is selected from every column of the problem. These variables are called basic variables. In this way we have n basic variables corresponding to their respective cells.

Since L.P (1) has total supply available at each origin is greater than or equal to the required demand. ie., $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$, we can create an additional destination D_0 with demand $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$, with shipping cost zero from each origin. Or we can say that, to make the problem balance we need to add m virtual cells having transporting cost zero. These cells are denoted as $(1,0)(2,0)\ldots(m,0)$ and treated as basic cells. Actually these cells do not exist in the original problem so called virtual cells.

In this manner total (m+n) basic cells are $(i_1,1)(i_2,2)\ldots(i_n,n)(1,0)(2,0)\ldots(m,0)$ and corresponding variables are x_{i1}, x_{i2},..., x_{ij}, x_{in}, x_{10}, x_{20},..., x_{m0} rest variables are non-basic variables.

Next L.P (1) is converted into standard L.P form then converted into matrix form. According to basic and non-basic data matrices and variables are partitioned into basic and non-basic respectively. To make the problem (1) in standard linear programming form by introducing only surplus variables we rewrite the problem as follows.

Maximize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$-\sum_{j=1}^{m} x_{ij} - x_{i0} = -a_i$$

$$x_{ij}, x_{0j} \geq 0$$

$$Min\ Z = CX$$
$$AX = \begin{bmatrix} b \\ -a \end{bmatrix} \tag{3}$$
$$X \geq 0$$

Partition the set of variables x_{ij} into x_{ij}^{B} and x_{ij}^{NB} and c_{ij} into c_{ij}^{B} and c_{ij}^{NB}. Where NB and B is defined as the set of indices for the non-basic and basic variables. A is partition in B and N matrices where B contains those columns of A which are associated with basic variables and N contains those columns which are associated with non-basic variables. Here B is square matrix of order (m+n) and N is of order $(m+n) \times mn$ then problem (3) can be written as

$$\begin{bmatrix} 1 & -c_{ij}^{B} & -c_{ij}^{NB} \\ 0 & B & N \end{bmatrix} \begin{bmatrix} z \\ x_{ij}^{B} \\ x_{ij}^{NB} \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ -a \end{bmatrix} \tag{4}$$

Initially all non-basic variables are at zero level therefore the standard problem is reduced to

$$\begin{bmatrix} z \\ x_{ij}^{B} \end{bmatrix} = \begin{bmatrix} 1 & -c_{ij}^{B} \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 \\ b \\ -a \end{bmatrix}$$

If $T = \begin{bmatrix} 1 & -c_{ij}^{B} \\ 0 & B \end{bmatrix}$ then $(T)^{-1}$ is calculated using (A-2-7) page no.822 from Taha [4].

$$(T)^{-1} = \begin{bmatrix} 1 & c_{ij}^{B}B \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} z \\ x_{ij}^{B} \end{bmatrix} = \begin{bmatrix} c_{ij}^{B}B \begin{bmatrix} b \\ -a \end{bmatrix} \\ B \begin{bmatrix} b \\ -a \end{bmatrix} \end{bmatrix}$$

Note that $B^{-1} = B$. Pre multiply both sides by $\begin{bmatrix} 1 & c_{ij}^{B}B \\ 0 & B \end{bmatrix}$ in equation (4) we get

$$\begin{bmatrix} 1 & -c_{ij}^{B} + c_{ij}^{B}B & -c_{ij}^{NB} + c_{ij}^{B}BN \\ 0 & B & N \end{bmatrix} \begin{bmatrix} z \\ x_{ij}^{B} \\ x_{ij}^{NB} \end{bmatrix} = \begin{bmatrix} c_{ij}^{B}B \begin{bmatrix} b \\ -a \end{bmatrix} \\ B \begin{bmatrix} b \\ -a \end{bmatrix} \end{bmatrix} \tag{5}$$

3.1. Analysis of Step by Step Procedure

For the convenience we are presenting the steps of algorithm in tabular form in detail. According to the set of equations (5) we form the simplex table as follows:

