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Lagrange Formalism for Electromagnetic Field in Terms of Complex Isotropic Vectors

Research Article

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Abstract: In previous works, Weyl's equation for neutrino has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. It has been proved, that the complex vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$ satisfies non-linear condition $\overrightarrow{F}^2 = 0$, equivalent to two conditions for real quantities $\overrightarrow{E}^2 - \overrightarrow{H}^2 = 0$ and $\overrightarrow{E}.\overrightarrow{H} = 0$, obtained by separating real and imaginary parts in the equality $\overrightarrow{F}^2 = 0$. Further, it has been proved, that Maxwell's equations can also be written through complex vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. However, in the general case, the solution of Maxwell's equations does not satisfy non-linear condition $\overrightarrow{F}^2 = 0$. In this work, in the development of this new tensor formalism, we elaborated the Lagrange formalism for electromagnetic field in terms of complex isotropic vectors.

Keywords: Electromagnetic field, Lagrange formalism, complex isotropic vector.

1. Introduction

In previous works, via Cartan map, Weyl's equation for neutrino has been written in tensor form, in the form of non-linear Maxwell's like equations through complex isotropic vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$, satisfying non-linear condition $\overrightarrow{F}^2 = 0$.

The last condition is equivalent to two conditions for real quantities $\overrightarrow{E}^2 - \overrightarrow{H}^2 = 0$ and $\overrightarrow{E}.\overrightarrow{H} = 0$, obtained by equating to zero separately real and imaginary parts in equality $\overrightarrow{F}^2 = 0$. It has been proved, that the vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ have the same properties as the vectors $(\overrightarrow{E}, \overrightarrow{H})$, components of electromagnetic field. For example, under Lorentz relativistic transformations, they are transformed as components of electric and magnetic fields.

In addition, it has been proved, that the solution of these non-linear equations for free particle as well fulfils Maxwell's equations for vacuum (with zero at the right side). Further, it has been proved, that Maxwell's equations can also be written through complex vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. However, in the general case, the solution of Maxwell's equations does not satisfy non-linear condition $\overrightarrow{F}^2 = 0$.

In this work, in the development of this new tensor formalism, we shall elaborate the Lagrange formalism for electromagnetic field in terms of complex isotropic vectors $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. We shall find the Lagrange function for the electromagnetic field in terms of complex isotropic vectors $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$ and from this Lagrange function we shall derive expressions for fundamental dynamical variables (energy, momentum, charge, spin) in terms of strengths $\overrightarrow{E}$ and $\overrightarrow{H}$.

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1.1. Research Method

In previous works, using Cartan map, Weyl's equation for neutrino has been written in tensor form, in the form of non-linear Maxwell's like equations through complex isotropic vector $\vec{F} = \vec{E} + i\vec{H}$. Further, it has been proved, that Maxwell's equations can also be written through complex vector $\vec{F} = \vec{E} + i\vec{H}$. Using the same method, based on Cartan map, the Lagrange formalism for neutrino field in tensor formalism, in terms of complex isotropic vectors has been elaborated. In this work, on the basis of the last result, we shall elaborate the Lagrange formalism for electromagnetic field in terms of complex isotropic vectors $\vec{F} = \vec{E} + i\vec{H}$.

2. Maxwell's Equations in Terms of Complex Isotropic Vectors

Weyl's equation for neutrino has the form

$$p_0\xi = (\vec{p}.\vec{\sigma})\xi. \tag{1}$$

With the help of Cartan map, spinor ξ is mapped on isotropic complex vector $\vec{F} = \vec{E} + i\vec{H}$ satisfying non-linear condition $\vec{F}^2 = 0$. Where $\vec{E}$ and $\vec{H}$ are components of the tensor $F_{\mu\nu}$ of "electromagnetic field". With the help of isotropic vector $\vec{F} = \vec{E} + i\vec{H}$, we can also define a current four vector

$$J_\mu = \begin{bmatrix} J_0 \\ \vec{J} \end{bmatrix} = \begin{bmatrix} |\vec{E}| \\ \frac{\vec{E}\times\vec{H}}{|\vec{E}|} \end{bmatrix} \tag{2}$$

representing the current density for neutrino. Using Cartan map, Weyl's equation (1) has been written in vector form as follows

$$D^0\vec{F} = i\vec{D}\times\vec{F} - (\vec{D}F_i)v_i, \tag{3}$$

where

$$D^0 = i\frac{h}{2}\frac{\partial}{\partial t} \tag{4}$$

$$\vec{D} = -i\frac{h}{2}\vec{\nabla}, \tag{5}$$

$$\vec{v} = \frac{\vec{J}}{J_0} = \frac{\vec{E}\times\vec{H}}{\lceil\vec{E}\rceil^2}. \tag{6}$$