It is well known that the optimality condition for primal simplex method is $(z_{ij} - c_{ij}) \leq 0,\ b \geq 0$. But here we see that all b are not greater than or equal to zero therefore applying dual simplex algorithm. From the simplex table-1 we have constraint equation as

$$Bx_{ij}^{B} + BNx_{ij}^{NB} = B \begin{bmatrix} b \\ -a \end{bmatrix}$$

Basic	x_{ij}^B	x_{ij}^{NB}	Solution
Z	$c_{ij}^B B - c_{ij}^B$	$c_{ij}^B BN - c_{ij}^{NB}$	$c_{ij}^B B \begin{bmatrix} b \\ -a \end{bmatrix}$
x_{ij}^B	B	BN	$B \begin{bmatrix} b \\ -a \end{bmatrix}$

Table 1. **Simplex**

P_{ij}^{NB} and P_{ij}^B are the column vectors corresponding to matrix N and B. The constraint equation is associated with the l^{th} basic variable can be written as

$$\left(Bx_{ij}^B\right)_l + \sum_{i,j\in NB} \left(BP_{ij}^{NB}\right)_l x_{ij}^{NB} = \left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_l \tag{6}$$

Because we are solving transportation problem using dual simplex algorithm first we select leaving variable. Suppose r^{th} basic variable is the leaving variable therefore corresponding constraint equation is written as:

$$\left(Bx_{ij}^B\right)_r + \sum_{i,j\in NB} \left(BP_{ij}^{NB}\right)_r x_{ij}^{NB} = \left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_r \tag{7}$$

If P_{st}^{NB} is a vector corresponding to the entering variable x_{st}^{NB} then the r^{th} constraint equation can be written as follows.

$$\left(Bx_{ij}^B\right)_r + \sum_{i,j\in NB-st} \left(BP_{ij}^{NB}\right)_r x_{ij}^{NB} + \left(BP_{st}^{NB}\right)_r x_{st}^{NB} = \left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_r$$

Thus entering variable x_{st}^{NB} is completed as

$$x_{st}^{NB} = \frac{\left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_r}{(BP_{st}^{NB})_r} - \frac{(Bx_{ij}^B)_r}{(BP_{st}^{NB})_r} - \frac{\sum_{i,j\in NB-st} (BP_{ij}^{NB})_r x_{ij}^{NB}}{(BP_{st}^{NB})_r} \tag{8}$$

Substituting the value of x_{st}^{NB} in equation (6) we have

$$\left(Bx_{ij}^B\right)_l + \sum_{i,j\in NB-st} \left[\left(BP_{ij}^{NB}\right)_l - \left(BP_{st}^{NB}\right)_l \frac{(BP_{ij}^{NB})_r}{(BP_{st}^{NB})_r}\right] x_{ij}^{NB} - \left(BP_{st}^{NB}\right)_l \frac{(BP_{ij}^{B})_r}{(BP_{st}^{NB})_r}$$
$$= \left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_l - \left(BP_{st}^{NB}\right)_l \frac{\left(B \begin{bmatrix} b \\ -a \end{bmatrix}\right)_r}{(BP_{st}^{NB})_r} \tag{9}$$

From the general matrix table, the z-equation coefficient associated with the variable x_{ij}^{NB} is $z_{ij} - c_{ij} = c_{ij}BN - c_{ij}^{NB}$. Nothing that $(z_{ij} - c_{ij})$ is always zero for all basic variables x_{ij}^B.

$$z + \sum_{i,j\in NB} (z_{ij} - c_{ij})\, x_{ij}^{NB} = c_{ij}^B B \begin{bmatrix} b \\ -a \end{bmatrix} \tag{10}$$

i.e., $z + \sum_{i,j\in NB-st} \left(c_{ij}^B BP_{ij}^{NB} - c_{ij}^{NB}\right) x_{st}^{NB} = c_{ij}^B B \begin{bmatrix} b \\ -a \end{bmatrix}$.