Equation (3), written through components $\vec{E}$ and $\vec{H}$, can be represented in the form of a system of non-linear Maxwell's like equations

$$\begin{cases} rot\vec{E} + \dfrac{\partial\vec{H}}{\partial t} = v_i(\vec{\nabla}H_i) \\ rot\vec{H} - \dfrac{\partial\vec{E}}{\partial t} = -v_i(\vec{\nabla}E_i) \end{cases} \tag{7}$$

Here we use the natural system of units in which $c = h = 1$. In the notations of complex isotropic vectors, Maxwell's equations for vacuum take the form

$$\begin{cases} D^0\vec{F} = i\vec{D}\times\vec{F} \\ \vec{D}\vec{F} = 0 \end{cases} \tag{8}$$

However, in the general case, the solution of Maxwell's equations does not satisfy non-linear isotropic condition $\vec{F}^2 = 0$, whereas the solution of Weyl's equation (3) always satisfies this condition.

3. Lagrange Formalism for Electromagnetic Field in Terms of Complex Isotropic Vectors

In spinor formalism, Weyl's equation (1) can be derived by variation principle from Lagrange function

$$L = \frac{i}{2}(\xi\sigma^{\mu}\partial_{\mu}\xi^{*} - \partial^{\mu}\xi\sigma_{\mu}\xi^{*}). \tag{9}$$

Transforming formula (9) according to Cartan map, we obtain

$$L = \frac{1}{2}\frac{\{[D_0\overrightarrow{F} - i\overrightarrow{D}\times\overrightarrow{F} + v_i(\overrightarrow{D}F_i)]\overrightarrow{F}^{*} - [D_0\overrightarrow{F}^{*} + i\overrightarrow{D}\times\overrightarrow{F} + v_i(\overrightarrow{D}F_i^{*})]\overrightarrow{F}\}}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^{*}}{2}\right)^{\frac{1}{2}}}. \tag{10}$$

Formula (10), written through components of vectors $\overrightarrow{F}$ and $\overrightarrow{F}^{*}$ takes the form

$$L = \frac{i}{4}\frac{\{[\frac{\partial F_i}{\partial t} + i\epsilon_{ijk}\frac{\partial F_k}{\partial x_j} - v_j(\frac{\partial F_j}{\partial x_i})]F_i^{*} - [\frac{\partial F_i^{*}}{\partial t} - i\epsilon_{ijk}\frac{\partial F_k^{*}}{\partial x_j} - v_j(\frac{\partial F_j^{*}}{\partial x_i})]F_i\}}{\left(\frac{F_iF_i^{*}}{2}\right)^{\frac{1}{2}}.} \tag{11}$$

In calculating variations in formula (11), expression $\left(\frac{\overrightarrow{F}\overrightarrow{F}^{*}}{2}\right)^{\frac{1}{2}}$ will be considered as a constant. From the Lagrange function for neutrino field (10), we can immediately derive the Lagrange function for electromagnetic field by cancelling the non-linear term in formula (10). We obtain

$$L = \frac{1}{2}\frac{\{[D_0\overrightarrow{F} - i\overrightarrow{D}\times\overrightarrow{F}]\overrightarrow{F}^{*} - [D_0\overrightarrow{F}^{*} + i\overrightarrow{D}\times\overrightarrow{F}^{*}]\overrightarrow{F}\}}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^{*}}{2}\right)^{\frac{1}{2}}} \tag{12}$$

3.1. Principal Dynamical Variables

Using Noether's theorem, we can derive from the Lagrange function (12), expressions for fundamental physical dynamical variables, conserved in time. Energy is determined by the formula

$$E = \int T^{00}d^{3}x, \tag{13}$$

where

$$T^{00} = \frac{\partial L}{\partial\overrightarrow{F},0}\overrightarrow{F},0 + \frac{\partial L}{\partial F^{*},0}\overrightarrow{F}^{*},0. \tag{14}$$