Substituting the value of x_{st}^{NB} (entering variable) from equation (8)

$$z + \sum_{i,j \in NB-st} \left[\left(c_{ij}^{B} BP_{ij}^{NB} - c_{ij}^{NB} \right) - \left(c_{ij}^{B} BP_{st}^{NB} - c_{st}^{NB} \right) \frac{\left(BP_{ij}^{NB}\right)_r}{\left(BP_{st}^{NB}\right)_r} \right] x_{ij}^{NB} - \left(c_{ij}^{B} BP_{st}^{NB} - c_{st}^{NB} \right) \frac{\left(Bx_{ij}^{B}\right)_r}{\left(BP_{st}^{NB}\right)_r}$$

$$= c_{ij}^{B} B \begin{bmatrix} b \\ -a \end{bmatrix} - \left(c_{ij}^{B} BP_{st}^{NB} - c_{st}^{NB} \right) \frac{\left(B \begin{bmatrix} b \\ -a \end{bmatrix} \right)_r}{\left(BP_{st}^{NB}\right)_r} \tag{11}$$

Now we claim that $\left(BP_{st}^{NB}\right)_r$ is equal to -1. Substituting the value in equation (8), (9), (11) the equations are reduces. With the help of the equations (8), (9), (11) transformed simplex table is given below

Basic	x_{ij}^{B}	x_{ij}^{NB}	Solution
Z	$\left(c_{ij}^{B} P_{st}^{NB} - c_{st}^{NB} \right) (B)_r$	$\sum_{i,j \in NB-(s,t)} \left[\frac{\left(c_{ij}^{B} BP_{ij}^{NB} - c_{ij}^{NB} \right) + \left(c_{ij}^{B} BP_{st}^{NB} - c_{st}^{NB} \right)}{\left(BP_{ij}^{NB}\right)_r} \right]$	$c_{ij}^{B} B \begin{bmatrix} b \\ -a \end{bmatrix} + \left(c_{ij}^{B} BP_{st}^{NB} - c_{st}^{NB} \right) (B)_r \begin{bmatrix} b \\ -a \end{bmatrix}$
$(x_{ij}^{B})_l$	$(B)_l + \left(BP_{st}^{N}\right)_l (B)_r$	$\sum_{(i,j) \in N-(s,t)} \left[\frac{\left(BP_{st}^{NB}\right)_l + \left(BP_{st}^{NB}\right)_l}{\left(BP_{ij}^{NB}\right)_r} \right]$	$\left(\begin{bmatrix} b \\ -a \end{bmatrix} \right)_l + (B)_r \begin{bmatrix} b \\ -a \end{bmatrix}$
x_{st}^{N}	$-(B)_r$	$-\sum_{(i,j) \in NB-(s,t)} \left(BP_{st}^{NB}\right)_r$	$-(B)_r \begin{bmatrix} b \\ -a \end{bmatrix}$

Table 2. **Simplex**

With the help of simplex Table 1 and 2 we can solve the transportation problems.

3.2. Algorithm

Step 1: Rewrite the problem as L.P form.

Step 2: Write the standard form of the problem by adding surplus variables.

Step 3: Generate the table and find basic feasible solution.

Step 4: Generate the initial dual simplex table.

Step 5: Find leaving variable from table in step 4.

Step 6: Find entering variable from table in step 4.

Step 7: Generate the optimal dual simplex table described in the procedure.

Step 8: Write the optimal solution from table generated in step7.

4. Illustration

We consider a numerical example and give complete illustration of our algorithm. Consider the following transportation problem.

	Destination			
Origin		D_1	D_2	Supply
	O_1	3	6	400
	O_2	4	5	300
	O_3	7	3	400
	Demand	450	350	

Step 1 : Rewriting the problem in the following form Maximize $Z' = -(3x_{11} + 6x_{12} + 4x_{21} + 5x_{22} + 7x_{31} + 3x_{32})$ subject to

$$x_{11} + x_{21} + x_{31} \geq 450$$
$$x_{12} + x_{22} + x_{32} \geq 350$$
$$-x_{11} - x_{12} \geq -400$$
$$-x_{21} - x_{22} \geq -300$$
$$-x_{31} - x_{32} \geq -400$$

Step 2 : Adding the surplus variables in each constraint

$$x_{11} + x_{21} + x_{31} - x_{01} = 450$$
$$x_{12} + x_{22} + x_{32} - x_{02} = 350$$
$$-x_{11} - x_{12} - x_{10} = -400$$
$$-x_{21} - x_{22} - x_{20} = -300$$
$$-x_{31} - x_{32} - x_{30} = -400$$

These equations can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ x_{31} \\ x_{32} \\ x_{10} \\ x_{20} \\ x_{30} \\ x_{01} \\ x_{021} \end{bmatrix} = \begin{bmatrix} 450 \\ 350 \\ -400 \\ -300 \\ -400 \end{bmatrix}$$

Step 3 : Selecting the basic variable from table given below

	D_1	D_2	supply
O_1	3 x_{11}	6 x_{12}	400
O_2	4 x_{21}	5 x_{22}	300
O_3	7 x_{31}	3 x_{32}	400
Demand	450	350	

$c_{rj} = \min_{i=1,2,3}\{c_{ij}\}$, $j = 1, 2$. $c_{r1} = \min\{c_{il}\} = \min\{c_{11}, c_{21}, c_{31}\} = \min\{3, 4, 7\} = 3$ for c_{11}, So x_{11} is the first basic variable, similarly x_{32} is another basic variable and other are surplus variables corresponding to supply constraints x_{10}, x_{20}, x_{30}. Hence basic variables are x_{11}, x_{32}, x_{10}, x_{20}, x_{30}.

Step 4 : Now we construct the initial simplex table with the help of simplex table -1 as follows:

Basic	x_{11}	x_{32}	x_{10}	x_{20}	x_{30}	x_{12}	x_{21}	x_{22}	x_{31}	x_{01}	x_{02}	Solution
Z	0	0	0	0	0	-3	-1	-2	-4	-3	-3	2400
x_{11}	1	0	0	0	0	0	1	0	1	-1	0	450
x_{32}	0	1	0	0	0	1	0	1	0	0	-1	350
x_{10}	-1	0	-1	0	0	1	-1	0	-1	1	0	-50
x_{20}	0	0	0	-1	0	0	1	1	0	0	0	300
x_{30}	0	-1	0	0	-1	-1	0	-1	1	0	1	50

Step 5 : To find leaving vector, $(x_{ij}^B)_r = \min[(x_{ij}^B)_l, (x_{ij}^B)_l < 0] = -50$. Here $r = 3$ indicates that the third basic variable x_{10} is the leaving variable so we remove x_{10} from the basis matrix.

Step 6 : To find the entering variable, $\min\left\{\frac{(c_{ij}^B BN - c_{ij}^{NB})}{(BN)_r}, (BN)_r < 0\right\} = \min\left\{\frac{-1}{-1}, \frac{-4}{-1}\right\} = 1$. So x_{21} is the entering variable.

Step 7 : Using the procedure we get final(optimal) transformed Simplex Table 2 as follows:

Basic	x_{11}	x_{32}	x_{10}	x_{20}	x_{30}	x_{12}	x_{21}	x_{22}	x_{31}	x_{01}	x_{02}	Solution
Z	1	0	1	0	0	-4	0	-2	-3	-4	-3	2400
x_{11}	0	0	-1	0	0	1	0	0	0	0	0	450
x_{32}	0	1	0	0	0	1	0	1	0	0	-1	350
x_{21}	1	0	1	0	0	-1	1	0	1	-1	0	-50
x_{20}	-1	0	-1	-1	0	1	0	1	-1	1	0	300
x_{30}	0	-1	0	0	-1	-1	0	-1	1	0	1	50

Step 8 : All b are greater than zero so the optimal solution is obtained with the objective Z= 2450 with $x_{11} = 400$, $x_{32} = 350$, $x_{21} = 50$, $x_{20} = 250$, $x_{30} = 50$.

5. Conclusion

Explanation gives us complete idea of the dual matrix approach by which the algorithm can be extended for various types of transportation problems. It can help us to develop the algorithm for the transportation problem having demand more than the supply as well as the supply more than the demand. It may be useful for developing the algorithm for multi objective transportation problems.

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