Replacing formula (12) into formula (14), we obtain

$$T^{00} = \frac{i}{4}\frac{[\overrightarrow{F}^{*}\frac{\partial\overrightarrow{F}}{\partial t} - \overrightarrow{F}\frac{\partial\overrightarrow{F}^{*}}{\partial t}]}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^{*}}{2}\right)^{\frac{1}{2}}}. \tag{15}$$

With consideration of expression

$$\overrightarrow{F} = (\overrightarrow{E}^{0} + i\overrightarrow{H}^{0})e^{-2ikt+2i\overrightarrow{k}\overrightarrow{r}}, \tag{16}$$

we find

$$T^{00} = k|\overrightarrow{E}|. \tag{17}$$

Similarly, for momentum we have

$$P^{j} = \int T^{0j}d^{3}x, \tag{18}$$

where

$$T^{0j} = \frac{\partial L}{\partial \overrightarrow{F},0} \overrightarrow{F}_{,j} + \frac{\partial L}{\partial F^*,0} \overrightarrow{F}^*{}_{,j}\,. \tag{19}$$

Replacing expression (12) in formula (19), we find

$$T^{0j} = \frac{i}{4} \frac{[(\overrightarrow{F}\overrightarrow{\nabla}_j \overrightarrow{F}^*) - (\overrightarrow{F}^* \overrightarrow{\nabla}_j \overrightarrow{F})]}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^*}{2}\right)^{\frac{1}{2}}}. \tag{20}$$

With consideration of expression (16), we obtain

$$\overrightarrow{P} = \overrightarrow{k}|\overrightarrow{E}|. \tag{21}$$

For charge, we have

$$Q = \int j^0 d^3x, \tag{22}$$

where

$$j^0 = i\left(\overrightarrow{F}^* \frac{\partial L}{\partial \overrightarrow{F}^*,0} - \frac{\partial L}{\partial \overrightarrow{F,0}} \overrightarrow{F}\right). \tag{23}$$

Using formula (12), we obtain

$$j^0 = \frac{1}{4} \frac{(\overrightarrow{F}\overrightarrow{F}^* + \overrightarrow{F}^*\overrightarrow{F})}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^*}{2}\right)^{\frac{1}{2}}}. \tag{24}$$

Replacing formula (16) into formula (24), we find

$$j^0 = |\overrightarrow{E}|. \tag{25}$$

The density of the spin pseudo vector is determined by the formula

$$S_k = \frac{1}{2}\epsilon_{ijk} S_{lm}, \tag{26}$$

where

$$S^0_{lm} = -\frac{\partial L}{\partial F_i,0} F_j A^j_{i,lm} - \frac{\partial L}{\partial F^*_i,0} F^*_j A^j_{i,lm}. \tag{27}$$

Here

$$A^j_{i,lm} = g_{il}\delta^j_m - g_{im}\delta^j_l. \tag{28}$$

Replacing formula (12) and formula (28) in formula (27), we find

$$S^0_{lm} = \frac{i}{4} \frac{(F_l F^*_m - F_m F^*_l)}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^*}{2}\right)^{\frac{1}{2}}}. \tag{29}$$

Thus, from formula (26) and formula (29) we obtain

$$\overrightarrow{S} = i\frac{(\overrightarrow{F} \times \overrightarrow{F}^*)}{\left(\frac{\overrightarrow{F}\overrightarrow{F}^*}{2}\right)^{\frac{1}{2}}}. \tag{30}$$

Using formula (16), expression for spin pseudo vector (30) can be rewritten through vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ as follows

$$\overrightarrow{S} = \frac{\overrightarrow{E} \times \overrightarrow{H}}{|\overrightarrow{H}|}. \tag{31}$$

4. Discussion and Conclusion

In this work, we elaborated the Lagrange formalism for electromagnetic field in tensor formalism, in terms of complex isotropic vectors $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. In previous works, Weyl's equation for neutrino has been written in tensor form, in the form of non-linear Maxwell's like equations, through complex isotropic vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. Further, it has been proved, that Maxwell's equations can also be written through complex isotropic vectors $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. In this work, we found the Lagrange function for electromagnetic field in terms of complex isotropic vector $\overrightarrow{F} = \overrightarrow{E} + i\overrightarrow{H}$. On the basis of Noether's theorem, we derived expressions for fundamental dynamical variables (energy, momentum, charge, spin) in terms of strengths $\overrightarrow{E}$ and $\overrightarrow{H}$.

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